Mathematical Methods in Physics-I
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Lecture – 13
Eigen Value, Eigen Vectors

So, welcome back student. In the last class we start a very important concept which is Eigen vector and Eigen values. So, we calculate some Eigen values, but you did not calculate the Eigen vectors also mentioned that Eigen values maybe distinct and it may be same. If it is distinct then I have some extra property with that. So, we will explode this thing in this class.

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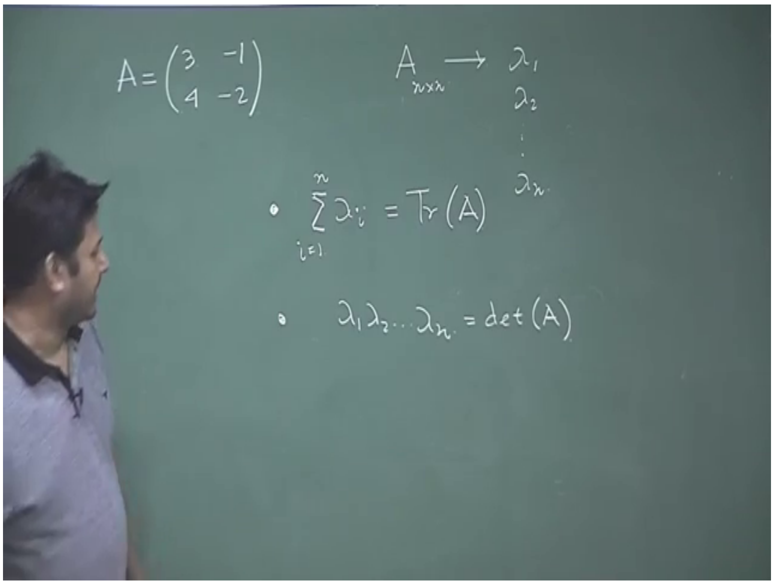

So, let us start with one matrix say A is 3 4 minus 1 minus 2. By the way I should mention at this point one important thing that the procedure that I am following is always for 2 by 2 matrix, but this procedure is exactly same for 3 by 3 and other dimension. This is not a very I mean different thing that is done in 2 by 2 it is easier, it is showing I mean I can get the result quickly that is why I am using always using in 2 by 2 matrix, but when you practice you should also practice for 3 by 3 matrix. It is important to calculate for 3 by 3 3 matrix how you calculate the Eigen values and Eigen vectors, but the procedure is exactly the same that I am doing here.

Apart from that I also like to mention before doing before calculating the Eigen vectors and eigenvalues for this matrix, I will also like to mention 2 important things. Say A is a matrix and I have the Eigen say this is A n by n matrix and I have in lambda 1 lambda 2 lambda n in number of eigenvalues. So, there is a relationship between last class, I mentioned there is a relationship between eigenvalues and Eigen this Eigen values with this matrix trace and all these things. So, please note it very important I will check anywhere I will check for this and all other cases, that summation of lambda i, i stand for 1to n is nothing, but the trace of a matrix A i believe many of you know that, but still it is my duty to show or the mention this this property this important property.

So, once you calculate the lambdas, and if you add them it will be the same as the trace of this matrix. This is one property and the second property is if you multiply lambda 1 lambda 2 lambda n, it will be equal to the det of this matrix. So, lambda 1 lambda 2 lambda 3, it is a n matrix that you multiply. And when you multiply the n number of lambdas whatever the lambda you are getting that will be equal to the determinant of matrix. So, with this 2 information very important and vital information, you can always cross check that whatever the lambda 1 lambda 2 lambda 3 is you are getting is a correct or not. I will going to do that for this matrix you can do that for other matrixes also. We should remember that.

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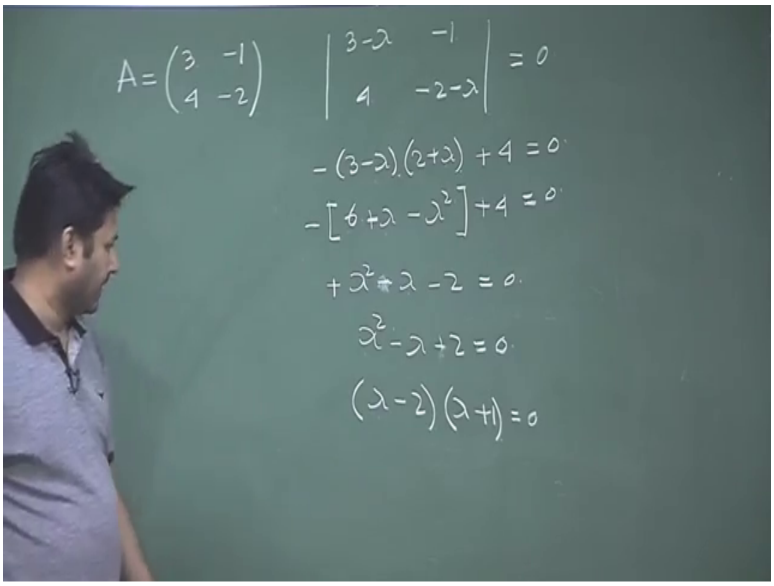

So, my secular equation for this matrix will be 3 minus lambda minus 1 4 minus 2 minus lambda is equal to 0. Determinant of this quantity is equal to 0. So, what I will have 3 minus lambda with a negative sign 2 plus lambda and plus 4 is equal to 0. So, first I need to calculate these things, that what is the value of lambda by using this equation. So, let us do. So, here I have 6 and then 2 minus and so 1 plus lambda. Sims and then minus lambda square plus 4 is equal to 0.

So, I have minus lambda square plus lambda and then here I have minus 2 is equal to 0. Or lambda square minus lambda plus 2 is equal to 0. It seems to it seems me that I can factorize that. So, lambda square I need to have a minus. So, plus. So, I am making one sign mistake. So, let me check where I am doing that is equal to 1. So, let me find that where I am doing the mistake. So, it is it is 3 minus lambda 2 plus lambda with a negative sign fine and then multiply. So, this is a minus. So, there should be a plus sign here. And if a plus sign is here then minus 6, and then minus 2 lambda and plus 3 lambda. So, one lambda is here and minus lambda square plus 4. So, this minus sign gives here I am doing one mistake.

So, this minus sign is plus this is plus. And this will be minus and this will be minus 6 and plus. So, it re minus 2. So, it is something like the here I am making small mistakes.

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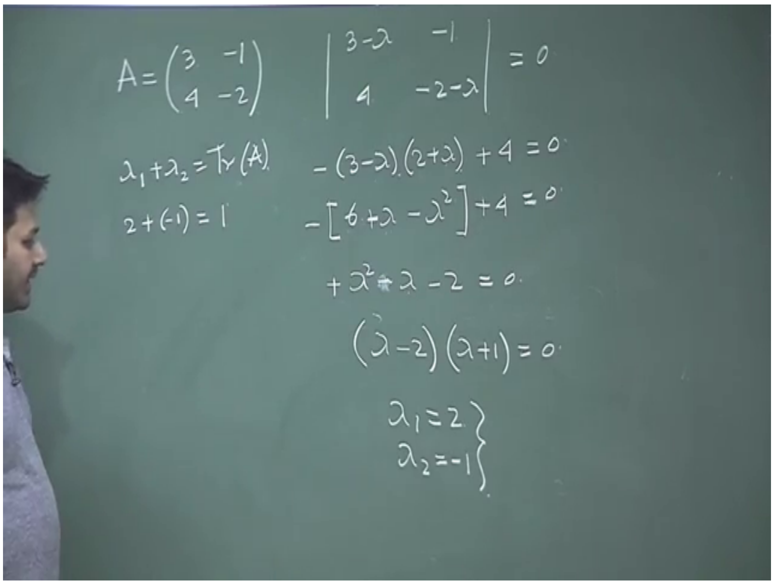

So, now I think I should factorize that. So, it will be lambda minus 2 lambda plus 1 is equal to 0. This I can factorize and I am leading this value. So, lambda square minus lambda 2. So, one lambda value I am getting lambda 1 is 2 and lambda 2, 2 Eigen values I am getting which is 1 and 2 and minus 1 this the first part is done.

So, now I am going to cross check whether these values what I am getting is correct or not. So, my lambda 1 plus lambda 2 should be trace of a here trace of a is 3 minus 2 1. So, lambda 1 plus lambda 2 is 2 plus minus 1 is equal to 1 which is equal to the trace of the matrix. So, first thing is correct second thing is the determinant of that. So, what is the determinant of A.

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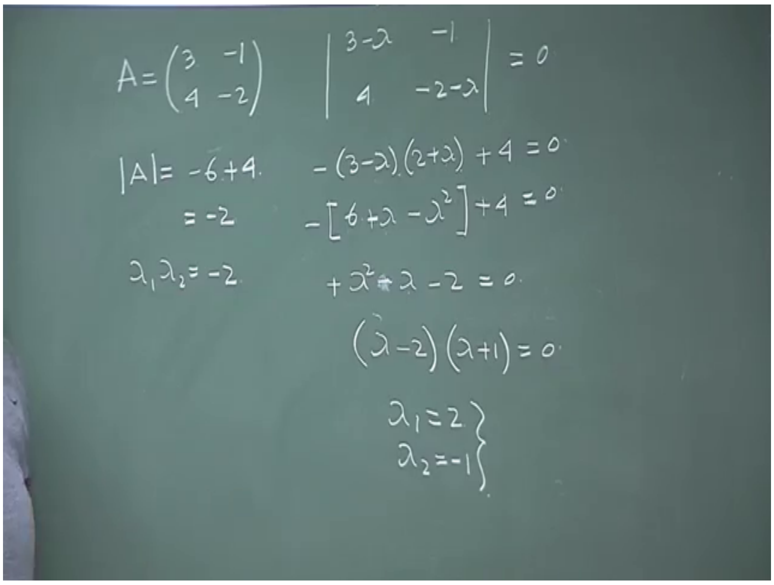

So, the determinant of a is 3 into 2 is a minus 6 and then minus with a plus sign and then 4. So, it will be minus 2 if I multiply lambda 1 and lambda 2 it will be also minus 2. So, my trace and determinant which is related to lambda 1 and lambda 2 is also I can verify that this is correct fine.

So, I can now calculate the Eigen vector. Because so far I didn't calculate the eigenvector, I just mention about the Eigen values.

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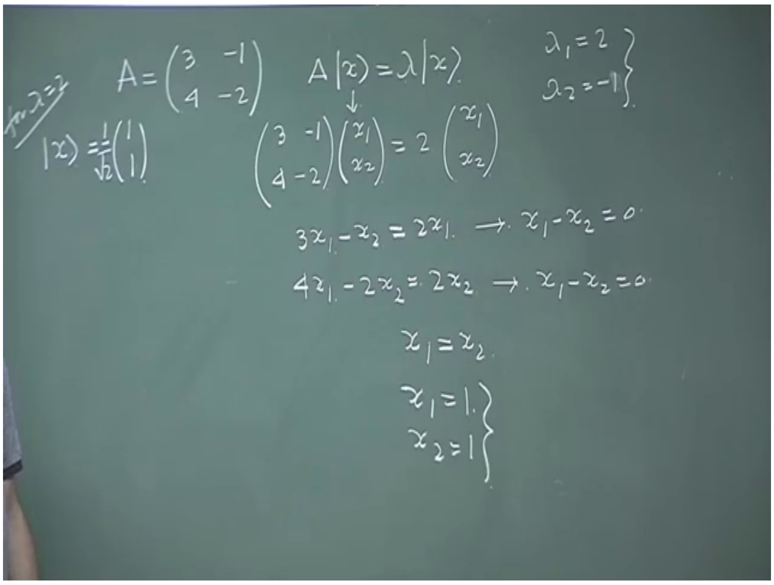

So, Eigen values let me write it here somewhere lambda 1 is 2 and say lambda 2 is minus 1. This now I will going to find out the eigenvalues. How the Eigen value will be there. So, if you remember my equation was something like this the Eigen value equation for 2 by 2 matrix. If I write. So, my this is x 1 x 2, that I need to find out a is this value. Now for 2 lambda I will should get 2 different Eigen vectors. So, lambda say for 2 x 1 x 2 I should have 2 equations, and if this 2 equation I can if I solve that then I will get the value of x 1 and x 2, but always you will get a relation why you are getting a relation I will explain. So, let me first find it 3 x 1 minus first equation minus x 2 is equal to 2 x 1.

So, from here I can have this 2 x 1 if I put it here then it will be x 1 minus x 2 is equal to 0. This is my one equation in simplified form. What about the second equation 4 x 1 minus 2 x 2 is equal to 2 x 2. If I put it here and take 4 common, I will get the same equation back x 1 minus x 2. Nothing new 2 equation I get, but the same 2 equation 2 equations are same to each other. So, I am not getting any unique result for that why I am not getting unique result I will explain. So, let me first find out. How to do this things always you will get this things. So, what you will do that you know x 1 is equal to x 2 now you need put your arbitrary value. So, x 1 I put one then x 2 as to be 1.

So, my first Eigen vector for lambda is equal to 2 is here. So, let me write it here. So, my first eigenvector for lambda equal to 2 is 1 1. I normalize that. So, I make a one over root over of 2 I make a normalize that. So, this is my first Eigen vector. Now why I am not getting these things if you if you check that you will find if this is a Eigen vector x and x, then any multiplication if I multiply any number with this side and this side that will be automatically the Eigen vector of a. So, there is a infinite number of possibility to have eigenvector, if I multiply any number with this 5 6 7. Whatever the number you multiply that will be automatically the Eigen vector of this. So, there that is why you will not getting any unique relationship between this 2, but the important thing is that you are getting the relationship between 2 components that is important always you will do and once you get the 2 components you can get the relationship hence this from this relationship you can find out what is the eigenvectors.

Now, I will do the same thing, I will do the same thing for another lambda. Which is minus 1 equation will be same except lambda should be replaced by minus 1. So, let me clearly write it will be minus 1.

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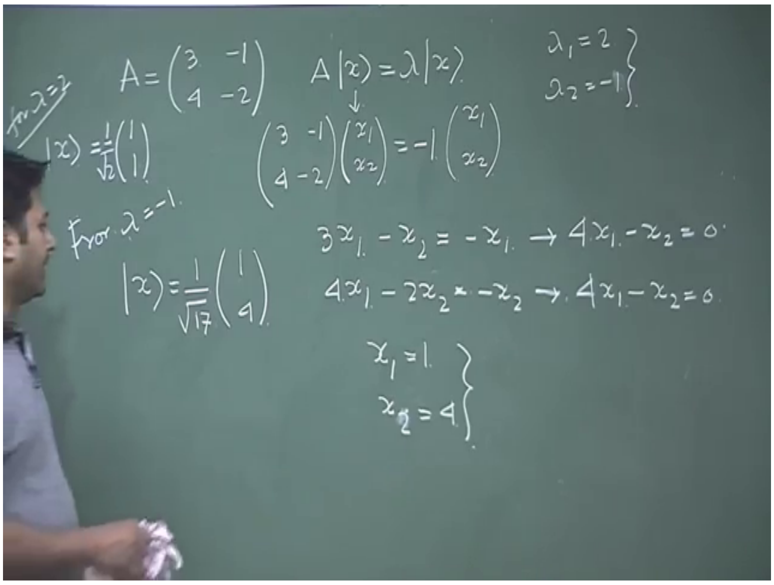

So, again I will have 2 2 equations for that. So, 3 x 1 minus x 2 is equal to minus of x 1 that is my first equation. From this equation I can, if I put it here then I will have from this I will have 4 x 1, 4 x 1 and then minus of x 2 is equal to 0. Again I have a relationship this for the second equation, if I do let us try to find out what is coming in the second equation is there is there anything new I can get from this second equation or not is minus of x 2. Now if I put this x 2 here 4 x 1 minus of x 2 I will get the same. So, 2 same equation I always get. And now again if I put one value then I need to find is what is the other value. So, if I put x 1 say equal to 1 x 4 x 2 will be equal to 4.

So, what will be the Eigen vector for that. So, for if I write for lambda is equal to say minus 1, I have x which is 1 4. And in order to normalize that I always as I mentioned if I multiply any number that will be automatically the eigenvector of that. So, I can normalize that. So, if I normalize it will be root over of say 17, 1 square plus 4 square root over of that. So, it will be. So, 2 Eigen vectors I can generate from that for 2 Eigen values and these 2 vectors are like this.

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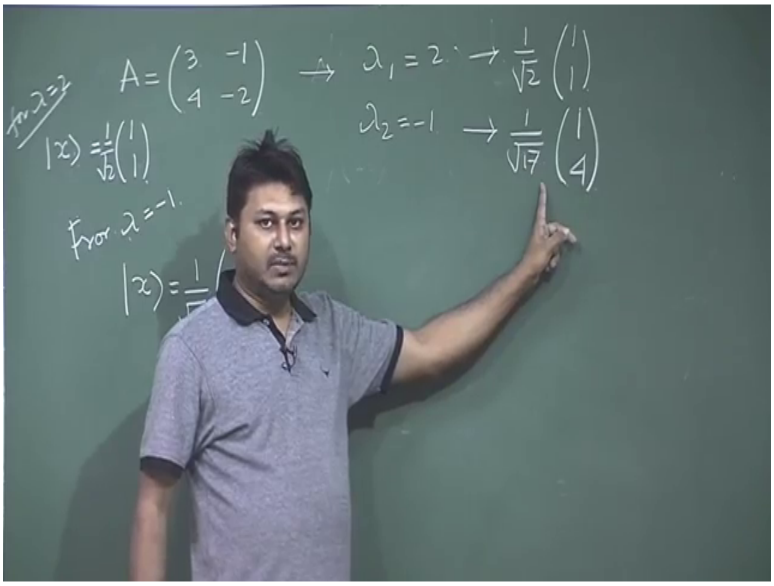

So, final thing is for, this I have lambda 1 is 2 lambda 2 is minus 1. And for that I have an Eigen vector 1 root over of this, 1 1. And for that I will have one then normalized from this fine.

So, next I will want to find out another important thing, that one very important theorem this Eigen vectors this eigenvalues are linear this Eigen values are distinct to each other; that means, they are not same they are not same. If they are not same if they produce the eigenvectors then this eigenvectors can be always linearly independent to each other, that is a very important thing. So, let me let me do this and write it clearly, but before that let me calculate another for another matrix.

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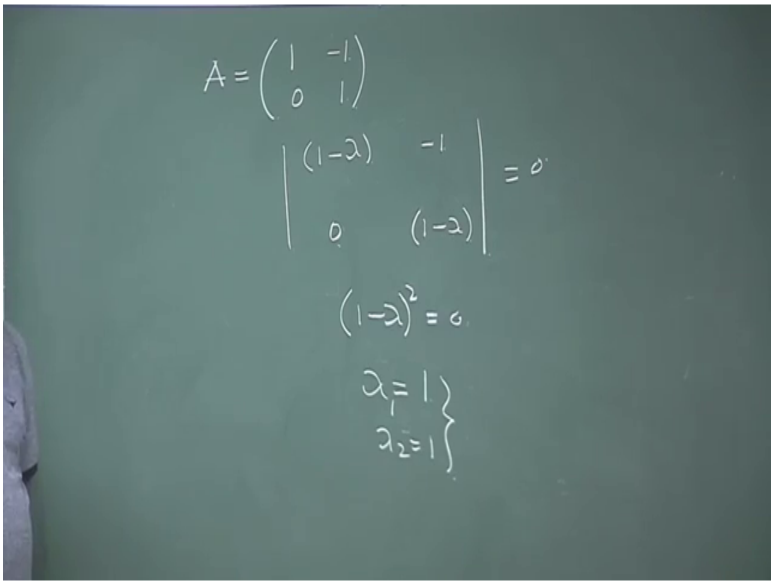

Let me calculate say 1 minus 1 0 1 what will be the eigenvalue and eigenvector for that. So, the characteristics equation will be 1 minus lambda minus 1 0 1 minus lambda is equal to 0. It will be 1 minus lambda square is equal to 0. So, this is a matrix 2 by 2 matrix where the eigenvalues are same 2 Eigen values are the same value. So, this Eigen values are not distinct. This is a different example where the eigenvalues are not same. That is done in the previous example where lambda 1 and lambda 2 are different, but here I have one matrix where Eigen values are same.

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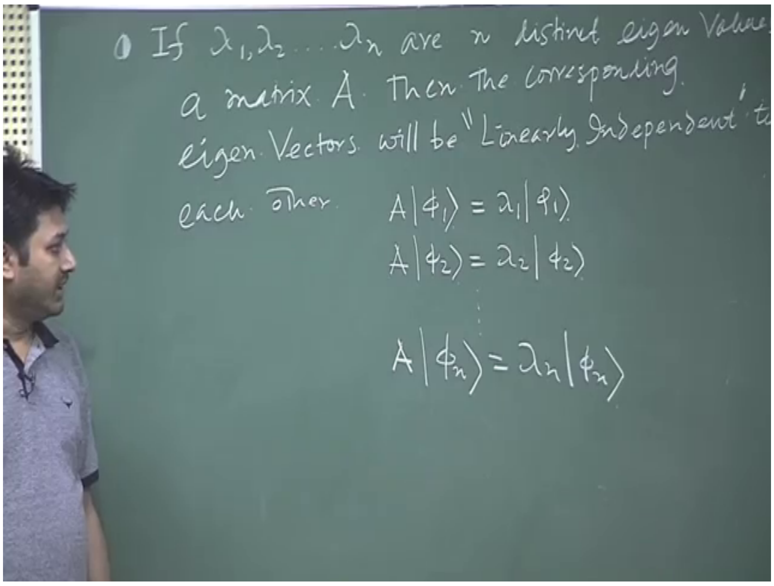

So, what is the difference between the same eigenvalue and different eigenvalue, that I want to find out. So, let us. So, the theorem suggest that, if lambda 1 lambda 2 lambda n are n distinct Eigen values of a matrix a. So, lambda 1 lambda 2 lambda 3 are n distincts Eigen values of a matrix a then the corresponding Eigen vectors will be linearly independent to each other. So, very important statement the statement suggest that for example, I have a and I have phi 1, vector I have lambda 1 phi 1 a phi 2 lambda 2 phi 2 and so on a phi n lambda n phi n.

So, I have these amount of equation every cases I use the Eigen value equation with different Eigen values for lambda 1, I have a Eigen vector phi 1 for lambda 2. I have a Eigen vector phi 2 and so on, for lambda n. I have a Eigen vector phi n and if lambda 1 lambda 2 is not equal so; that means, lambda 1 lambda 2 lambda 3 are distinct they are not equal to each other. So, no lambda i is equal to any of lambda j they are not equal any of them are not equal to each other all are distinct all are distinct then phi 1 phi 2 phi n, this Eigen vectors are linearly independent that is the meaning of these things.

So, I will going to try to prove that. And then we will see this proof this is the tricky proof, but it is a important conclusion that you should note that if a eigenvectors whatever I am Eigen, whatever the value of eigenvalues are there if they are distinct then from that I can always generate the Eigen vectors which are linearly independent to each other. So, let us try to prove that, which is a tricky one, but it is important the conceptually this is important.

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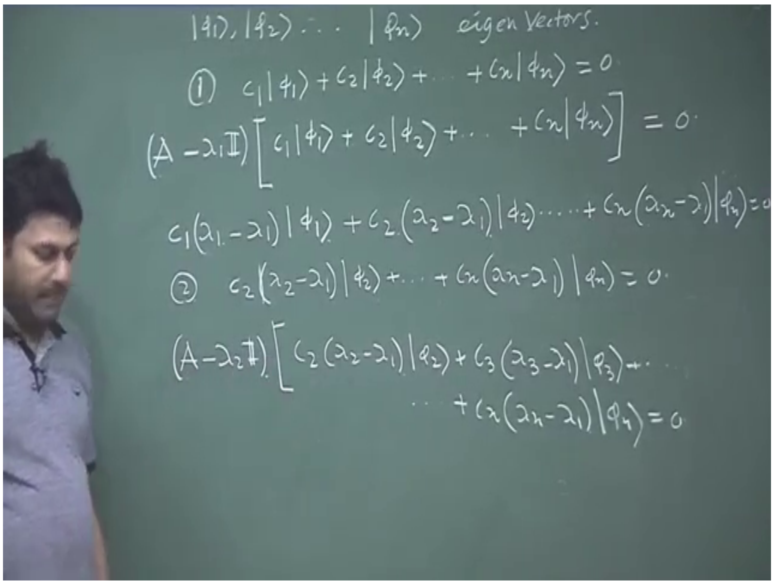

So, let us start with this equation i. So, phi 1 phi 2 phi n as I show are the Eigen vectors, are the Eigen vectors are the Eigen vectors, phi 1 phi n this are the Eigen vectors. So, I am demanding that these should be linearly independent; that means, c 1 phi 1 plus c 2 phi 2 plus c n phi n is equal to 0. I can put this equation if I put this equation then if phi 1 phi 2 phi n are linearly independent, then the only trivial solution is c one is equal to c 2 is equal to c n is equal to 0. So, I need to prove that if these are the Eigen vectors of a with distinct Eigen values then this is 0.

How to do that. So, what I will do that say this is equation 1. I will operate a minus lambda 1 I over this, c 1 phi 1, plus c 2, phi 2 plus c n, phi n is equal to 0 I will operate these over that when you operate. So, a if I operate over that, then what is the value I will have c one first I operate these over that. So, first thing I have is c one lambda 1 minus lambda 1 phi 1, because if I operate over that c a phi 1 is lambda 1 phi 1 and lambda 1 phi 1 is lambda 1 phi 1 with i. So, I will have my first expression this for first value what will be the second value, the second will be c 2 I operate a over phi 2. So, I will have lambda 2 minus lambda 1. Then phi 2 if I go on then I will have my final term c n lambda n minus lambda 1 phi n is equal to 0.

Now, the first term you can see that this is 0 lambda 1 minus lambda 2 is 0. So, what equation after putting this operation over that, what equation I am getting I am getting a new equation which is c lambda 2 minus lambda 1 phi 2 plus c n lambda n minus lambda

1 phi n is equal to 0. This equation is coming from this to this after having this operation. So, next thing what I will do that I will operate. So, this is my equation 2 I will operate a minus lambda 2 i over that. So, a minus lambda 2 i, I operate over my new equation fine. So, if I operate then what happen what will be my first term. So, let me erase this also erase this this I do not require that.

So, what will be my first term my first time will be c 2 multiplied by lambda 2 minus lambda 1 then a will operate to over that and lambda 2.

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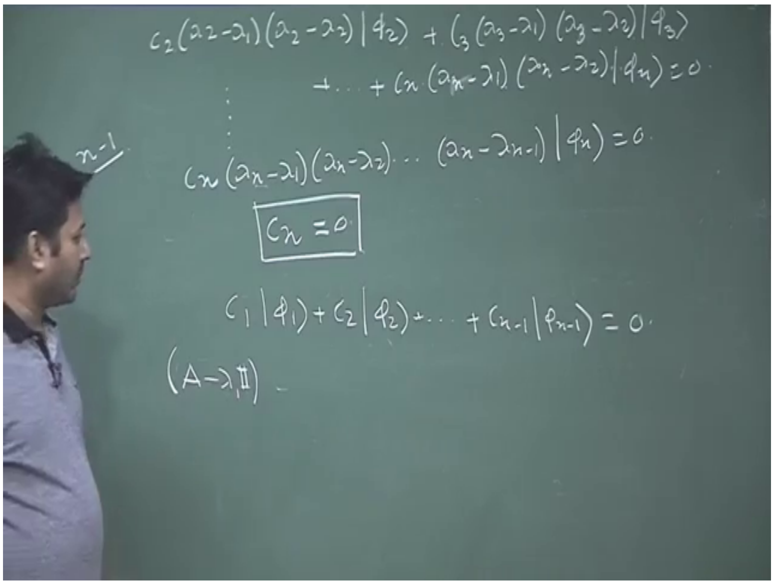

So, I will have lambda 2 minus lambda 2 phi 2, that I have then what will be my second term these over c. So, c 3 lambda 3 minus lambda 1 then a over that. So, lambda 3 minus lambda 2 and then phi 3 and so on. And finally, I will have c n lambda 3 lambda n minus lambda 1 lambda n minus lambda 2 phi n is equal to 0. Now I believe now again this c 2 term will vanish because lambda 2 minus lambda 2 is here. So, if I go on with that if I go on with that with n minus 1 operation. Then finally, what I am getting if I go on I go on with this things and at n minus 1 operation if I operate if I do that n minus 1 time. Then I will have one equation final equation, because c one vanishing in the first operation c 2 vanishing in the second operation third operation, you will find c 3 will going to vanish and so on. So, in the last operation you will have an expression like this this is my operation this is my operation at n minus 1 n minus 1 operation I will have.

Now, important thing is all the lambdas are distinct. So, here you will find lambda n minus lambda 1 lambda n minus lambda 2 lambda n minus lambda n minus 1. So; that means, if I say that since it is a distinct. So, it will not be 0 phi 1 is not be 0. So, what will be 0 c n will be 0 that is my first conclusion c n is equal to 0. If c n is equal to 0 whatever the equation I have start with the equation is something like this. I should stop here and say my new equation is this because c one I calculate this c 1 is 0.

Now this is a one part next part is that I start with this equation and again operating with that a minus lambda 1 i, I operate over this and do that in n minus 2 time. And then find c n minus 1 is 0. If I do that again, I will have one equation and do the same thing, I find c n minus 2 is equal to 0 and gradually I find c 1 equal to 0.

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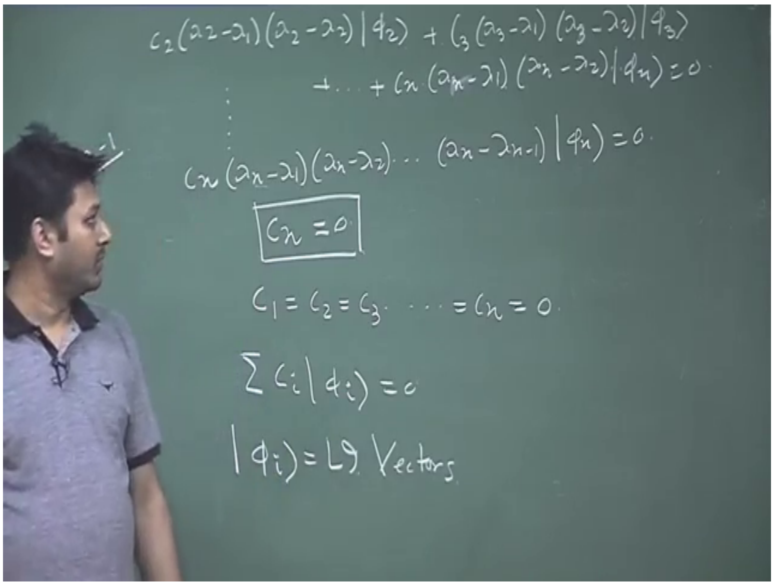

So, all the c I s you can say with this operation with this operation, if I do that this operation repeatedly I can show that c 1 is equal to c 2 is equal to c 3 c n is equal to 0. So, my starting equation was something c i phi i is equal to 0. Now I am saying if it is 0 then this equation only satisfied for all c I s equal to 0; that means, phi i all are linearly independent vectors. So, whatever the phi i, I have is a linearly independent vectors. When I am getting things when all the lambdas are distinct to each other. So, with this proof it is a very important concept.

We know that if a matrix have a Eigen values which are distinct which is not same to each other, then you can find a Eigen corresponding Eigen vectors they are linearly independent vectors. This is a very important concept and you should remember this concept it will help you in future. So, with that today I will like to conclude that, in the next class we will learn more interesting things related to matrix which is diagonalization of this matrix. So, with that let us conclude this. So, in the next class we will start from this we may use this concept little bit and then find how to diagonalize a matrix. So, this is a very important thing also with that. So, see you tomorrow in the next class.