Mathematical Methods in Physics-I
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Lecture -12
Unitary Transformation, Similarity Transformation

So, welcome back student. So, in the last class, we were talking about the unitary transformation through which you can transform one set of orthonormal basis to another set of orthonormal basis and the condition was T matrix should be this.

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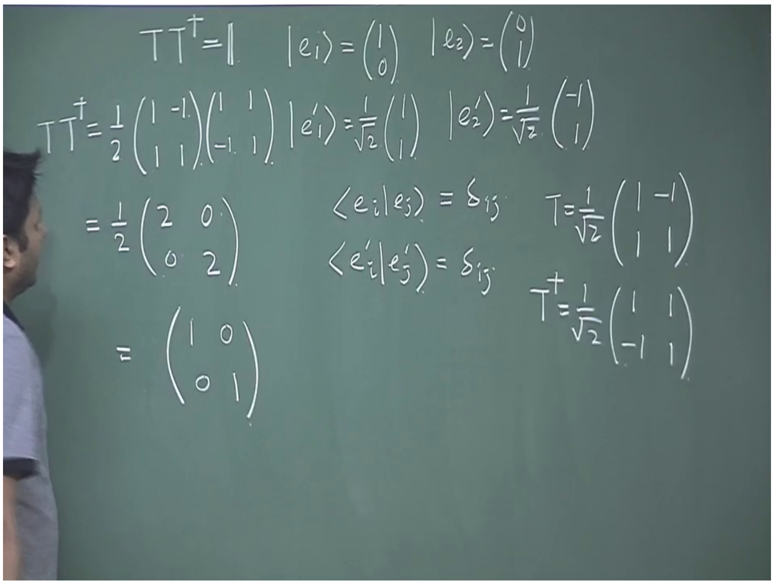

Now, let us go back to the example that I showed in the last class that my e 1 vector was something like this, e 2 vector was something like this and also I have a prime basis which was this. This is another basis.

So, now if you note that e i e j for the first basis, non-prime basis, it is orthonormal and the new one is also orthonormal. You can check that. You can confirm that whether this is orthonormal or not and you will find that both the things are orthonormal. Now, we know that this is my prime basis, this is my non-prime basis and this is my prime basis. I have the basis in 2 d. Two vectors are given to me; here also two vectors are given to me. I know that both the cases, the basis that is forming are orthonormal to each other and also, I know most importantly T matrix. The transformation matrix I know which was something like this.

Now, according to our theory that we developed in our last class suggest that this condition has to be fulfilled. So, just check in this. This small calculation I want to just check whether this T matrix which is transforming this basis to this basis are following that thing or not. So, if I do, then T matrix is there, T dragger matrix is how much? In the last class, I mentioned this is a transpose plus complex conjugate. So, if I make a transpose, it will be something like this.

Since all the elements here are the real elements, there is no need to make a complex conjugate because complex conjugate gives you the same thing because they are real. So, now I need to just multiply these two things; T, T dragger and check whether this is a unit matrix or not. So, let me find whether this gives you half. So, straight way I can write half here and 1 minus 1 1 1. So, half 1 1 minus 1 minus 1 it will be 2, 1 1 minus 1 1 minus 1 it is 0, 1 1 it is 0, 1 1 1 1, it is again 2 divided by 2. So, it will be just 1 0 0 1.

So, that means this transformation which I was doing in the last class, the T matrix they are forming in such a way that their following these things. So, obviously this is a unitary transformation and quite obvious because whatever the vector I am getting here as a basis is orthonormal. So, we just cross verify that this is true, just for checking. So, whenever you transfer one matrix, one set of basis and the set of basis with some transformation, if the transformation matrix is known to you, you can readily verify with this that they are forming a unitary transformation or not. If they are not forming any unitary transformation, then you can ensure that the new set of basis may not be orthonormal, ok.

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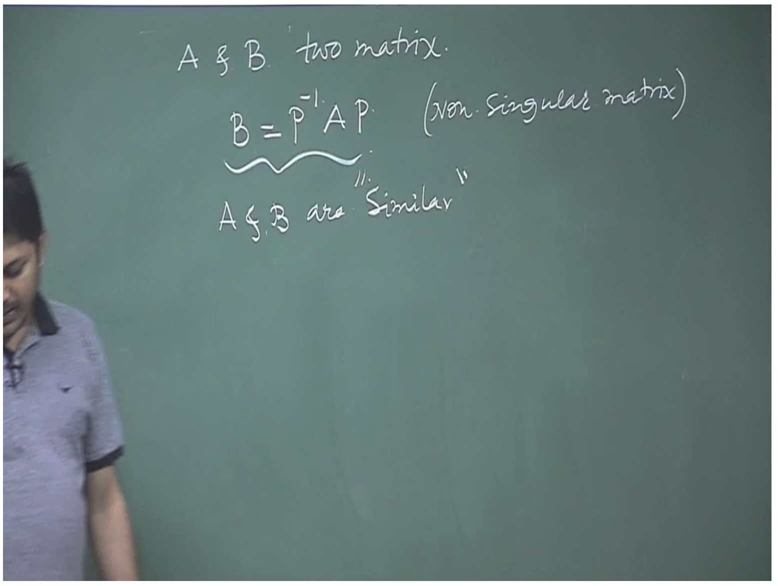

After that we will say something another important thing which is say A and B are two matrix. If A and B are related something like say B is equal to P inverse AP, where P is a non singular matrix, then we say if they are related like this relation we can say A and B are similar. So, by definition I have one matrix A, I have another matrix B, and A and B somehow related to this kind of thing P inverse A P. If I have one matrix P and then, if I do these things I will get something; another matrix A and B is related like this. So, then I called that A and B are similar.

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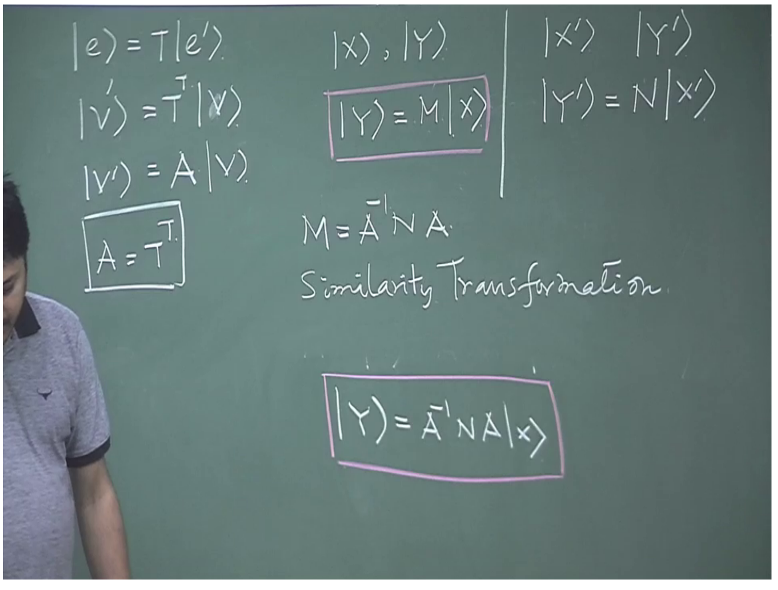

Why it is required? So, let us try to find out why it is important. So, the same old rule that if e is related to e prime with some matrix say T, then the vector in prime basis is related to this matrix. So, T transpose matrix let us put a name on that. So, let me write it is say A, where A is this up to this is fine. So, now the thing is in non-prime basis. I have two vectors say X and Y and they are related like Y is equal to sum matrix say M and sum matrix M and X. Please note that M is a matrix. It is operating over a vector when M matrix is operating over a vector whatever it is giving is also a vector, another vector.

The simple example is a rotation matrix. So, if I have a rotation matrix and if I rotate, if I operate over sum, I will have say another matrix like this, another vector like this. So, rotation matrix is rotate one vector to another vector. So, this is an operation. So, you can consider different kind of operation through which you can change one matrix to one vector to another vector here. For example, I am doing the same thing. I have a vector given XA vector giving Y in prime basis. I operate M over the vector X. I am getting new vector Y. In other word, X and Y are related to this operation in non-prime basis.

What about in prime basis? In prime basis, that means I transform this matrix to that. Also I transform Y matrix to Y prime like this. Then also I can get in the new prime basis. Also, I can write a similar kind of equation in the prime basis. I can have also a similar kind of equation. Now, the question is what is the relationship between M and N? Try to understand the problem. Once again I have X and Y in my hand which is in prime basis which are related to some operation M which is A matrix. I operate over x and I am getting Y. This X and Y which is in non-prime basis can be represented in prime basis also. That is why I put the prime here and in that prime basis also, I can define a matrix N such that I can operate over that and I am getting this matrix back.

So, now the question is, what is the relationship between this and this. So, now I will be going to use my this rule. This rule suggest that V prime and V, this is two vector which is related to A. So, I can write it readily here. So, here if I use this Y prime, let me write it here. Y prime is nothing, but A into Y. The relationship between Y and Y prime is related to A like this N X prime is also related to A like this.

Now, I make a operation A inverse from the left hand side. So, A inverse A Y is equal to A inverse N A. X A inverse A is 1. So, I will have one equation which is Y is equal to A inverse N A X. Now, this equation and this equation are same. This equation and this equation are essentially same equation. If I compare these two equations, then I find a relationship between N and M and the relation is let me erase this part. My M is nothing, but A inverse NA. If you remember just I mentioned earlier about the similar matrix. It is nothing, but the same thing.

So, M and N are similar here because they are related with another matrix non-singular matrix A which is nothing, but the relationship which is nothing, but T transpose. T is the relationship between e and e prime. That means, if I know the relationship between the basis, if this matrix information of this matrix in my hand I can make a similar transformation, that means I can have an operation N which is on prime basis. So, the similar operation I can have M in non-prime basis which is related in this way with the similar way. So, that is why this operation and this operation are eventually the similar operation because they are related to this because they are similar. So, with this A, I can find that these two things are similar. This is called the similarity transformation.

So, I can transform one matrix to another matrix in the operator form because matrix or now deal work as an operator because it operates over the vector. So, this is called the similarity transformation. After the knowledge of the similarity transformation and all these things, now we will go to a more important thing which is Eigen value and Eigen vector of a matrix. So far I am mentioning that A matrix say A is a matrix and it is operating over A X and as a result, it is giving something like Y.

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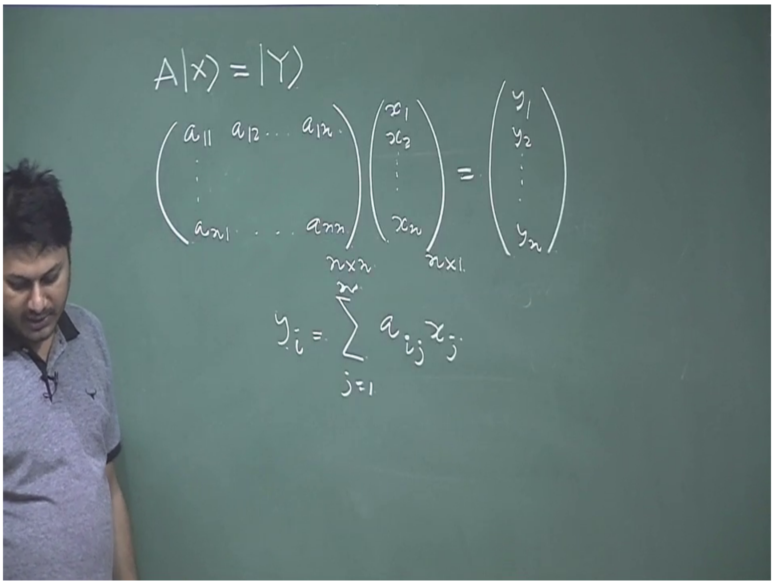

So, I operate this over this and I am getting. So, in my matrix notation, I should write a 11 a 12 a 1n a n1 to a nn. This is my matrix as an operator x, I should write x 1 x 2 x n. If I do this operation, then I will get this as a n by n matrix, this is a n by 1 matrix. So, if I do the operation, I will get another column matrix here. So, this column matrix, the elements I should write y 1 y 2 y n, where the general element y i. How I am getting the general element? It is by the multiplication of this. So, this equation, I will be a i j x j j is transform 1 to n.

This is the general notation of these things. The important thing is that I can operate this over A matrix which is represented in N tuple notation or in column matrix. This will operate and gives a new vector and they are related with this. It is known. I mean it is not a very new thing here. Now, important thing is let me now write here Eigen value and eigen vector.

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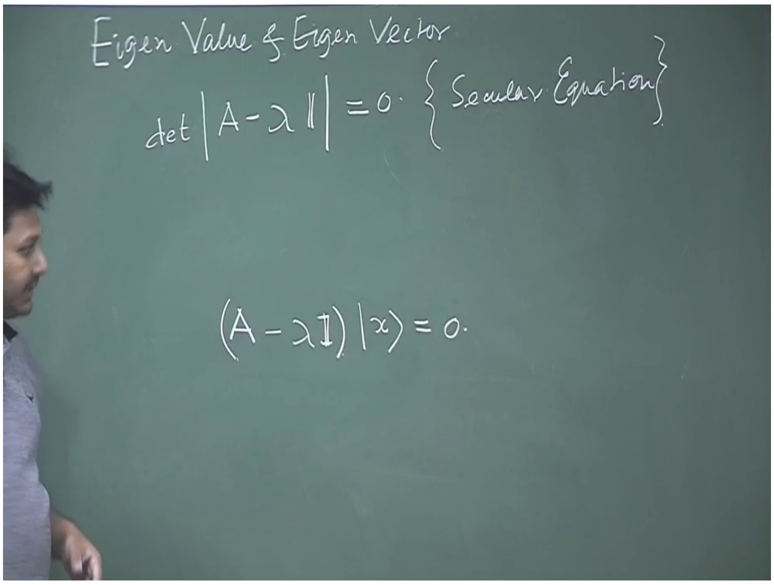

So, now the important thing is that A is an operator that is operating over some vector x. As a result I am getting something like this. What is the difference between these and the previous thing? In the previous thing what I am getting that A is operating over x and I am getting something called y. This vector and this vector are assumed to be not the same thing. It is possible that they are not the same thing. Normally it happens, but here we have something special. I operate A over x. After having the operation, I am getting some vector which is the same vector multiplied by some constant. So, essentially what is the meaning of that if this is a vector, if I operate over this vector, the vector is not changing the length of the vector rather changing.

So, I am just changing operating over these things and I am just changing the length of the vector because lambda is characterized by the magnitude of this vector and this right hand side is changing means I am changing lambda. So, that means, I am operating over that where the vector, the direction of the vector is not changing or what is changing is the length of the vector. So, I am not generating any new vector. What I am generating is vector which is the same vector, but whose length is different. This is very important in quantum mechanics because in quantum mechanics, we will learn that. That is why this concept is important that in quantum mechanics what happened that I am operating this over this A state.

Now, in the quantum mechanics, this vector are represented by state. So, I will operate an operator over A state, so that the state will not going to change. So, right hand side here the vector is not going to change. In quantum mechanics, I will have the state here in state of vector, the state will not going to change. So, this kind of operation is very important in quantum mechanics where you operate over a state with some operation and as a result, you are getting the state which is the old state, but with some value here, these values essentially eigen value of the operator. So, let us write at x is called the eigen value and x is called the eigen vector.

Eigen means characteristics. So, this is a characteristic value of an operator A. This is a characteristic vector of the operator A. So, one is vector, one is value. Eigen value, eigen vector also I can write this in a different way if I take this right hand side. So, it will be something like this. I need to multiply because lambda is a constant. I need to multiply by the unit vector this. Now, from that I can generate one equation and the equation is mod of A minus lambda of i is equal to 0.

This mod sign is not a mod sign rather this is a determinant of that. So, I write and when I am writing that I say mod of this, it is not a mod in that sense. It is just its determinants. So, better to write a date of these things is equal to 0; this is called the secular equation. All the characteristics equation from the secular equation I can get the value of lambda. So, let us go to some examples. If I give some example, then it will be helpful and you will understand few things.

So, let us start with that one. So, I know the basics of eigen value and eigen vector and now, I will try to apply that over some vector.

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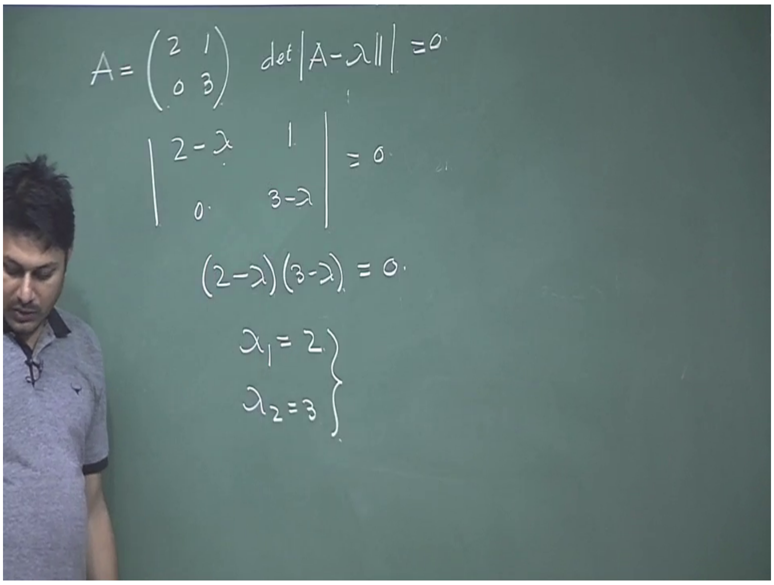

So, let us say 2 1 0 3, this is the matrix. Now, you are asked to find out what is the eigen vector or eigen value of these things. So, I need to find out the secular equation. Secular equation was something like this. So, if I do that, then det of this is 0. So, if I do that, so 2 minus lambda into i. That means, let me write it more clearly. 2 by 2 matrix how it looks like? So, it is a 11 minus a 11 a 12 a 21 a 22 minus lambda 1 0 0 1. I want to find out det of entire matrix. So, what I do? So, it will be a 11 minus lambda a 12 a 21 a 22 minus lambda and determinant of that is equal to 0. The structure will be something like this. So, I will be going to use the same structure. So, 2 minus lambda 1 0 3 minus lambda is equal to 0.

If this is the case, then I will have 2 minus lambda 3 minus lambda is equal to 0 because the determinant, this multiplied by this and this is 0. So, 1 lambda value I will have lambda 1 is equal to 2 and lambda 2 is equal to 3. This is so quite easy that if a matrix is given, then you can figure it out.

So, now I will take another example to show that it is not necessary that I was always get say 2 1.

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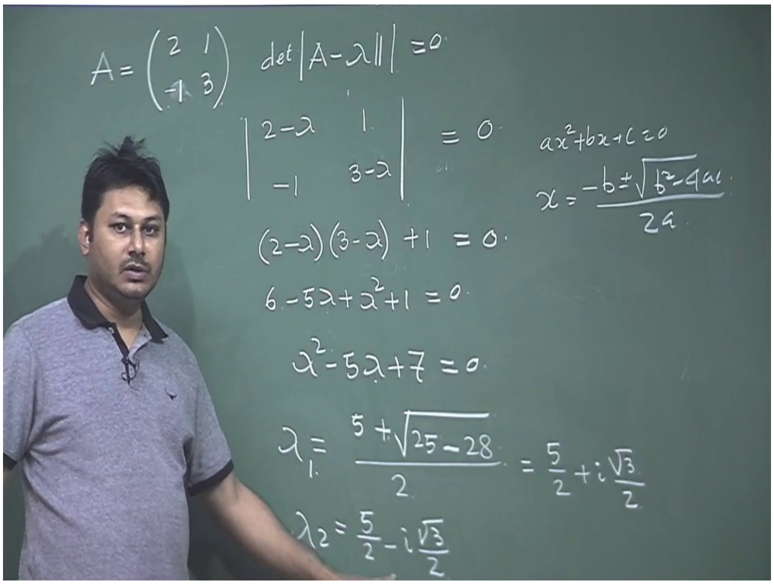

If I change this 2 minus 1 say what happened? So, this was 0. So, that is why I am getting some value which are always real, but what happened in this case 2 1 minus 1. So, let us again try to find out the secular equation is 2 minus lambda 1 minus 1 3 minus lambda. Because of this non-zero value, now I am getting something 0.

So, 2 minus lambda 3 minus lambda and then, this is minus 1. So, I will get A plus 1 here. So, my question is now slightly different. So, now I need to solve this. This will be a quadratic equation. I need to solve this. So, it will be 6 minus 5 lambda plus lambda square plus 1 is equal to 0 or in other word, it is lambda square minus 5 lambda lambda square minus 5 lambda plus 7 is equal to 0.

So, now I know how to solve that. So, lambda 1 is minus b plus root over of b square which is 25 minus 4 a c. So, minus 4 a into c, here a c is I will just use a x square plus b x plus c equal to 0. The solution I am using I believe all of you are aware of these things. I am just using this one. So, here 4 a c is 28 and 2. So, 1 lambda 1 is this. So, it is nothing, but 5 by 2 plus i. This quantity is minus 3. So, I will put a i here.

So, root over of 3 by 2 and lambda 2 is nothing, but the complex conjugate of this. This sign is changing. One is plus and one is minus, fine. So, with these two examples, I show that there is a possibility that eigen value maybe complex and I mean not an issue.

So, I will be going to stop here. In the next class, I will again start with this point and try to find out what happened when the eigen values are same. Here in both the cases, both examples I find two eigen values which are different and there is a possibility that I have 1 1 matrix where the eigen values are different or same.

So, what happened it is different and what happened it is same? It is important thing that we will be going to discuss in the next class. Also, there is a relationship between eigen values and the trace of the matrix and determinant of the matrix that I also show in the next class. So, let us stop here. In the next class, we start from here and show how the different eigen values that is I am getting which is if it is degenerated, then I have something extra with that. So, what extra I will get from that? So, in the next class, we will discuss that. We will stop here. So, see you in the next class.