Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 10
Transformation of Basis

So welcome back to the next class of linear vector space.

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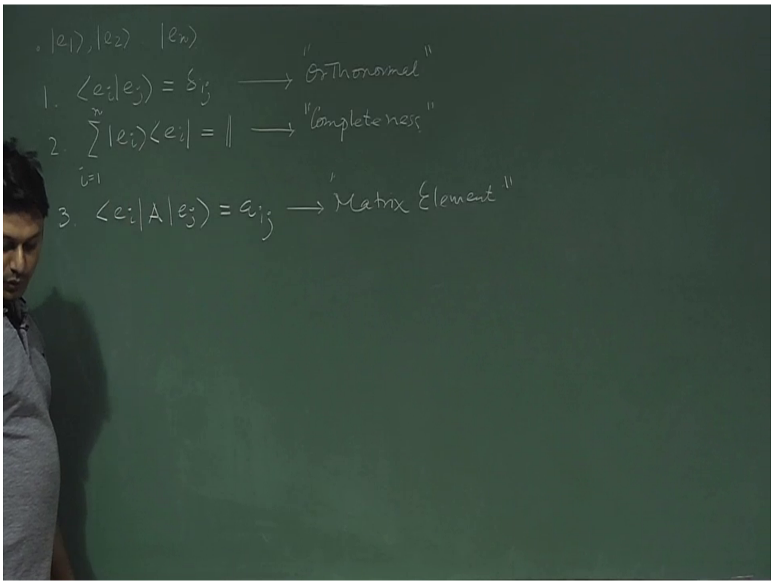

So, first let me write what I find in my last class. That, e 1 e 2 e n, if these are the basis then they are following few important properties. One is they are orthonormal, if they are orthonormal basis, then mathematically this is the first property. Second property, this quantity which I say the projection operator is this, this is called completeness. And finally, this is the matrix elements. So, first one is orthonormal. Second is completeness, and third is matrix element. So, these 3 properties the basis hold and this is very important to know, that was my last class.

So, now in today's class, we will try to understand few important aspects which is called change of basis or transformation of basis. We know what is basis.

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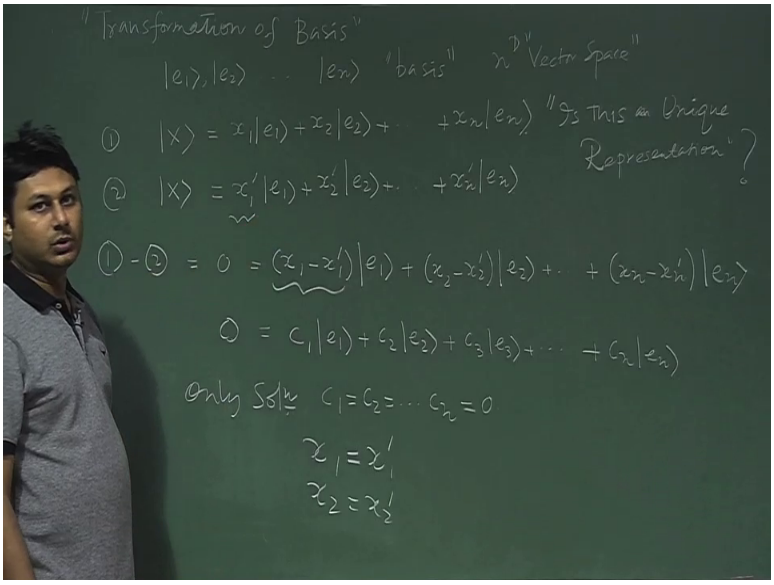

Now, you need to find out what happened when the basis is transformed. Let me explain that in a more clear way before that. So, say e 1 e 2 e n are basis forming a basis in n dimensional vector space. So, they are farming a basis in n dimensional vector space n D vector space. When they are farming the basis in n dimensional vector space that eventually means that any vector x, can be represented in terms of these basis, fine. No problem in that. So, I have a basis. And what I am trying to find out a vector, a vector it decomposes into these bases, where these are the components. These are the components. Now the question is this representation, unique is this, in is this unique representation, is this a unique presentation. The question is the unique representation; that means, is it possible to write these things like this; that means, I am writing the same vector using the same basis, but I am now changing the component. Is it really possible? That is the question, and the answer is no.

When I am having a basis in my hand, I can able to represent this vector uniquely and this components are unique. There will be no other components and this component if I write, try to write it will not be possible. So, let me do that in a more convincing way. So, say this is my equation 1, I can write the same vector with this this is my equation 2. If I subtract equation 1 minus equation 2. If I subtract equation 1 and equation 2, then my left hand side will be 0. What will be my right hand side? Right hand side will be x 1 minus x 1 prime e 1 plus x 2 minus x 2 prime e 2 plus xn minus xn prime en, I will have a the expression like this.

Now, this is the component; that means, this is a scalar quantity. I can write this scalar quantity in this way. So, on when c 1 is x 1 minus x 1 prime. Left hand side it is 0. Now you should remember this expression. This is a very important expression; this is a linear combination. This is a linear combination of the vector e 1 e 2 e 3 en linear combination. And now e 1 e 2 are basis. So; that means, e 1 e 2 e 3 en, these are linear set of linearly independent vector. Now if I have a equation with the linearly independent vector like this form, then the only solution is only solution that is possible is c 1 is equal to c 2 is equal to cn is equal to 0. If c 1 c 2 c 2 c 3 cn is equal to 0, then only we can have this equation to be satisfied. If this is 0, then readily I find that x 1 is equal to x 1 prime, x 2 is equal to x 2 prime and so on; that means, I cannot write if vector when a basis is given to me with another component.

So, the components are unique. You can prove that components are unique, when we are not using any other set of basis. So, first thing I learnt, we learnt that, when a basis is given to you, I can expand a vector. And this vector can be represented uniquely for that basis only. Now it is not necessarily that you will have only one basis in n dimensional vector space there should be another basis also.

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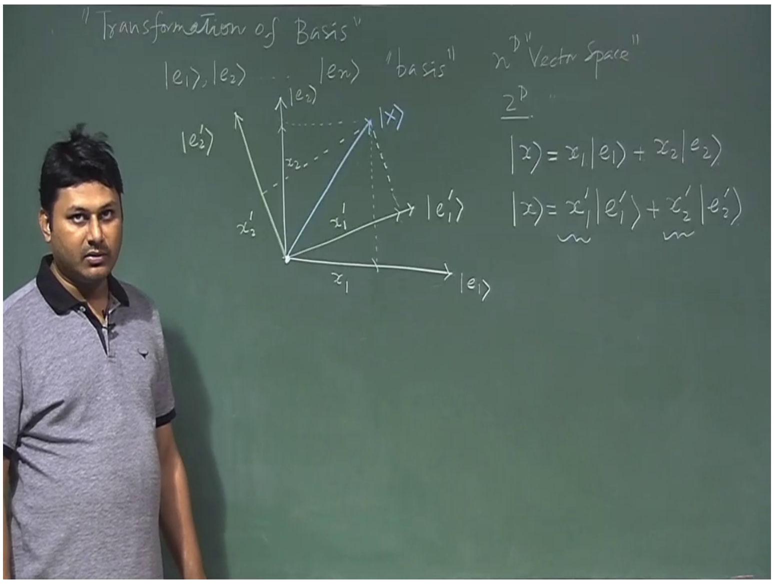

So, let me give you one example. Say this is my preferred basis, natural basis. Normally I took this natural basis in this way e 1 and e 2, perpendicular to each other. And along this and this direction which is perpendicular to each other. Now I am taking another basis, which is also perpendicular to each other. This is my e 1 prime and this is my e 2 prime. These 2 are another basis in 2D mind it this is for 2D for 2D, I can take these 2 different basis. Now if a vector is given to you, this is a vector a general vector x is given.

So, I can represent this vector either this component or this component. So, same vector is still, vector is same. I can represent the vector according to the choice of the basis. For example, here I can write this take a projection here, take a projection here the normally, the process we use. And then this vector will be represented in terms of e 1 and e 2. If I write this vector x, say my component is x 1 which is this one, my component is x 2 which is this one. So, I can write it this way, but also I can write this vector for this basis. This component say it is my x 1 prime. This component say x 2 prime. If I do that, then the same vector I can write in prime form. This prime form.

So; that means, if a vector is given to me, that when I represent a vector, I can represent a vector according to my likings. And this likings is totally depends on how I choose my basis. If this is my natural basis, I can readily write my vector. If this is not a natural basis it is different basis, but still it is a basis, then this component will going to change. In this particular class, we need to find out how these things happen, I mean how I can find out these components in a different basis. If I change the basis how this component I will going to figure out that we need to learn. So, before that let me go back to some basics which is also required here, some simple matrix relationship or matrix notations which is useful here.

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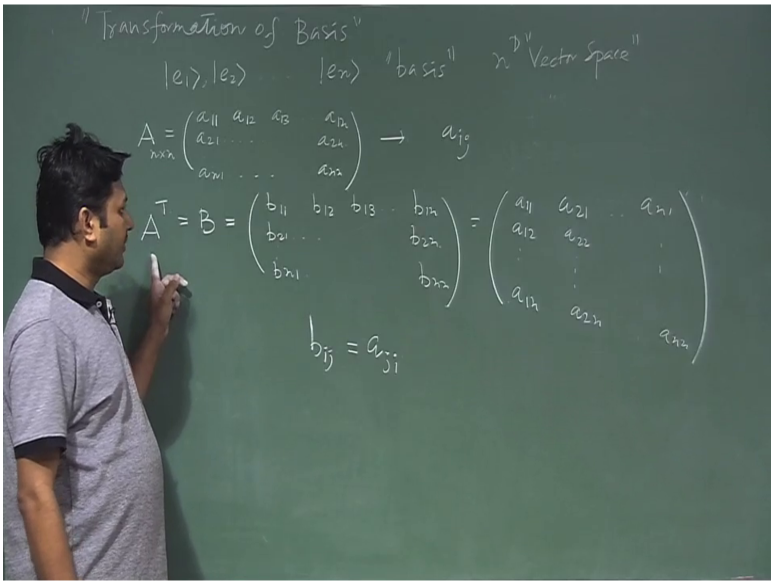

For example, a matrix x A is represented by this, 1 1 say n by n matrix make it simple 1 2 1 3 1 n, then a 2 1, a 2 1 and then n 1 n 1. So, the general component is a ij. General component of this matrix is aij. If I transport this matrix, what is the meaning of transposing? If I transpose this matrix; that means, my say when I transport, this matrix this component, the I write it as a another vector B. So, my B, this is b 1 1, b 1 2, b 1 3, b 1 n as usual b 2 1 b 2 nm as usual b n 1 2 b nn, but since it is B transpose of this, this is nothing, but the transpose of this quantity. So, a 1 1, a 1 2, a 1 n, a 2 1, a 2 2, a 2 n, a n 1 a nn. The general component of this matrix is b ij, no problem with that which is basically equal to a j i.

Because now, it is changing, B transpose, if I make B transpose of a. Then eventually what I am doing that I am interchanging that I am thus interchanging that, this is one important aspects that not a very trivial thing, but still you need to remember that, that I am just changing that that is first. Second thing is what is the matrix multiplications say AB are 2 matrix.

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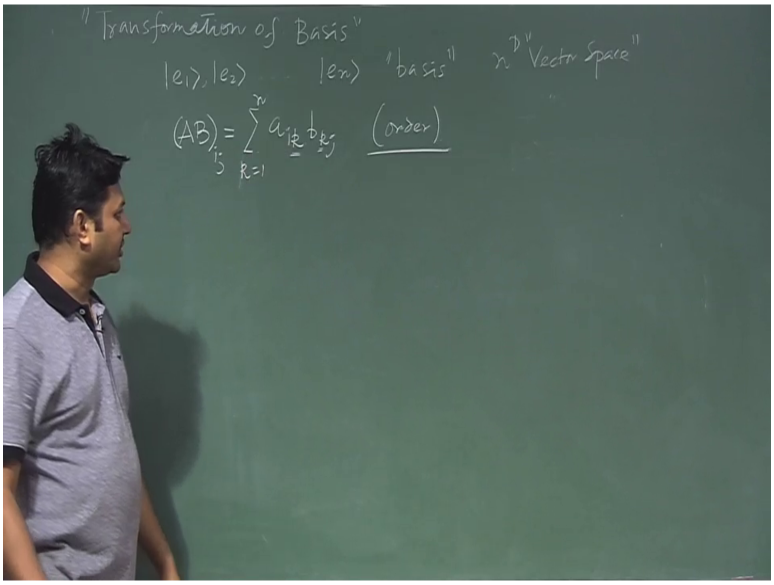

When I multiply this A and B, the general component ij general component of this matrix is represented bya i k bkj, i k is the (Refer Time: 14:43) running from say 1 to n. If it is n by n matrix, here one important thing you need to note that the position of k a valid matrix multiplication can possible when this 2 k are sitting in this this order. So, order is important. So, order is something, which is important in matrix multiplication a valid matrix, multiplication 1, when I try to multiply 2 components. So, this k and this k should be sitting side by side so that they can be multiplied with this notation. This is another important thing; you need to know. Apart from that there are some matrix relation.

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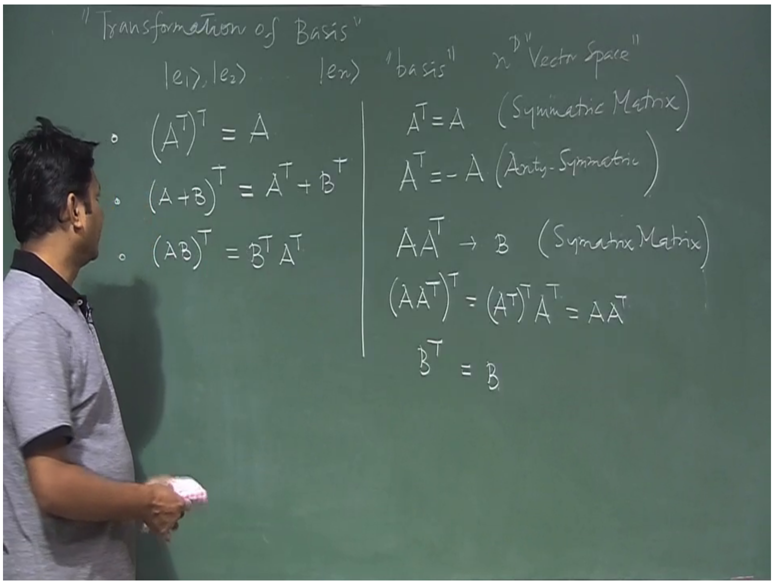

For example, B transpose, if I make B transpose of this quantity, it will give me the same matrix A is a property of transpose which is also important. Then A plus B, 2 matrix A and B.

When I make B transpose, it is same as B transpose plus B transpose. Third is the multiplication AB whole transpose. If I want to make this, it is B transpose, A transpose. Few things also you need to know regarding the matrix, which is if A transpose is equal to A, because this matrix is symmetric matrix. If A transpose is equal to minus A, we call this anti symmetric or skew symmetric matrix.

Now whatever A, any matrix if I multiplied with another matrix, which is A transpose, this matrix, this matrix is automatically symmetric matrix. Why? This is say B. This A B transpose matrix, I say this is another matrix B. So, when I make A A transpose whole transpose, it will come like A transpose, whole transpose A transpose. Which is A A transpose. So, this is my B transpose, what I am getting is nothing, but B. So, B transpose B; that means, A A transpose whatever the matrix I am taking is a symmetric matrix always.

Well, after having a very brief idea what is symmetric matrix, how the transpose is working because transpose is important in transformation of basis. So, let us go back to our problem our original problem transformation, basis transformation. So, this is the basis that is given. I want to find out for another basis, this is one basis. And this is another set of basis. This is another set of basis. This is one basis; this is another set of basis.

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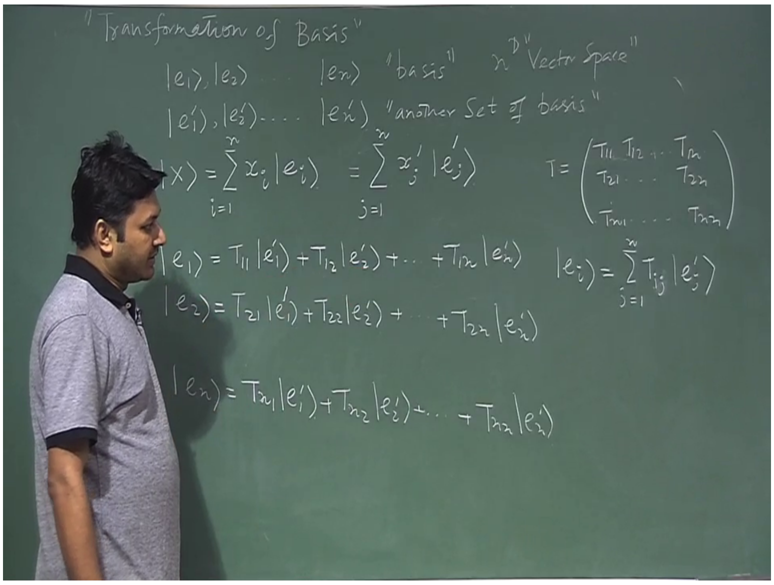

If 2 basis are given, then how to find out the components from this basis to this basis that is the thing I need to do. So, my vector x can be represented in this basis like xi ei, i tends to 1 to n. In a similar way, I can write it that is vector can also be represented as this. I am just changing, this component. This is dumb index I can change that. So, I need to put a prime here sorry. So, one vector is represented in this. Same vector can be represented in this. Now what I am doing my e 1, vector this is also a vector in a vector space. And when the vector space is there I can write this vector in terms of the other basis also. If I write then let me write the vector like 1 1, e 1 prime these are the components 1 2 e 2 prime T 1 n en prime.

I can write a vector e 1 in terms of other vector, where these are the components of this. That I always do T 1 T 2. In the same way I can write e 2 as T 2 1 T 1 plus e 2 2 e 2 prime left hand side all these things will be prime and so on, en will be T n 1 e 1 prime T n 2 e 2 prime T n n en prime. So, I can find a general matrix form for that. For e. So, this T is nothing, but the matrix T 1 1, T 1 2, T 1 n. T 2 1 T 2 n and T n 1 T nn. This is a matrix n by n matrix I have. And my ei can be represented in general ei, can be represented in general summation of T ji ej prime, where j runs from 1 to n. From this expression I can write the general form. So, for example, e 2, T 2, T 2, T 2.

So, this let me check whether I am doing. So, no here should be i j, any e vector, e 1 e 2 or e 3 can be represented in inform of the matrix like this. Now what I will do that if this is known, this is known I will just replace this thing here. It is replaced this things here.

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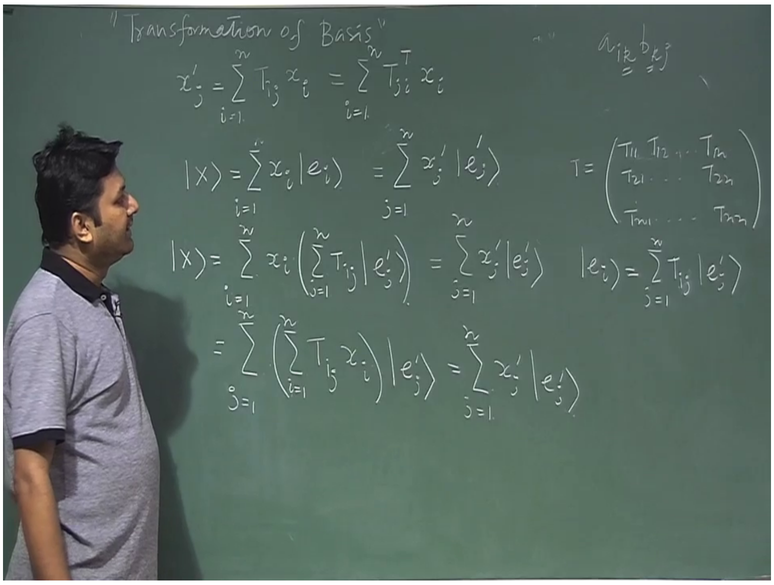

So, my x. So, this I am going to replace ei, I will replace this quantity. This in the right hand side it is already there. So, this I can rearrange slightly. So, j is equal to, I can shift this summation sign bring this summation sign here. And then I put this summation sign i, here 1 to n. Then tij xi bracket e j prime, which is fine. So, left hand side and right hand side if, I compare the component of e j prime and e j prime I can figure out. So, component is now in my hand. So let me so, component in in in my foam. So, x prime j is nothing, but summation of i 1 to n, summation of i 1 to n let me check. Once again if it is let me check once again everything is correct or not. Then I go to the next thing summation of i T i j ej prime.

I just replace this. And tij, I just replace here eij is there. So, let me put it here. This form and then let us try to find out how the things is going on. Tij xi, this is the thing summation of these things. This now one thing you should note here which is important that it is over sum i, which is running from 1 to n tij is here. And i which is running j is fixed j th component, I tried to find out, but this is the order of i is mismatched. In the just a few minutes ago. I show that for valid matrix multiplication, the order is important aik bkj. This order one has to maintain in order to have a valid matrix multiplication here.

This is a vector components, these are the tij is forming a matrix. When they multiply to each other again this is a matrix find the multiplication, but the order is not satisfying. In order to make this order change, I can do a trick. And the trick is that I put a transpose and then I put tij. So, whatever the value I have if I make a transpose over that then j and I will be interchange that I just showed before and if I do that, then I have a expression where I have a valid matrix multiplication form. So, eventually when I have these things, when, I have these things. Then T matrix if T matrix is given to me. Then my x vector in prime in prime frame the components of if x vector in prime frame x vector component.

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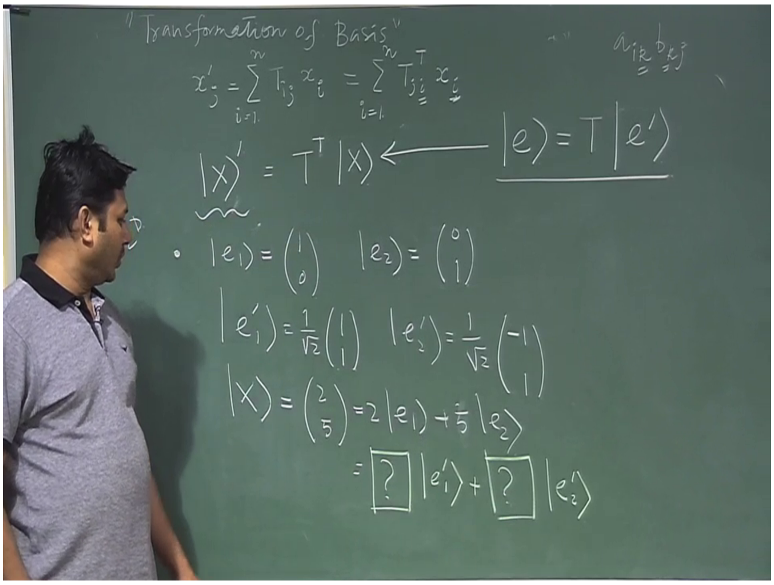

I should write component of vector X in prime frame can be represented by T transpose and X this, X is X vector in non prime frame or no normal frame. So, what I am getting? So, let me conclude with that. So, next day I will again come back to this this issue. My e vector and e prime vector is related to a matrix like this. If my T vector is known if I able to find out my T vector from that, I can find the x vector which is a given vector whose component; how this component is related to the previous component this, utilizing the matrix T.

So, I know the T and make it inverse I know what is the component of X vector in non prime, I calculate that and I readily find out what is the thing in X prime. So, I will I will show a very simple example with that the problem. The problem just me write down the problem. Then today I am not going to solve that I will call going to solve that in the next class, the problem is 1, e 1 vector is given natural basis this e 2 vector is given basis this.

And prime basis is given like 1 by root over of 2 1 1.

So, I have 2 vectors in 2D. I have 2 vectors basis vectors like this, another basis vector like this. The problem is now a general vector X, which is say 2 5 can be represented as e 2 e 1 plus 2 e 2 5 e 2. The problem is fill in the blanks. If I want to write in terms of e prime, what should be in the blanks? This is the problem. This is the definition of the problem. So, using this whatever I figure out here, using this treatment or this recipe I will fill up these blank. What should be the value here, I will going to fill up. So, with that. So, let u stop here because it will take some more time to calculate all these things and I need to discuss few other things also.

So, with that with this problem with this definition, I will just finish. In the next class. I will start with this problem which is very important in understanding the transformation of basis very simple, in a very simple 2D structure, it can be done in 3 d structure 4 d structure n D structure. But the rule is that if e 1 and e 2 is related to a matrix T, if you know this matrix T then using this matrix, you can figure out your vector in e prime frame also because this is known. For example, here 2 5 in non prime frame is known. What you do not know, that how to write this, in the prime frame, where the prime frame is given by these 2 vectors that we will do using this treatment.

So, with that let me conclude here. So, see you in the next class. And then we will discuss all these things. In the next class, also we will try to understand what is the transformation and what is the orthonormal transformation, what is called the similarity transformation. There are different kinds of transformation that you can find with this formalism. So, that all these things, we will discuss in our next class, with that see you in the next class.