Astrophysics & Cosmology
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Lecture - 08
Fluid Mechanics

Welcome, we have been discussing the fluid equations and fluids flow and in the last class we derived the 3 equations essentially, that we used to govern, to determine the flow of fluids.

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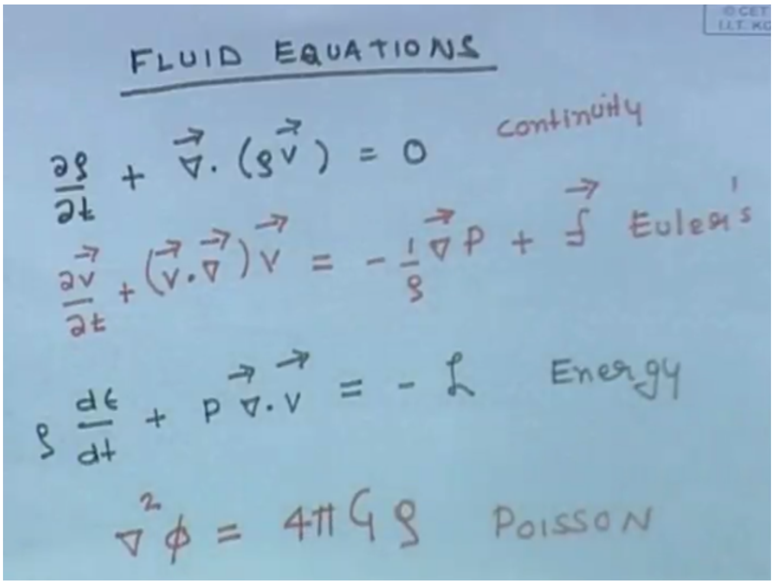

So we had first the continuity equation and then we had Euler's equation, the Euler equation and we had the energy equation. And in some of the situations that we are interested in we also the fluid is self-gravitating. So the body force comes from its own gravitation potential, from the gravitation it generates on itself. So f is -grad phi and that phi is governed by this equation, the Poisson equation.

It tells you how the matter, the fluid itself produces a gravitational field. So these are the fluid equations and in today's class we shall consider a few examples of solutions and implications of the fluid equation. So let us start off with the simplest possibility which is hydrostatics.

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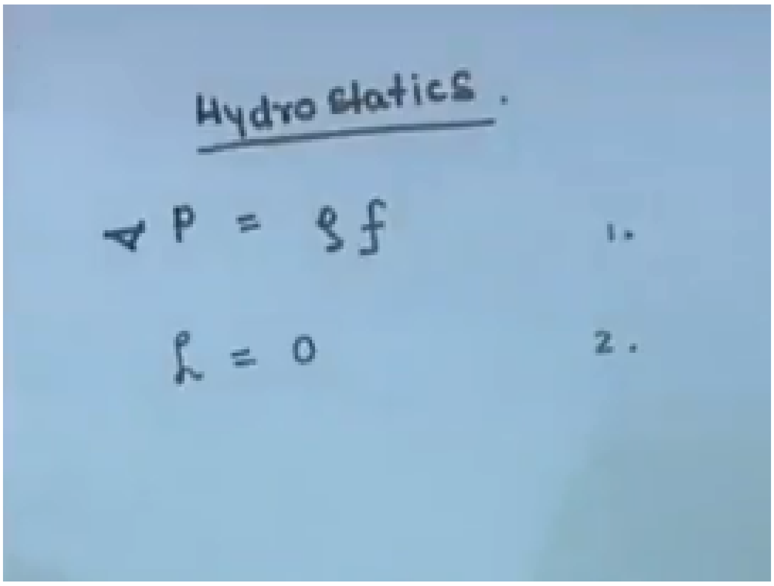

So we shall discuss hydrostatics first, and what we mean by this is that the fluid is at rest. So there are no time derivatives and the velocity of the fluid is also 0. So all time derivatives vanish and the velocity is also 0, so under this condition, the continuity equation is trivially satisfied, because all time derivatives vanished and the velocity is 0. The Euler's equation, the first 2 terms involving the velocity do not contribute.

And we have essentially grad P = rho*the force per unit mass or the acceleration, so that is the first equation, that is the Euler's equation. So the 2 forces, the pressure gradient force and the body force they have to cancel out. And the energy equation again the time derivative is 0 the velocity is also 0, so it tells us that the heat loss, the rate of at which the heat is lost has to be 0.

So hydrostatics we have 2 very simple equations which have to be satisfied. Let us consider a few examples of hydrostatic equilibrium where we have a fluid which is in hydrostatic equilibrium. So let us first a fluid which has uniform density more or less and same temperature also, like water.

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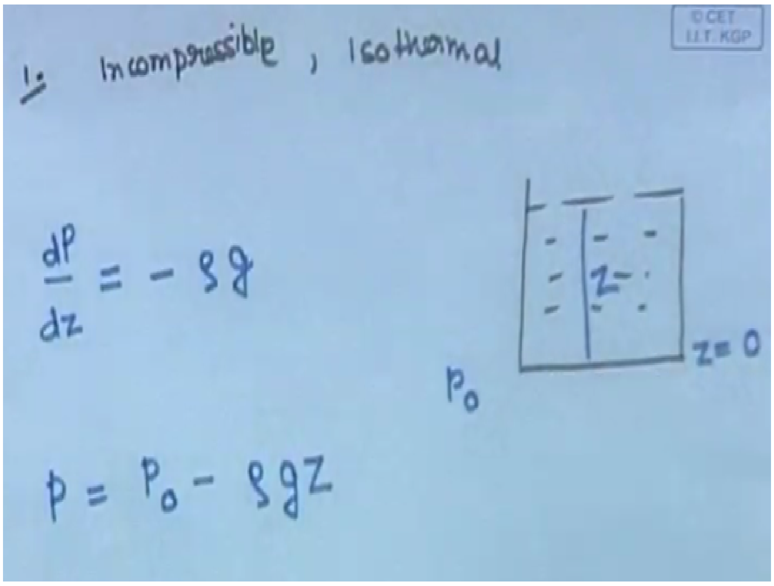

So let us say it is incompressible, and the temperature isothermal. Temperature is also the same throughout. So the example is a tank full of water or some fluid like water which is nearly incompressible for the many purposes here. And this is the z direction let us say that choose the z direction upwards. Let us call this the z direction and this z = 0, and the pressure here is let us say P0.

So we have this equation it is now, since the temperature is the same throughout there will be no heat loss from one part of the fluid to another so we did not bother about this. Let us look at this equation, this equation tells us that dP dz =, now we have the density of the fluid and this is force per unit mass, the body force and in this case it is just the gravitational acceleration, so this is minus it acts downwards, the force acts downwards the gravitational acceleration so -rho*g along the z axis.

And the solution to this straight forward, so we have P = P0 -rho gz. As you go up the pressure keeps on falling, or there you may say that there is pressure due to the column of the fluid above the particular point that you are looking at. This is a very simple application of the fluid equations, hydrostatic.

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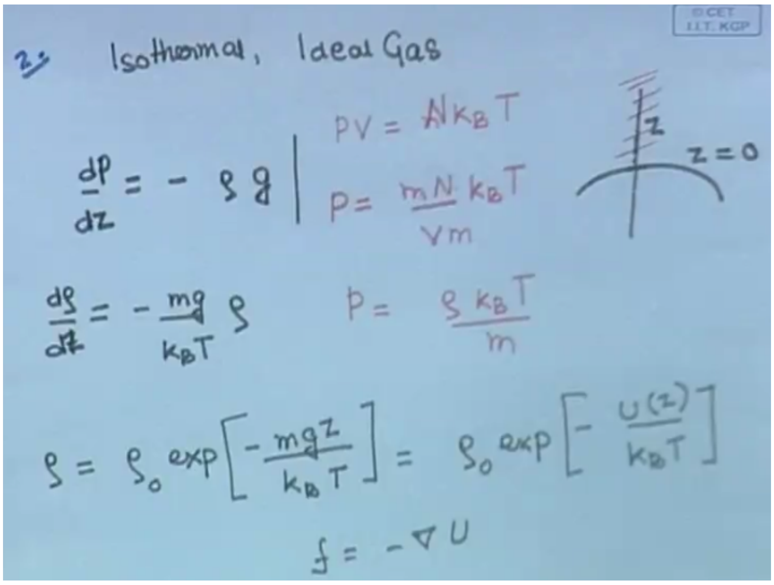

Next let us consider same thing. Let us consider a gas an ideal gas which is isothermal. So we can think of it, an ideal gas so it is isothermal at the same temperature and we have an ideal gas. So the same thing, there is a gravitational force acceleration and we can assume that the gravitational acceleration does not change. An example of this application of this could be the surface of the earth.

And we are interested in how the, there in the atmosphere, earth atmosphere, we are interested how the pressure of the atmosphere changes as we go along the z direction, again. Let us that z = 0 over here at the surface of the earth. And we are not concerned with the change in the gravitational acceleration that the distance that we are interested in are relatively small compared to the radius of the earth, few kilometres.

So suppose I go from here to the top of mount Everest, my height will change by roughly 10 kilometres, > 10 kilometres. And the radius of the earth is, let us assume that the gravitational acceleration is more or less same, temperature also let us assume that it is more are less same, though we know that at the top of mount Everest is going to be much cooler. But let us make some simplifying assumptions like this.

And go ahead and see what happens if we assume that the whole atmosphere is in hydrostatic equilibrium. So again since the temperature is the same everywhere, there will be no heat loss from individual fluid elements. So the heat loss is 0 and we have dP dz = -rho*g, well that is the first equation. Now we now this is an ideal gas for an ideal gas we also have that the pressure is PV =NKBT.

Pressure * the volume of the fluid = the number of atoms or molecules in the fluid * the Boltzmann constant * the temperature in Kelvin, absolute temperature. That is the relation between the pressure volume and the temperature for an deal gas, PV = NKT, N is the total number of or we can use capital N for this total number of atoms or molecules in the fluid.

Now this we can write as P =, so I can divide N/V multiply by the mass of each particle, N/VKBT and I should divide by mass of each particle. So we have the pressure =, this we know is the density mass * number of particles divided by volume, so it is rho KBT/m. So with this relation between pressure and the density they are not independent now, temperature is the same throughout the atmosphere.

So this is the atmosphere we are dealing with, the atmosphere we are dealing with is here, and the temperature is the same throughout. So this equation now becomes d rho dt =, so let me put all the actors on the right hand side, we have –mg/KBT*rho, d rho dz, sorry it is a dz derivative d rho dz = this. Now we can integrate this, integrate this is straight forward we have d rho dz = a minus constant*rho.

So the integral of this is straight forward and what we have is that rho = some rho0 exponential –mgz/KBT. And you can check it for yourself in general that if the body force is the gradient of a potential - the gradient of a potential acceleration, then we can write this as rho0, you can check this that it comes out with this where f. So if the acceleration f, which we have here is going to be constant is the gradient of some potential.

Then it can in general be written like this. This I am not sure, but you can check it out for yourself. So what do we see over here, what do we find is that if you assume that the atmosphere has a constant temperature, and the gravitational field is a constant as we go up, then the density of the air falls exponentially with the height and the density falls exponentially implies the pressure also falls exponentially with height.

So this gives you a reasonable approximation of the density of air and the pressure decrease as you go up above the earth surface and the pressure and the density, for example on the top of mount Everest are considerably lower than what they are here, which is why people require oxygen and you would be severely sick if you were there in that altitudes > 7000 or

6000 metres for considerably long time. So you have different mountain sickness, altitude sickness arising from this.

So we have considered 2 examples of hydrostatic equilibrium, let us consider a third example, which is astrophysical in nature. The third example of hydrostatic equilibrium where we will apply this concept of hydrostatic equilibrium is the solar corona. So let us take a look at what we mean by solar corona first.

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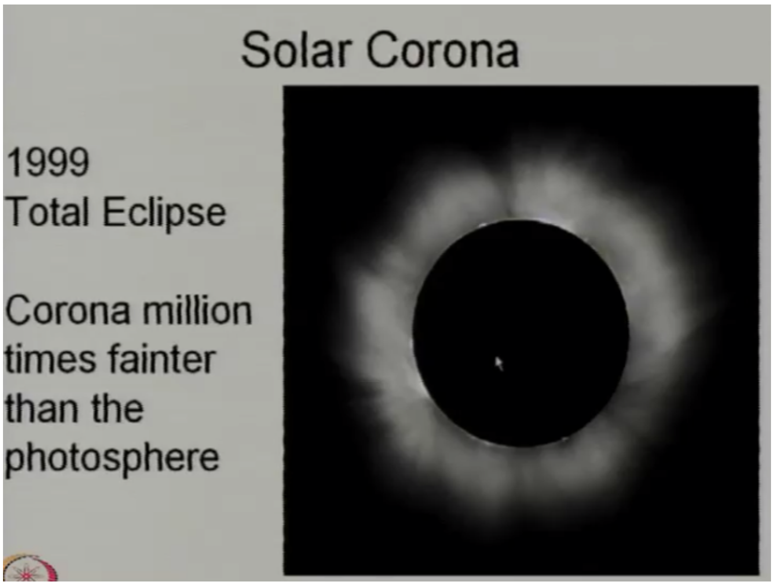

The solar corona, this is the picture of the sun taken during the 1999 total solar eclipse. And the sun here is blocked out by the moon. I have shown you this picture already. Now the interesting thing here is that when the sun is blocked out, when the sun the dazzling light from the sun is blocked out, then you see this very faint and tenuous bright thing around the sun, this is called the corona, it comes from the word crown.

It surrounds the sun, and the radiation from this is 1 million times fainted than the radiation from sun, the radiation from the sun originates in the photosphere, so it is a million times fainted. So we can only see this in the visual light, if the sun is blocked out which happens during the total solar eclipse. So the corona is seen only during the total solar eclipse, that is the first thing.

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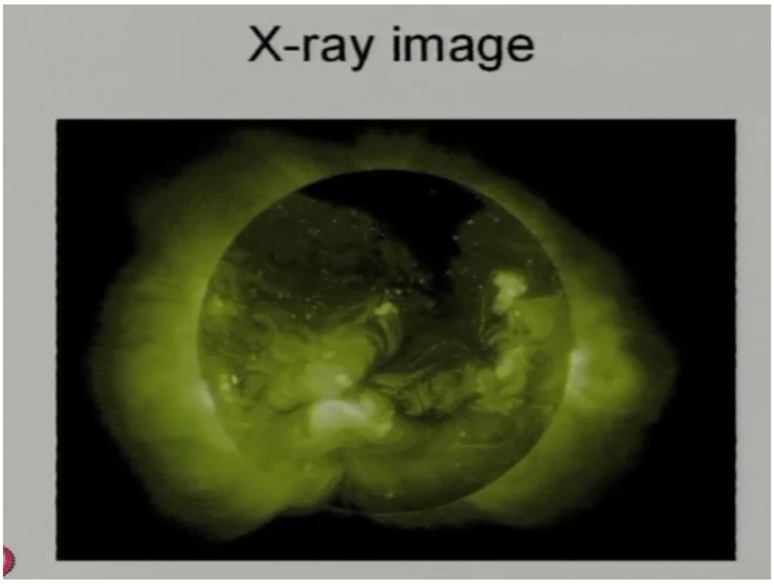

But if instead you were to make an x-ray picture, x-ray image of the sun. This shows you an x-ray image of the sun taken from a Japanese x-ray satellite. Now in this image, what you notice is that the sun is darker and the corona which is outside the sun is actually brighter. In an x-ray you do not need a solar eclipse to see the corona you can see the sun even otherwise in any x-ray image of the sun that it is the corona which is brighter than the sun itself.

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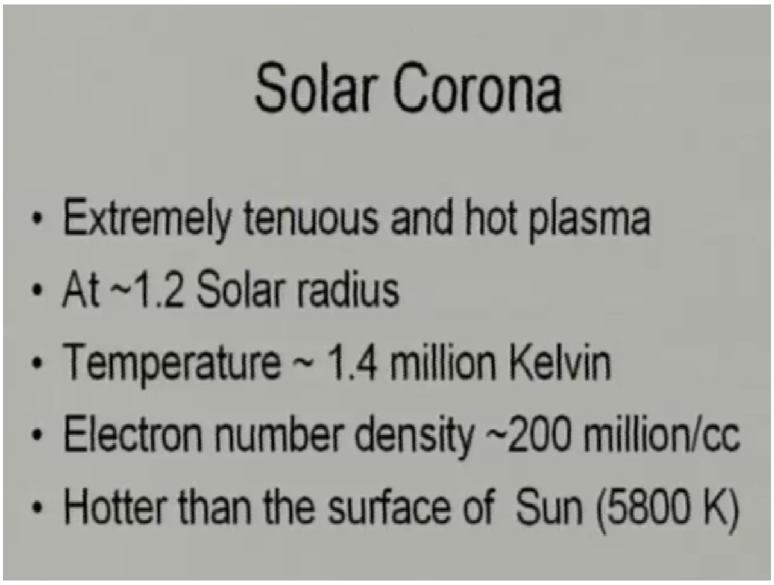

Now that is very surprising but it is a fact. So the solar corona is extremely tenuous. See the density is very low, hot gas, hot plasma essentially it is ionised the plasma we mean is ionised gas. So it is an extremely low density and extremely hot plasma surrounding the sun, that is called the solar corona. And in visual the radiation is a million times fainter, but in x-ray it is brighter.

The temperature at around 1.2 solar radius is a rough value is around 1.4 million Kelvin. And the density is around 200 million per electron, electron number density is 200 million particles per cc, and it is mainly electrons and protons, some amount of helium ions also. But mainly helium, hydrogen, electrons and protons, hydrogen ions. So the sun is surrounded by this very hot, very low density plasma around the sun which surprisingly is much hotter than the sun.

And it is a very puzzling matter, how is this solar corona which is around more than a million Kelvin, how is it hotter than the surface of the sun which is just 5800 Kelvin, what maintains the corona at higher temperature, it is rather puzzling issue.

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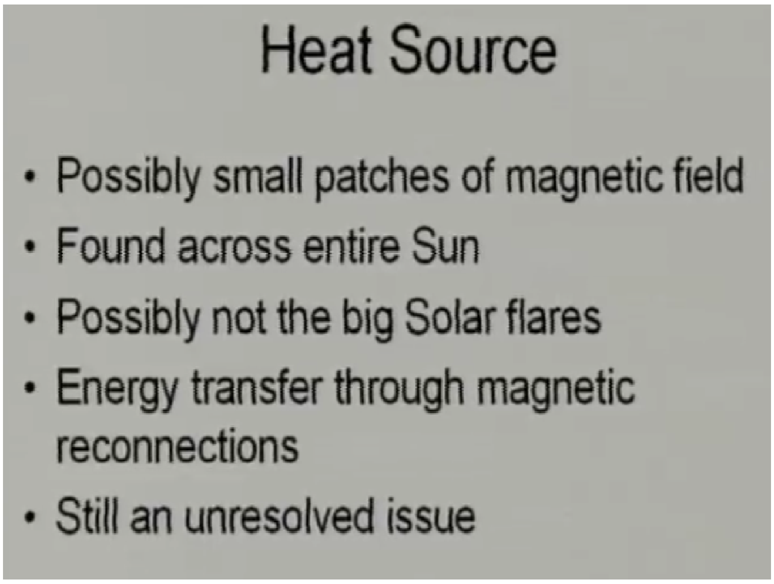

It is now believed that it is possibly originating from small patches of magnetic fields that are found all over the sun. The source of the energy is now believed to be small patches of magnetic field found all over the sun. And possibly if you look at the pictures of the sun I have shown you one, right in the beginning you have these solar flares. These are gas which come out from the sun.

So you have these solar flares also coming out. Now it is widely believed accepted that the corona is heated by magnetic fields. When the magnetic fields come out of the sun, these magnetic fields are carried by the motion of the plasma. Now when the magnetic fields gets intertwined then what happens, they reconnect, you may have by the motion of the plasma very complicated pattern of the magnetic fields then they reconnect so that you have very simple pattern than some magnetic field blob may just separate out.

And during this reconnection energy is pumped in to the plasma outside the sun, that is the picture. So the flares also do carry out energy from the sun, but it is now believed that the solar flares which are seen only during the active period of the sun and they do not occur during passive period of the sun, the sun has these periods. So they are not the source, the source is possibly the small patches of the magnetic field.

Anyway the general picture is that it is transmitted to the plasma through magnetic forces which the reconnection of magnetic fields, but it is still remained as a unsolved issue. For our purposes let us not bother about it, let us not bother about what heats the solar corona, what we are concerned with is as follows. So we have, let us try to model the solar corona.

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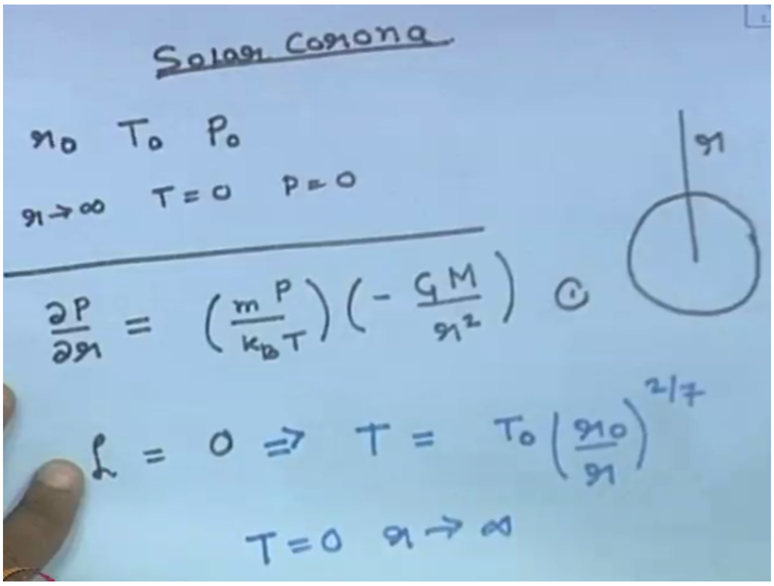

We are going to try make a model for this solar corona. So this is the surface of the sun and at the radius of the sun r0. We will assume that it has some temperature, very high temperature T0, and it has some pressure P0, and you can then work out the density. Further we will assume that it is in hydrostatic equilibrium, so this gas outside the sun we will assume the plasma is in hydrostatic equilibrium.

And as you go to, as r goes to infinity, we expect the temperature to go to 0, the pressure go to 0 and everything vanishes. So this is the gas which is there in the hydrostatic equilibrium. There is a temperature T0 which is very high, which is maintained by the sun at the centre and the temperature falls off as you go far away at infinity it should be 0, the pressure should also be 0.

That is the model which was worked out by Parker, famous plasma physicist at the University of Chicago called Parker, he worked out this model. Now let us go back to our equations of our hydrostatic equation, so these are the 2 equations that we have. And the first equation here everything varies only the radial direction r, so the first equation is that the radial derivative of the pressure del P/del r = we have the density*the body force, the body force here is acting inwards so we have = rho.

Now let us write rho in terms of the pressure so we know that if you write rho in terms of the pressure, we have worked it out in the last example. So we can write rho in terms of pressure so this will be mP/KBT. So we have the mass of the particles by P/KBT. And then we have the gravitational force, the gravitational force is minus GM/r square. So we have this that is the first equation.

We also have the condition that the heat loss L should be = 0 for any unit of the fluid, per unit volume of the fluid should be 0. Let us consider the heat loss first.

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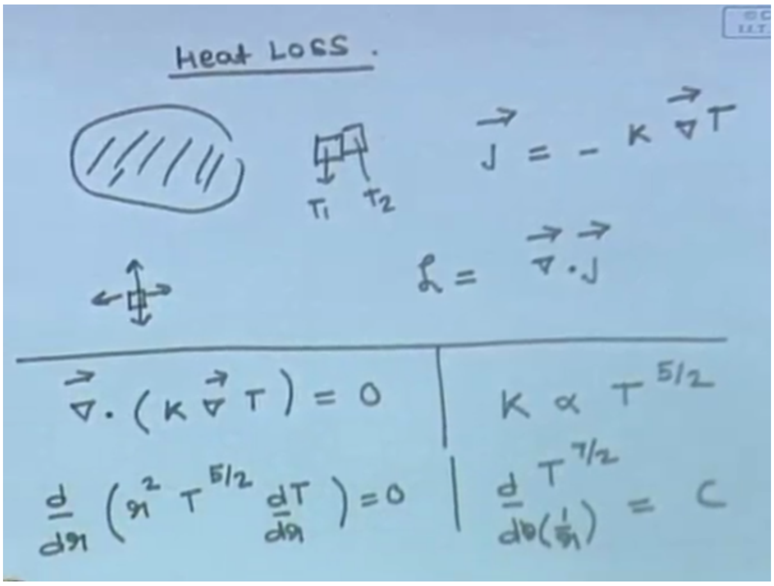

Now if you have a fluid, this is the fluid let us say, whose different parts or at different temperature or anything like any medium where different parts are at different temperature we know that heat will flow from the hotter to the cooler part through conduction. And the conduction current that flows out from this, let us just consider this, this is under temperature T1, this is at the temperature T2 slightly different.

The conduction current that flows out from this is J, it is proportional to the rate which the temperature changes with position. So the conduction current is -K grad the temperature. There is a temperature difference then you have a conduction current and if it is in opposite direction as the temperature gradient. And this case, the heat conductivity which is dependent on the material, so that is the conduction current.

And if I now have a unit small volume here and ask what is the total heat that flows out. Then you have to take this current and do a surface integral over this and the surface integral can be converted into a volume integral through Gauss theorem. So we know that the heat loss rate is essentially the divergence of the conduction current. That is the rate at which the heat is flowing out.

So in this case different parts of the solar corona, the temperature is going to be hottest near the surface and then the temperature expected to fall off. So we are going to have heat conduction and we will have a current, but the total heat loss per unit volume should be, total heat loss should be 0. So it is essentially what it tells us is that in spherical polar coordinates that the condition is.

Let me first write the condition del dot K grad temperature, this should be = 0. So this is the condition, further we also know it is known I am not going to go through it here, it is known that the heat conductivity in a plasma K is proportional to T to the power 5/2, it increases by temperature, the hotter the plasma, the better it conducts heat. So this is known.

So what it tells us for our problem is that this in spherical polar coordinates this becomes d/dr r square T to the power 5/2 d/dr T = 0, or what it tells us or we can write this as follows we can write this as d/d of 1/r, so if you do the d/d of 1/r, then you will get r square/dr with negative sign, that does not matter because it = 0. So d/d of 1/r of T to the power 7/2 is a constant, this is what it tells us, this equation will give us this, the solution to this.

So the fact that the energy equation here finally tells us that the temperature scales. So I can write down the form of the temperature over here. So what it tells us is the temperature T scales as r to the power -7/2. So let me put it here it implies that the temperature varies with r as = T0 r0/r, this should be 2/7 not 7/2 right. So the T to the power 7/2 = this so this should be 2/7. Let me just write down the solution here.

Now we have to next use this in this equation. We now know the temperature so we have to put this back put this into the equation for the pressure, so let us do that.

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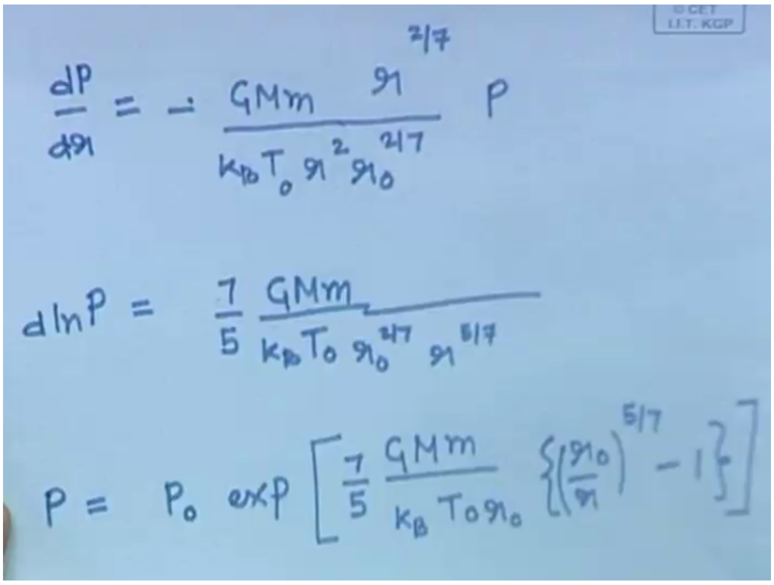

So let me do that next, so the Euler equation what it tells us is that dP/dr = let me put all the factors correctly, so we have m we have a minus sign and then we have g capital M small m and here we have KB r square/KBT, the temperature we will write as T0, so T0 and we have r square here and then I will r0 to the power 2/7 and r to the power 2/7. So here I have r0 to the power 2/7, in the numerator I have r to the power of 2/7 and then we have the pressure itself.

So we can now integrate this equation if I integrate this equation what I will get is dlnP =, and I have to integrate this term now, this term let us see what the we have GMm/KBT0 and what is the combined r dependents of this term, let us see. So the combined r dependents of this term r to the power of minus 12/7, this is 14/7 this is 2/7. So it is r to the power of -12/7 and if I integrate r to the power of minus 12/7, then I will get 5/7.

So I will have 7/5 here and I will have r to the power of -5/7. So I have r0 to the power of 2/7 and r to the power of 5/7, which we can write the solution straight away now P =, so there will be a constant when I integrate this which I can write as P0 exponential, I can write this whole thing as follows, exponential then I have 7/5th GMm/KBT0 and so we have r to the power of 5/7 here.

So let me write it as r to the power, so what we can do is, we can write this as r0 over here. So I can write it like this r0/r to the power 5/7-1 * this factor. This -1 into this is a constant, which I can multiply P0 and it will give me the constant of integration here. So the pressure can be written like this. That is the solution to this. Now the point to note over here is that if you, that r=r0 this gives me exactly P=P0.

Here we have put the boundary condition that the temperature goes to 0 as r goes to infinity. We have imposed the boundary condition that the temperature goes to 0 as r goes to infinity. So this ensures that T goes to 0 at r tends to infinity. Now we would also like the pressure to go to 0 as r goes to infinity, but notice here that at r = infinity the pressure does not vanish, you have a finite value for the pressure.

And this pressure turns out to be more than the pressure of the interstellar medium. So between the stars there is some gas so if this pressure where equal to that the pressure of the interstellar medium, then we could say that there are equilibrium over there but this pressure turns out to be considerably more than the interstellar medium. So what Parker showed that you cannot have a static solution.

If you want where the temperature and pressure both go to and the density everything goes to 0 at infinite. So if you want to hold the solar corona in place in hydrostatic equilibrium, then there must be something exerting pressure at infinity and we know that there is no such thing exerting pressure at infinity, so he concluded that the solar corona must be expanding out.

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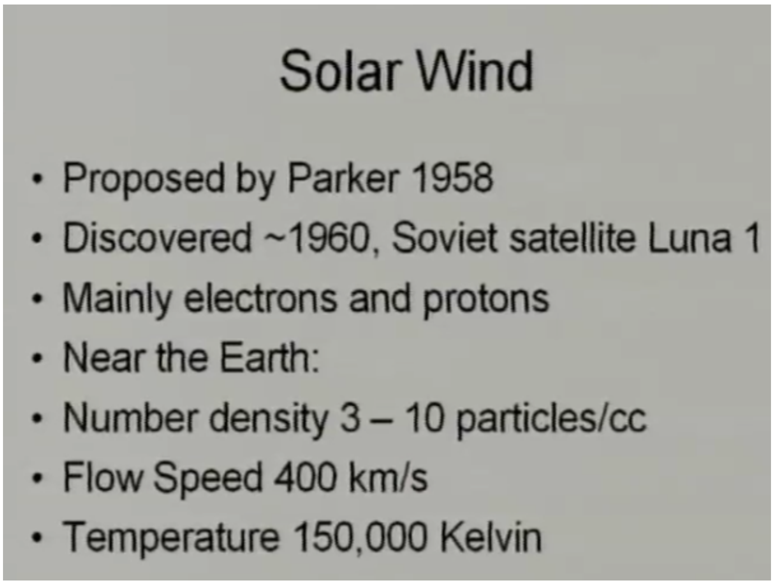

So this is the proposal made by Parker in 1958 based on this simple calculation. And initially this calculation was not accepted. His paper was actually rejected from FJ by 2 referees. Fortunately, Chandrasekar was the editor and he accepted it. But this proposal was verified by the soviet satellite in 1960, satellite Luna 1 and now it is well known that there is a solar wind.

So the gas coming out, this gas outside the sun is not in hydrostatic equilibrium it is actually blowing out and it is made up of electrons and protons like the corona, now at the location of the earth, near the location of the earth, the number density is around 3-10 particles/cc, considerably smaller. The flow speeds are of the 400 kilometres per second, and the temperature is around 150,000 Kelvin.

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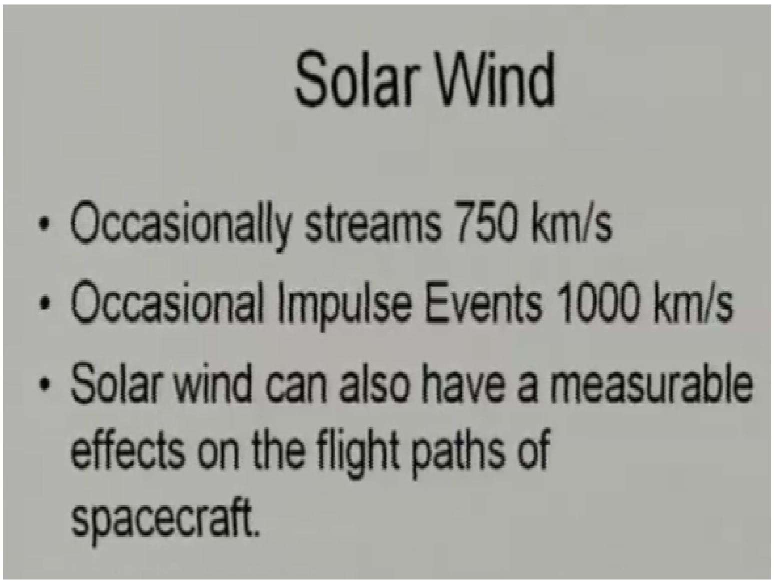

There are occasional streams which have 750 kilometres per second speed. And there are impulse events which can have speeds as high as 1000 kilometres per second. Solar winds are very important for satellite communication and they can also have measureable effects on the flight paths of the spacecrafts. So wind, constant wind blowing out from the sun.

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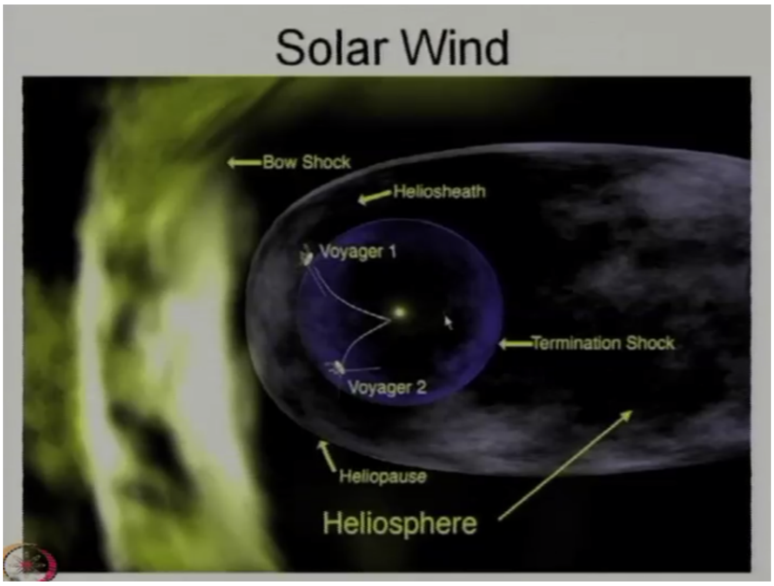

This shows you a picture, not a real picture, it is an artist impression of the solar wind. So this is the sun at the centre of this picture and this is heliosphere, this is the region which is filled by the solar wind, and here it is actually interacting with the interstellar medium and you have this bow shock and this whole thing is the heliosphere up to here and this also shows you the trajectory path of voyager. So the voyager is the man-built satellite that has covered the largest distance till now, so it is still within the solar wind.

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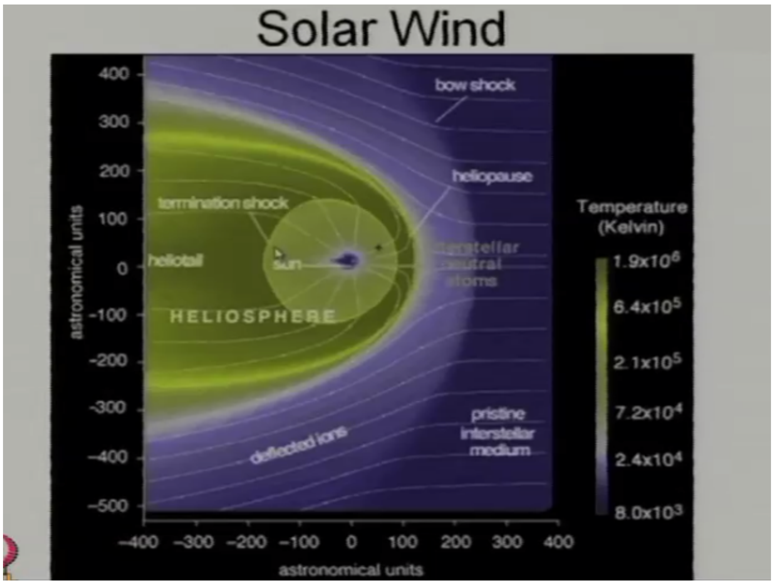

To put this in scale, this picture shows you the same thing and you can see that the solar wind extends to hundreds of astronomical units. Now let us get back what we were doing, we were discussing fluid mechanics and we just took a diversion and discuss the solar corona one astrophysical application and its consequence the solar wind. The solar wind can also be described in fluid mechanics terms, maybe we shall discuss it later on in this course.

So let us get back to what we were discussing. So we were discussing the fluid equations, and we until now have discussed the hydrostatic situation. Let us go back to the fluid equations and apply it to flows where there are velocities, the situation that we consider till now there were no velocities. Let us no go back and look at fluid flows.

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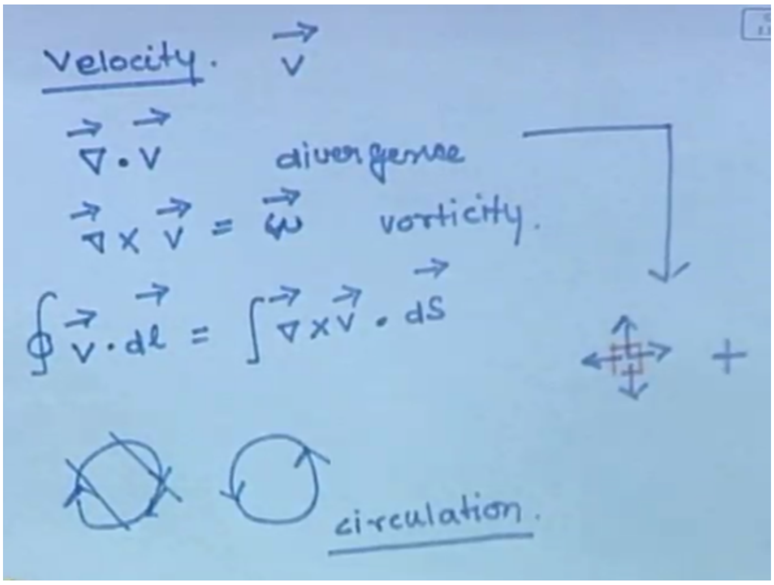

So before we start discussing flows let me briefly discuss the nature of the velocity field V. Now it is a well known fact in vector calculus that this vector field V can be decomposed into 2 parts, one part which has a divergence, this has a divergence and another part which has only a curl, this is called the vorticity. This is usually denoted by omega, the curl of V and it is called the vorticity and what do these 2 represent.

So we know from Gauss theorem that del dot V, so the divergence, if the divergence is positive it tells us that, this is my volume. It tells us that fluid is flowing out, a vector field which looks like this, will have positive divergence. And the opposite situation where it is converging will have negative divergence, that is the simple application of Gauss theorem. This can be converted into a surface integral and if we take as infinity as volume it tells us that the flux out of that volume or into that volume.

So the divergence represents this kind of a motion. Now we also know the Stoke's theorem, the Stoke's theorem tell us that take V dot dl around a closed loop in the anti-clockwise direction so this is a loop. Take a close loop like this, and integrate V dot dl around such a loop. So dl is the tangent vector to the curve. So you integrate this V dot dl and from Stoke's theorem this = the surface integral of curl of V dot ds, so ds is now pointing inwards.

Sorry this is clockwise, I have drawn the picture incorrectly, it should be other way round anti-clockwise. Now ds points outwards and you take the curl of V and do dot ds and integrate over this, then whatever it gives you is essentially the circulation the V dot dl. So a velocity pattern that looks like this goes around in the circle is what is quantified by the vorticity, the circular motion is also called circulation. This is also called circulation.

This is vorticity and this integral is also called circulation, the surface integral of the vorticity. So the velocity field can be uniquely broken up into these 2 parts, a part that has the divergence and a part that has a vorticity, the curl. And conversely if I told you these 2 components you could reconstruct the entire vector field. So if I told you what the divergence was and what the curl, vorticity is.

If I tell you these 2 components, you can reconstruct V. These are all well known things in vector calculus. Now let us go back to the Euler equation. So this is the Euler equation. It is often convenient to write it in a slightly different way, so let me do that. So we will use the vector identity.

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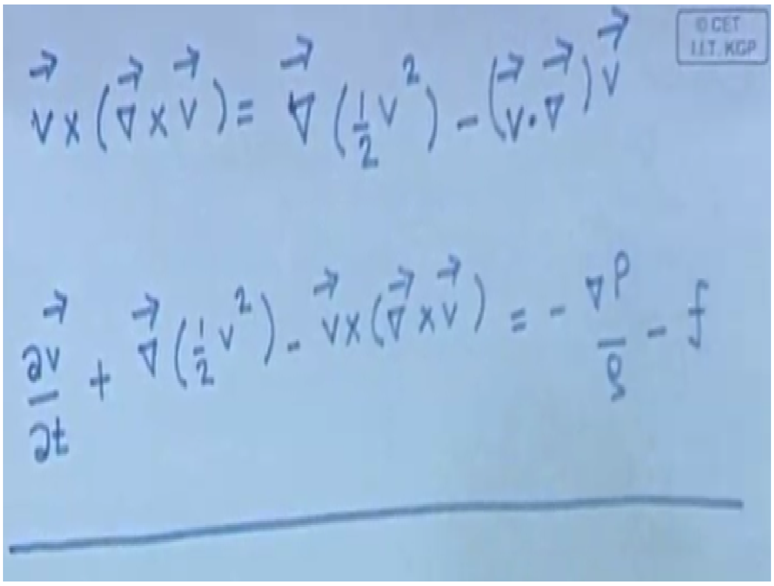

So you can use the identities relating A cross B cross C, which you all know, you can use this to simplify this expression and if you do that what you will find is that you can write this as the gradient of half V square. It is just A cross B cross C nothing more, apply that identity. So this turns out to be the gradient of half V square -V and this is the term that appears in the Euler equation. So we can replace this in the Euler's equation.

And what we have now is, so that is the Euler equation written in a slightly different way. Now let us we can do a few things using this, so let us look at flows.

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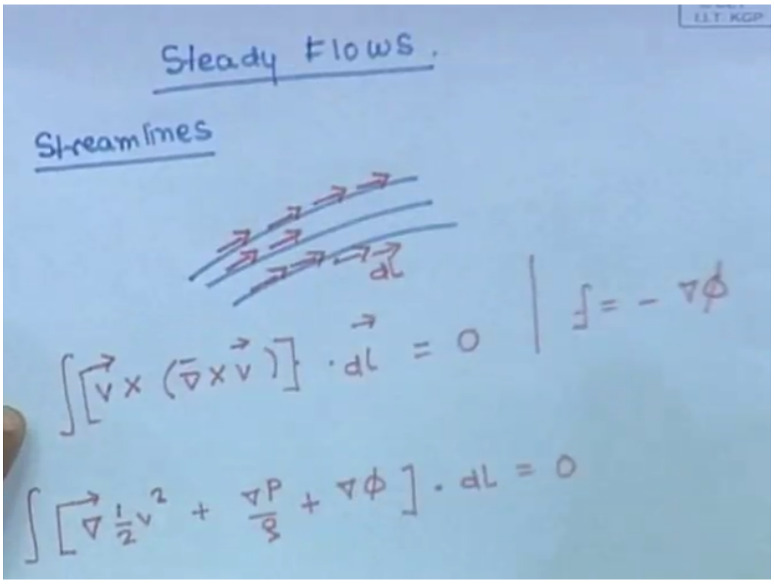

Let us consider steady flows, what do we mean by a steady flow, steady flow is a situation where we have a velocity and we have a density but everything is time independent. Now for steady flows you can define something called streamlines. These are curves, the tangent to which denotes the direction of the flow, so the velocity, the tangent to which the velocity at any point is the tangent to this curve.

And you can think of it like the electric field or magnetic field. These are also some field lines. The velocity of field at any instance is tangent at any position is the tangent to this curve, these are called streamlines, and the fluid flows along these streamlines. Let us now integrate this equation. We are considering a steady flow, so there is no del V del T. Let us integrate this equation along a streamline.

So we are going to integrate this equation along a streamline and let me write it down. Before we write down let us just look at this term if I integrate this along the stream line, then let me write down the term. So we have V cross curl of V dot this dot dl, where dl is, that is what we mean by integrating along a streamline. Now dl is parallel to V at this point. This is V cross something. So it is obvious that this is going to be 0, this does not contribute.

So what we are left with is basically the integral of this, the other 3 terms, what we have is that the integral gradient of half V square plus grad P/rho minus F, now we are also going to assume that the body force f acceleration force per unit mass, can be written as the gradient of some potential. So we are going to assume that f = - grad phi, could be gravity, could be electric field something.

So then this becomes +grad phi dot dl = 0. Now these 2 terms are gradients so the dot dl is essentially just the difference between the velocity and the phi.

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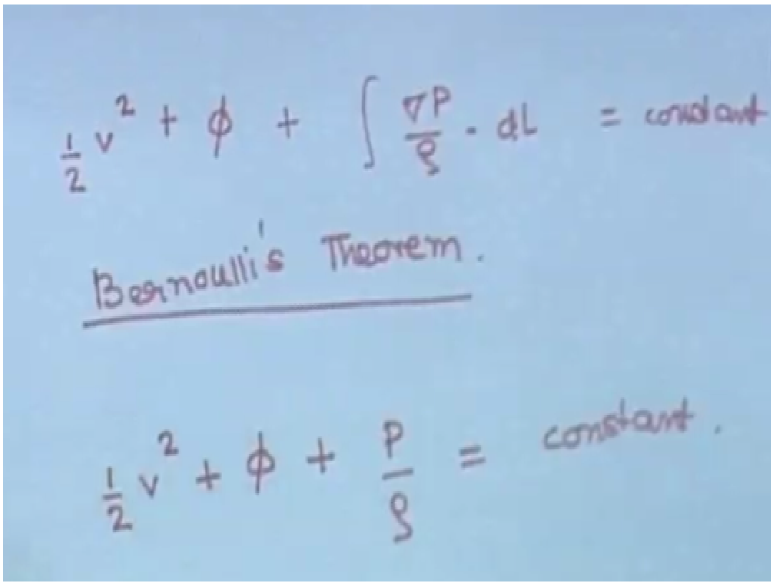

So this tells us that, so when I integrate this I will just get the constant. I will just get the value of half V square+phi at this value minus the value of half V square+phi at this value. So what we can say from this is that this + this + the integral grad P/rho dot dl should be a constant, and this is what is known as Bernoulli's theorem. And if your fluid happens to be incompressible the density is the same throughout.

Then it simplifies even further what it tells us is that half V square+phi+P/rho is a constant, which is Bernoulli's theorem. Let us work out one quick application of the Bernoulli's theorem before we end today's lecture.

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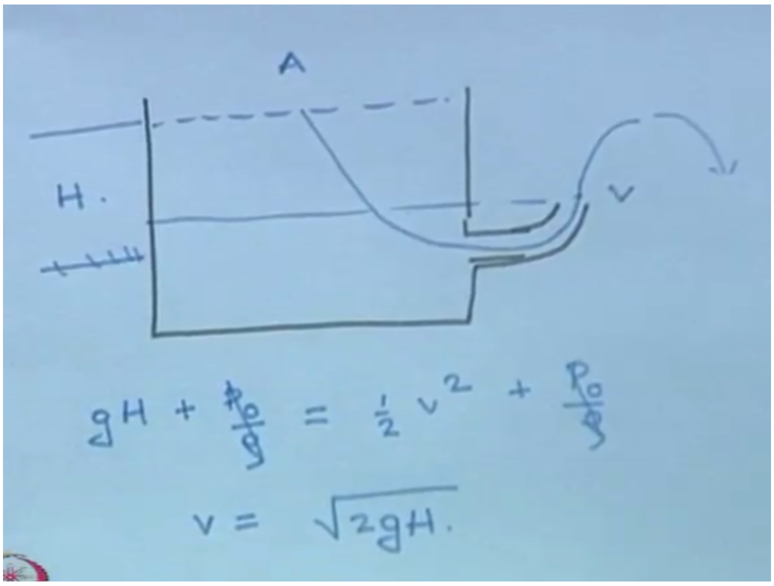

So we have a container like this with a small outlet over here, and let us say the outlet points upwards and it is filled with some fluid. So the fluid is going to obviously flow out, it is going to flow out something like this, then it will fall. And let us say that this difference in height is H and as the fluid flows out the height, the width is so large let us assume that the water level here does not fall very appreciably.

And we would like to calculate the velocity of the fluid at this point. So the velocity here is 0, the atmospheric pressure is P0 let us apply this equation. So at the top of the fluid at the point A, the velocity is 0 there is a gravitational potential, which is MgH, gravitational potential is gH and we have the atmospheric pressure/rho. So gH+P/rho, this should be equal to, now at the outlet over here, at the tip of the outlet we have half V square.

This is the height difference, sorry between the tip of the outlet and this. At the tip of the outlet we have half V square, this = half V square+phi, so the potential is 0+P0/rho. So what we see is that these 2 terms cancel out and what we get is that V = the square root of 2gH. So we have used Bernoulli's theorem to calculate the speed with which the water comes out of the outlet over there.

So let me end our discussion of fluid mechanics over here and we shall move on to some other topic in the next class.