Astrophysics & Cosmology
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Lecture - 07
Tidal Forces and the Earth Moon System

Welcome to today's class. Before we start on fluid mechanics, there is a small point from the last class. So in the last class, we were discussing the following situation. There is big massive object over here.

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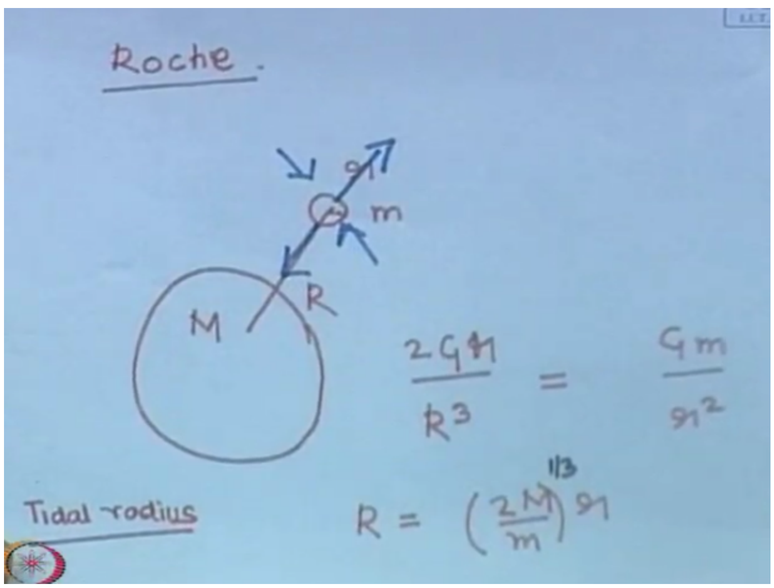

And there is a smaller object, which is in the gravitational field of this massive object and the small object we are assuming is gravitationally bound and the question that we were addressing is how close can the small object get to this big massive object, before the tidal force produced by this disrupts this small guy over here and we had found that the closest distance is capital R, which is given by twice the ratio of the masses.

This is the mass of the big object. This is the mass of the smaller object. This to the power 1/3*the size of the smaller object. In the last class, I had forgotten to put this the power 1/3. So if this object gets any closer than this, it will be tidally disrupted by the gravitational field of this big massive object. This is tidal disruption. Now in today's class, we are going to discuss a bit of fluid mechanics.

Let me tell you why we are going to discuss fluid mechanics. Take for example, the sun. The sun we know is a nearly spherical ball of gas and if you want to study the structure of the sun, one describes it as a fluid. So it is a gas, which can flow. If there is pressure gradient, then the gas can flow. So the simplest possible approximation is to describe it as a fluid. There are various components in the universe, which are conveniently described as fluids.

Which is the reason why we are going to discuss a little bit of fluid mechanics. It is a tool, a theoretical tool that is extensively used in astrophysics and cosmology.

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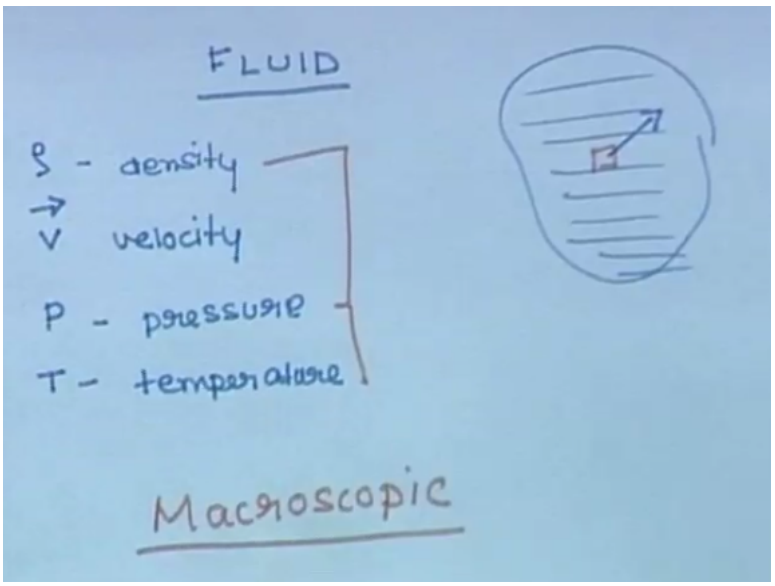

What is the thing that we are trying to describe. You can well take water as an example. There is water. We know that water is made up, any substance is made up of atoms and molecules. So there really are, if you look at very closely water or any substance for that matter, there are particles. One can write down the Newton's laws of motion for these individual particles or if you have a gas, we know that there are particles and one can write down the Newton's laws of motion for these individual particles.

But for many purposes we are interested in the motion of the individual particles. We are interested, it is more convenient to think of it as some kind of a continuum. So when we think of water, it is convenient to think of it as something which is continuous. The discrete nature of the particle is not very important for many situations. This is what we mean when we do fluid mechanics, we treat water or gas as a continuum.

It is something, which is continuous and we shall use then a macroscopic description of the substance that we are dealing with. So we shall deal with quantity. So let us just draw a picture for example. This is a region of space, which is filled with some medium and we will describe it in terms of the density, rho. This is the mass density. So the density tells us the mass inside a small volume.

If I take a small volume here, the density tells us the mass in this small volume. That is one of the things that we used to describe any fluid. It is a macroscopic quantity. If you look at it microscopically, you will find that there are discrete particles located, but we are going to average the mass and use the density to describe it. Similarly, we will also use the velocity. We will ascribe one velocity to this region of the water in this region.

That is the mean velocity of all the water particles in that region. This is the other macroscopic quantity in this fluid picture. That is the velocity and then we have the pressure and we have the temperature. These are all thermodynamic quantities and we know that the density, pressure and temperature are related through an equation of state. So they are not all independent. So if you know the equation of state, for example for a gas, we know the equation of state of an ideal gas.

These are not independent. They are related through some equation of state. So we shall be dealing with these quantities. So the entire state of the fluid is described by telling its density, its velocity, pressure, temperature, actually any 2 of them will do at all points. So this is how we describe this in a macroscopic fashion. So we shall deal with these variables. That is the first thing. This is the macroscopic picture.

The second point is that there are 2 approaches to fluid mechanics.

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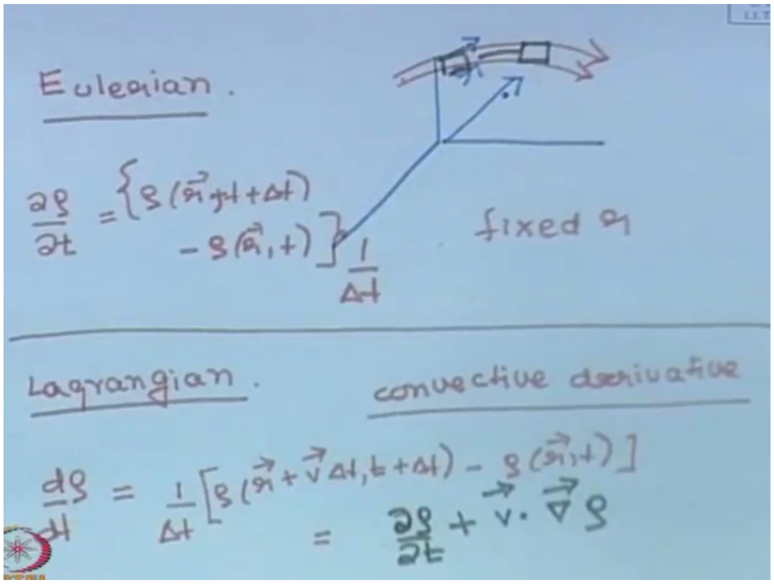

The first approach is called Eulerian approach. In this approach, we look at the fluid at a fixed point in space. So when we want to study the density for example, we will focus our attention in the Eulerian approach at the value of density at a fixed R. so R will remain fixed. If the fluid flows, the different elements of the fluid will come to this R at different times. Say, the fluid flows like this and I look at it at 2 different instance of time at a fixed R, then I can ask how does the density change.

So we have del rho, del t, which is at a fixed R. So R is fixed position. This is rho at R+R, t+delta t-rho at R, t, the whole thing divided by delta t. We can look at the derivatives and things like that in the Eulerian picture, and when you evaluate this derivative, you are essentially looking at 2 different bits of fluid because the fluid is flowing, so if I sit, I will see the fluid that is flowing past me.

So if I look at it one instant, and I look at it the next instant, a different fluid element will be here. The one I was looking at in the past would have shifted somewhere else. This is the Eulerian picture. We also have the Lagrangian approach and in this approach, what we do is we follow a fluid element as a single fluid element as it flow. So here suppose I am looking at the density, then I will follow this fluid element, as it flows.

So if it moves from here to here, I will follow this. So my attention here is on a particular fluid element. We have something called the conductive derivative here. So when I look is, if I ask what is the time rate of change of density, in the Lagrangian picture, I do not sit at a fixed place and look at the fluid flowing past me. I move with the fluid and then ask the question. So I follow one piece of fluid, as it flows from here to here and ask how does its density change.

And there will be a change here due to 2 reasons, so the position also will change and the change in the position is R+V delta t*t+delta t-rho Rt. So that is the conductive derivative and if you do a Taylor expansion in this t, you will have 2 terms, the first term will be del rho del t, it is a partial derivative at a fixed R. That will be one term and there will be one more term, which is V dot grad rho.

So the spatial variation in rho at a fixed time will also come in, because the R has changed. So if I write this in terms of partial derivatives, I will get 2 terms, one is the partial derivative of rho with respect to time and the other is V dot the gradient of rho. This is called a conductive derivative. So this tells you the rate of change of the density or any quantity for that matter. If you move along with the fluid. So these are the 2 pictures.

Now let us determine the equations that govern fluid motion, the flow of a fluid. So all of the equations that we are going to consider, they are essentially guided by conservation principles, something conservation principles. So let us first consider the mass continuity equation.

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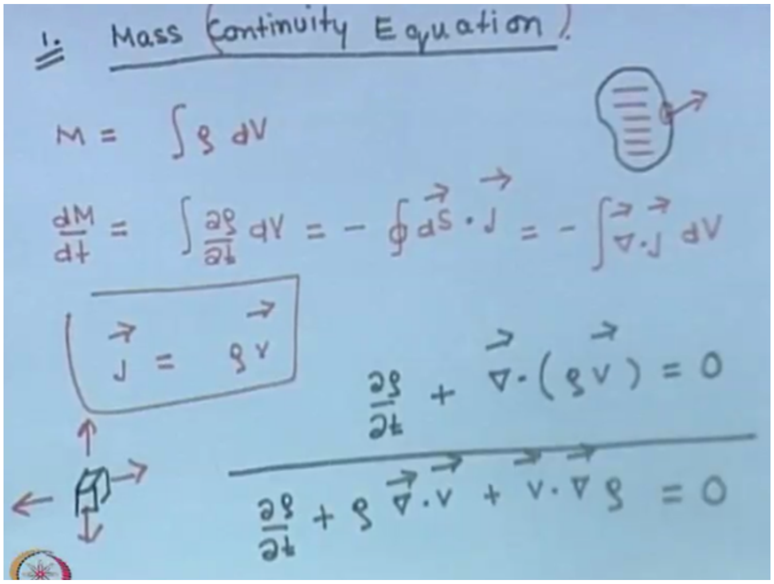

The way we arrive at this equation is as follows: Consider an arbitrary volume like this and let us ask the question what is the mass inside this volume. Now that is straight forward to calculate the mass is the density. So the mass in this volume M is the integral of the density over the volume. This will give me the mass. Now let us write the rate of change of mass and the rate of change of mass and if I differentiate this with respect to time, the volume remains fixed.

I am working in the Eulerian picture, so the volume remains fixed. The mass in this volume can change only because of the change in the density. Because the mass is the integral of the density. So the rate of change of the mass will be del rho del t dv where the integral is over this volume. Now we know that mass conserved. So if any mass changes inside this volume, it could occur either because the mass has flown out or mass has flown in.

Now if the mass flows out, then there is a net flux of the mass which is positive. So this is going to be equal to minus the mass flow out from this volume. Now the mass is going to flow out of this volume to the surface and the mass that flows out, the rate at which the mass flows out is given by the surface integral of ds that is the surface element area pointed like this dot j. this is a general kind of thing as a flow.

So that the rate at which the mass flows out is the surface integral of the mass current, j is the mass current. That is the rate at which the mass is flowing per unit area. The current we mean the rate at which it is flowing per unit area. If I integrate this over the entire area, I will get the total mass rate flowing out. So the rate at which the mass changes is minus this, because any mass flows out, its mass will come down.

Now how much is the mass current? The mass current or for that matter any current is the quantity into the velocity with which it flows. So if I want to calculate the mass current, the mass current J will be the mass density rho, mass per unit volume into the velocity with which it is flowing. Now if I take a unit area and ask the question, how much mass is crossing it per unit second, it will be given by j dot the area. So this is the mass current.

So I have this over here. So if I use this, let me now do it in a more general term. So this term using Gauss law, we can write it as minus, now this is a surface integral that can be converted into the volume integral of the divergence of this current, so I can write it as del dot j dv. Now equate the left hand side with the right hand side, I have 2 volume integrals, one on the left hand side and the other on the right hand side.

And these 2 should be equal, so it tells me that the integrants should be equal. So I am led to the equation del rho del t+ the divergence of the mass current, which is rho into the velocity, this should be = 0. So this is continuity equation. It tells me that matter is conserved locally. So if the density goes down somewhere that means that the matter is flowing out from there. We can look at it like this.

This is a small unit volume and if the density goes down over here, it tells me that there is a flux of matter flowing out. So if this term is negative, then this term has to be positive. The divergence of mass current is essentially a measure of the flux flowing out. That you can easily verify from the Gauss theorem. Similarly, if the mass is increasing in this unit volume, then there should be a flux inwards. This is the continuity equation and we can expand it and write it out also.

So let me do that here. So if I expand it and write it out, then I have del rho del t+ then I will have one term, which is rho*divergence of v+v dot gradient of rho that should be =0. That is the mass continuity equation. It tells us that matter is neither created nor destroyed locally. This is the first fluid equation. Let me put down the equations over here in one place so that we have it for future reference. So the first fluid equation:

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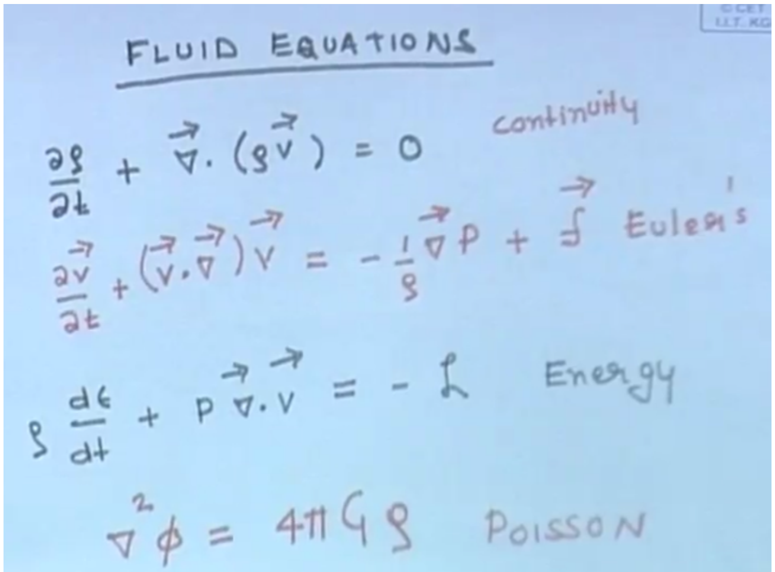

Del rho del t + the divergence of the mass current is =0. This is the continuity equation. Let us take up the next fluid equation. Next fluid equation is the conservation of momentum. So we have looked at the conservation of mass. Let us now look at the conservation of momentum. We can use the same picture. You have this volume. Let us write down the momentum inside this volume, total momentum.

For simplicity, we will write down only the x component first and then we can generalize it to all the components. So let us take the x component of the momentum. The total momentum inside, momentum is mass*velocity. I have to integrate mass*velocity. So this is the momentum equation.

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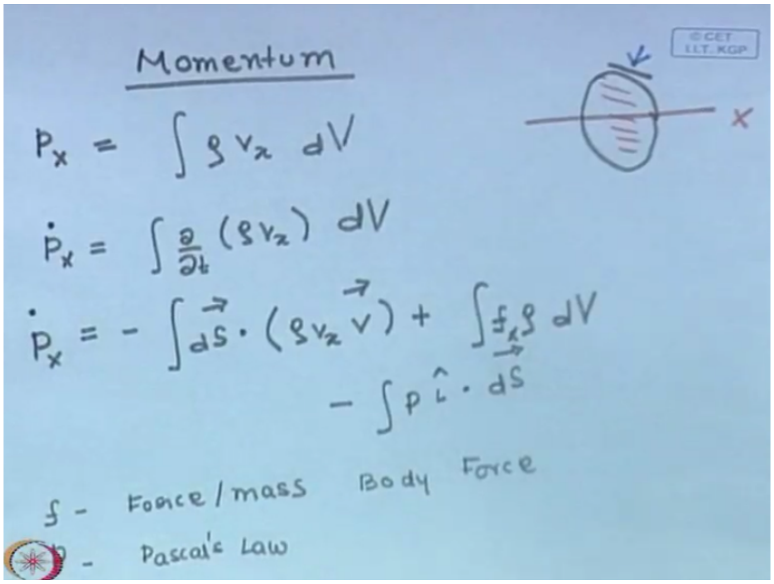

I am interested in the x component of the momentum and this will be given by the integral of the mass rho per unit volume * the velocity of that volume in the x direction, which is vx*the integrated over the volume V, total volume. This is the velocity; this is the volume. Fine, so that is the momentum in the x direction. Now let us ask the question that what is the time rate of change of momentum in this volume, p dot.

Let me draw this volume here and that is my fluid. This is the x direction. So P dot along the x direction will be again the time rate of change of rho*vx integrated over the full volume. That is the rate of change of momentum in this sphere, along the rate of change of the x component of the momentum. Now let us apply the law of conservation of momentum, actually it is the Newton's second law of motion.

So this momentum can change due to various reasons. So the first reason is that the momentum can either flow in or flow out. So what is the rate at which momentum flows out. Let us ask the question, what is the rate at which momentum flows out. So again, we have to do how do we calculate that, we have to calculate the momentum current. The surface integral of the momentum current will give us the rate at which the momentum flows out.

So we are interested in the x component of the momentum. So how do you calculate the current just multiply it with the velocity. This will be equal to one term which is the integral ds dot the momentum current and the momentum current, the mass current was rho*v, so the momentum current is going to be rho*vx that is the x component of the momentum into v. Rho*vx is the quantity we are looking at, the rate at which it flows is rho*vx*V, just like the mass density.

If I ask the mass current, it is rho*v, the momentum current is the x component of the momentum into v. So this is just the flow in or out of the momentum. Momentum can flow in, momentum can flow out. That is one of the possible reasons why the momentum can change. Now there is another possibility, we know that is Newton's second law of motion that the rate of change of momentum is equal to force. So we have to calculate the total force acting on this fluid.

Now there can be 2 kinds of forces, one is called a body force. A body force acts all through the fluid. So let us assume that there is a body force and the force per unit mass, we will call it f. So force per unit mass is f. So if I want to calculate the total force, the rate of change of momentum will have a term, which is + the total force. Total force is the force per unit mass, which I am calling f into the mass, which is per unit volume, which is rho integrate over the entire volume which is dv.

If I do this integral over the entire volume, I will get the total and I am only interested in the x component of the force, so I will write fx. So f is the force per unit mass, which we also call acceleration and it is acting on the entire fluid. So this is the force per unit mass. This is the mass per unit volume. So this gives me the force/unit volume. I integrate over the entire volume; I will get the total force acting on this fluid.

And I am only interested in the x component, so I have taken fx. That is the next term and we have one more term, which is possibility, which is the surface force. So there may be external forces acting on the surface of this fluid. So we could have forces, which act on the surface, so we could have surface forces here. They act on the surface and we are going to consider a particular kind of a surface force. Let me write these terms over here, what these terms are.

So f is force per unit mass, it is a body force and we will consider a particular type of surface force, which is pressure p, so if this surface is immersed in a fluid, we know that there will be pressure exerted by the fluid outside on this and we know Pascal's law, so this is called pascalian pressure, which states that the pressure is isotropic, same in all directions. So if I put a surface over here, if I take this surface, the force due to the pressure is going to be normal to the surface.

So the force due to the pressure is going to act in the same direction as ds, the unit surface area. So if I want to calculate the total external force due to the pressure and I am interested only in the x component of that, then I can write that term as there being one more force term, which is the integral of pressure into the x component of the surface area. Because the pressure is always along the surface area and we are interested only in the x component of the force.

So I will write this as i that is the unit vector along the x direction dot ds and this will have a negative sign because the pressure acts inwards opposite to the surface area. So I have p dot which is equal to this + this + this, the 3 terms. Now again, p dot we have a volume integral here, when I write it in terms of the rho*v, this term is a surface integral, which we can convert into a volume integral. This term is a surface integral, which we can convert into a volume integral.

So let me write down the final equation and the final equation is going to be. So we first have this term here. So the integrants of all the volume terms should be same. So we have.

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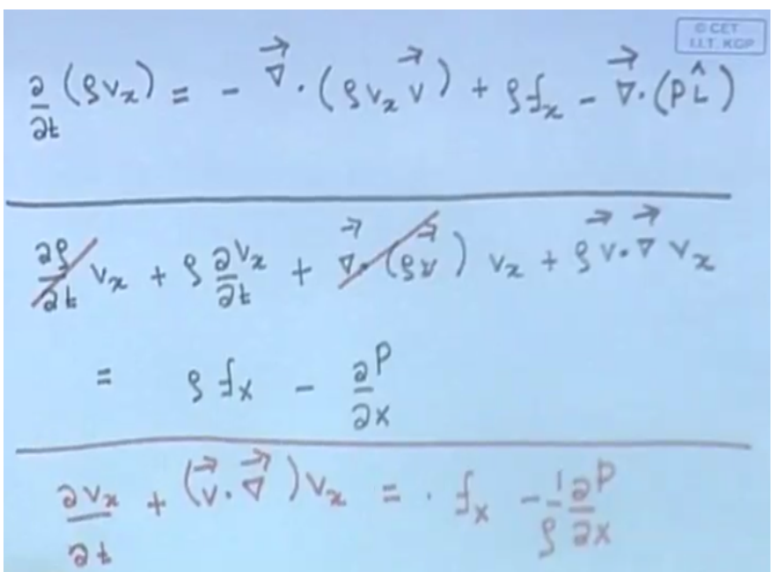

Del/del t of rho vx, that is this is = minus the divergence of this term. So this will be = - the divergence of rho vx*V, that is the divergence of this term and then I get a volume integral that is Gauss law and here I already have a volume integral, so what I have is +rho*fx and then I have to take the divergence of this term and that is going to be minus the divergence of p*i.

So this is the conservation of the x component of the momentum. Now we can simplify this a little further. So let me simplify it a little further so I will have the left hand side, let me do this simplification here, it is a bit of an algebra, but let us go through it. So I will have 2 terms here, one term is del rho del t*vx+ a second term which is rho* partial derivative of vx. This is equal to, I can take this on to the left hand side, so I will have +.

Again there will be 2 terms, one term is going to be the divergence of rho*v into this whole thing into vx and there will be another term which is +rho*v dot the gradient of vx. That is how I have written this term taking it to the left hand side and I have written it as 2 terms. I have broken it up. And I have this term, so this is going to be = rho fx and look at this term now. This is the divergence of the pressure into i, the unit vector.

When I take this divergence, there will only be a del del x term, partial derivative with respect to x. There will be no derivative with respect to y or z, because j and k do not appear over here. So I will have one more term, which is minus the x derivative of the pressure. Now you see we can simplify this, look at this term. This is del rho del t. del rho del t+ divergence of rho*v is 0. So we can combine this term del rho del t and here we have the divergence of rho*v.

Both of them multiply vx, so the sum of this and this is 0 and we are left with, let me write the remaining terms. So the remaining terms, we can cancel out rho throughout. Rho occurs everywhere, so we can cancel it out throughout except for this term, where there will be one by/rho now. Let me write down this equation. So the equation is del vx del t+v dot the gradient acting on vx. So I have written this term, I have written this term = fx-del/del x of p.

This is only the x component of the momentum. I have similar equations for the y component and the z component of the momentum also. There will be a one/rho here. So there will be 3 such equations, one for the y component, one for the z component also and I can now combine all 3 equations and get my final equation. So let me write down the final equation, which I get when I write. I can write this for a vector v.

The final equation is del v del t+v dot grad v = -1/rho*grad p+f. So that is my final equation and this is called the continuity equation and this is called Euler's equation. So this tells me how the velocity will evolve. This tells me that the mass is conserved here. We have used momentum conservation to calculate the evolution of the velocity. Now there is a third equation, which is the conservation of energy.

So we have looked at the conservation of mass, we have looked at the conservation of momentum, let us next look at the conservation of energy and so this is the energy conservation equation.

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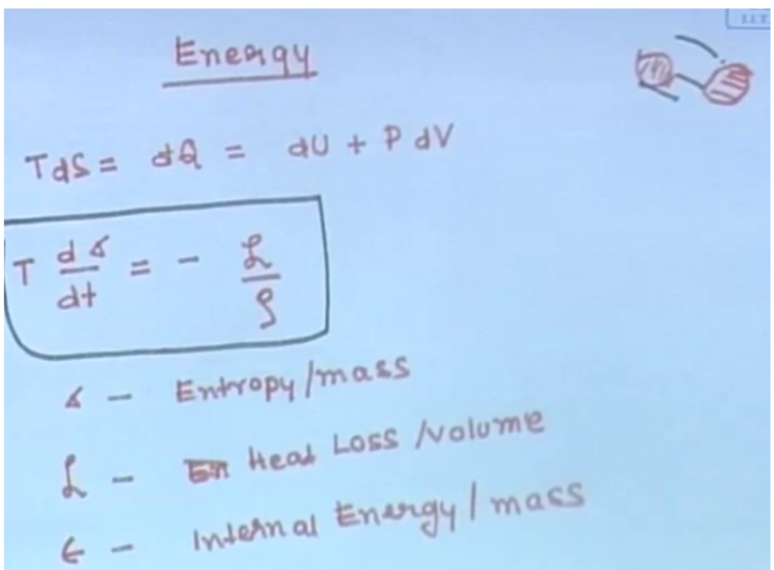

When we discuss the energy conservation equation, we start from the first law of thermodynamics. So let us consider some fluid element like this. If I compress or rarify this fluid element stretched, then we know that the energy exchange is going to be governed by the first law of thermodynamics and the first law of thermodynamics tells us that dQ = dU+p dv. So the heat flow is equal to the change in the internal energy.

The heat that this system exchanges that flows across the boundary of this system is equal to the change in the internal energy + the work done. This is the first law of thermodynamics. It is essentially the conservation of energy and we know that the entropy ds is defined as dQ/t. So we can say that this is = t ds. Now in our discussion the fluid element that we are looking at not only can it get compressed or stretched out and exchange heat, but it can also flow.

So we have to now consider this fluid element as it flows, as it moves. So it is first of all, these thermodynamics holds for a fixed system. My system cannot change. Since my system cannot change, it is obvious that I should look at this fluid mechanics not in the Eulerian picture, but in the Legrangian picture. So it is very clear that we have to work in the Legrangian picture. We have to follow the same fluid element as it flows.

We cannot look at different. If you work in the Eulerian picture, then we will be considering different fluid elements at different instances of time. So it is clear that we have to first write down the equations in the Legrangian picture. So we are following the evolution of one fluid element. That is the first thing. Second thing is that let us consider a fluid element of unit mass. So we will work with the entropy per unit mass sigma.

And the rate of this entropy per unit mass, so my system has unit mass, the rate of change of entropy per unit mass is equal to. We are interested in the rate of change of entropy per unit mass and that is going to be the heat so t, so we are interested in this. So t times the rate of change of entropy per unit mass is going to be equal to the rate at which heat flows in or flows out of this mass element. This we can write it like this. So sigma is entropy per unit mass.

And L is the heat loss per unit volume. So we would like to convert the heat loss to per unit volume to the heat loss per unit mass. This is the heat loss per unit mass, heat change or heat flow per unit mass. So we just divide it by the density. Density is mass per unit volume and this will give you the heat loss, rate per unit mass, and since it is the loss we put a minus sign and this tells you the heat gain, which is td. So I have essentially written this equation in this way for a unit mass and differentiate it with respect to time.

Now this can also be written. Let me also introduce the internal energy. This is the internal energy per unit mass. So the energy conservation can either be written like this and if it is an idiomatic situation, then the right hand side will be 0. There will be no heat loss or gain if it is an idiomatic situation and we have the fact that the energy per unit mass is going to be conserved.

In case it is exchanging heat, it is not idiomatic, then there will be a heat loss or heat gain, which is given by this. We can also express this in terms of the internal energy per unit mass. So let me do that. We know that td sigma, so this will be written as. We are going to use this expression now. I am writing this in terms of these 2. So I will have one term, which is the rate of change of internal energy per unit mass.

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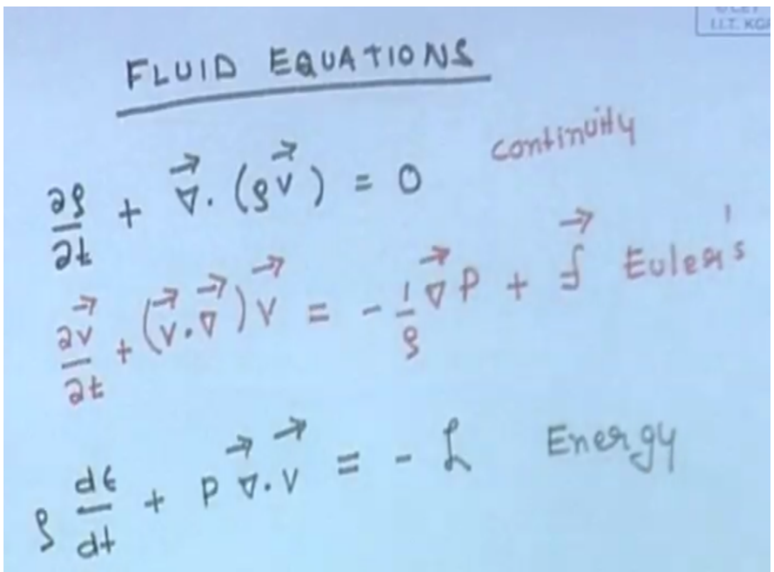

So I will have one term which is this term. This is going to be pressure into the rate of change of volume per unit mass. So I will have one more term, which is +p*d/dt. Now the volume per unit mass is going to be one by the density. Density is mass per unit volume. So volume per unit mass is going to be one by density. So the same equation now becomes this and this is = - the heat loss per unit mass.

We can further simplify this term here, so let me just simplify this term. So this term can be written as -1/rho square d rho dt. I am just simplifying this term and this becomes = -1/rho square. Now this is the conductive derivative d rho dt. I a following the same fluid element as it moves. So we can write this as del rho del t+v dot grad rho, which we can use the continuity equation. Look at the continuity equation.

If I break this up, there will be one term, which is v dot grad rho and there will be one more term, which is rho*divergence of v. So I can write this as one/rho square, rho*divergence of v using the continuity equation. I have just simplified this term over here. So now I can plug this back in here and multiply it throughout by rho and what I get then at the end of the day is the following equation. I have rho*the conductive derivative of the internal energy + p*the divergence of v.

One of the rho cancel out from here and one of the rho cancels out when I multiply it with rho= - the heat loss/unit volume. So that is the conservation of energy. So let me write it down here. This is energy. These are the 3 fluid equations. In addition to this, we shall be using quite often a 4th equation and that is the fact that the force that we are dealing with is gravitational.

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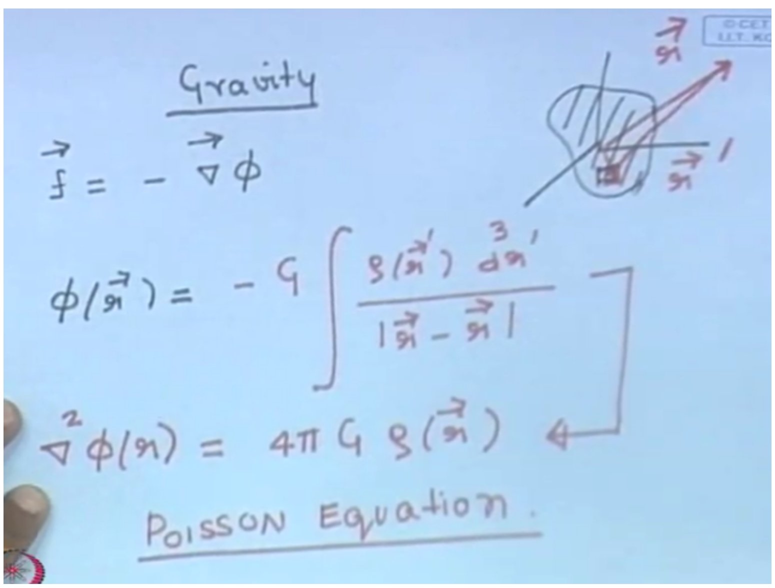

So we have a gravitational force. So the body force that we are dealing with, so if I have a fluid here in a gravitational field, then it will experience any element of the fluid will experience an acceleration force per unit mass, F, which can be written as minus grad of some potential phi. Phi is the gravitational potential. For a gravitational force, we know that the acceleration can be written as minus grad of, where phi is the gravitational potential. That is the first thing.

Second thing is given the mass distribution, the same mass distribution is going to produce a gravitational field. So I am interested, let us say in calculating the gravitational field produced by this at some point r over here. So I am interested in calculating phi at some point r produced by this mass. How do I calculate it? What we do is we divide this up into small, small mass elements. So let us ask the question how much is the gravitational potential produced by this mass element at this point.

So we know that this will produce a gravitational potential here which is -gm by this distance. So if I say that this is located at r prime, then this is going to be -g*rho at r prime dQ r prime that is the volume element over there divided by the distance between these 2, which is r-r prime. So this tells me the gravitational potential produced at r by a particle at r prime. The mass of that particle being the density into the volume.

-gm/r and to find the gravitational potential by the entire object I have to integrate this over the whole of space. So this tells me the gravitational field produced by a mass distribution and here we shall be dealing with self gravitating system, so the field acts back on the same object itself. It is an extended object and the field acts back on the same object itself. Now without going through the calculation, it is a very simple thing. I shall just write down.

So this is the way you can calculate the gravitational potential from the mass density. It is convenient to write it instead of this integral expression, write it as a differential expression and that differential expression can be obtained if I act with the Laplacian. If I calculate del square phi, so I act with the Laplacian on this del square phi and this comes out to be 4 pi G*rho r. So this is the same equation as this. These are identical.

So if I take the Laplacian operator and act it on this phi, I will get this or if I integrate this equation, I will get this other solution and this equation is called the Poisson equation. Del square phi is 4 pi G*rho. So for a self gravitating fluid in addition to the 3 fluid equations that we had, we also have the Poisson equation, which is del square phi = 4 pi G*rho. This is for a self gravitating fluid.

So the fluid exerts a gravitational force on itself, different parts of the fluid exerts gravitational forces on the other parts. This is called the Poisson equation. This is essentially the Newton's law of gravity written in the form of partial differential equation, nothing more than that. Today let me stop here. Today we have introduced the fluid equations and in future classes, we shall apply these fluid equations to astrophysical situations.