Astrophysics & Cosmology

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Lecture - 03
The Solar System

Welcome, we were discussing the motion of 2 particles that interact with one another through the gravitational interaction and we started off with discussing the generalized 2-body problem and we saw that the angular momentum is conserved, so it has to be, no, okay, the generalized 2-body problem, we reduced it to a one-body problem of the reduced mass and the motion of the centre of mass.

The centre of mass moves like a free particle, so it is not of interest. So we are now interested in the motion of the reduced mass around the centre of mass and this is the central force problem. The angular momentum we saw is conserved in such an interaction which is isotropic, does not depend of the direction. So we have the conservation of angular momentum and we also have the conservation of the energy because the interaction is not explicitly time-dependent.

So we were using these 2 conserved quantities and the fact that the orbit will be in a plane to determine the orbit. So we have the conservation of angular momentum, let me write that down again.

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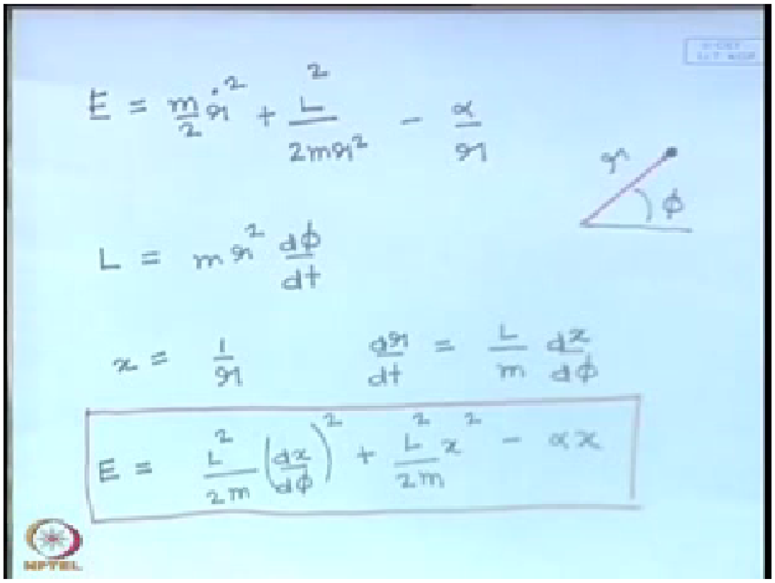

Let me start with the energy, the conservation of energy E, we had calculated this, this was m/2 r dot square+L/2m r square, L square/2mr square-alpha/r. So let me remind you what each term stands for, this was the kinetic energy of the radial motion, this was the kinetic energy of the tangential motion which we wrote in terms of the angular momentum. Please note that I have changed the symbol for the angular momentum.

We had capital M earlier but then I realized that it could be confused with the total mass, so I have replaced with capital L. Capital L is the angular momentum. So this is the kinetic energy of the tangential motion and this gives rise to the centrifugal, this potential -alpha/r. This is the gravitational interaction. So we have this and we also have the angular momentum L which is mr square d phi dt.

And we are interested in calculating the motion, the orbit of the reduced mass around the centre of mass, this is the centre of mass, that is the origin and this is the particle reduced mass. This distance represents the separation between the 2 particles that we were originally interested in. So we are interested in the motion of this r as a function of phi, that is the orbit. So we would like to calculate the orbit in the plane of motion right now and we proceed.

I had outlined how we proceed in the last class. We proceed by eliminating time from this whole thing and it is also convenient to introduce x which is 1/r, okay. And using x which is 1/r and using this, the time derivative, so the time derivative of r can be written as L/m* the phi derivative of x. Okay we did this algebra in the last class. So this is straightforward algebra, nothing much to it.

So we have to just replace dr dt in terms of dx dt and you can use this to replace derivatives in terms of t with derivatives in terms of phi, okay. So doing this, you are led to the relation that dr dt which occurs over here can be written as the angular momentum by mass * dx d phi, okay. This is straightforward algebra and we substitute this here then we have the energy E. The energy = L square/2 m dx d phi, whole square+L square/2m*x, where x is 1/r-my x square that right-alpha*x.

So the energy, the conservation of energy can now be written in this way and the same exercise can also be done if I had some other interaction instead of gravity, the only change would be that I would have, instead of alpha x, I would have the other interaction that was there, okay. So in case you are interested in some other interaction, if you just replace this term, -alpha x with something else, okay.

So this is the final expression for x in terms of phi, you can see that it is a very, solving this is quite straightforward. So what we can do is, we can take all the terms involving x on to one side and there is only then going to be, so we can take E, okay let me just write down this equation in a convenient way, what we will do is we will multiply this whole thing, this whole equation with 2m/L square, okay. We will multiply it with 2m/L square.

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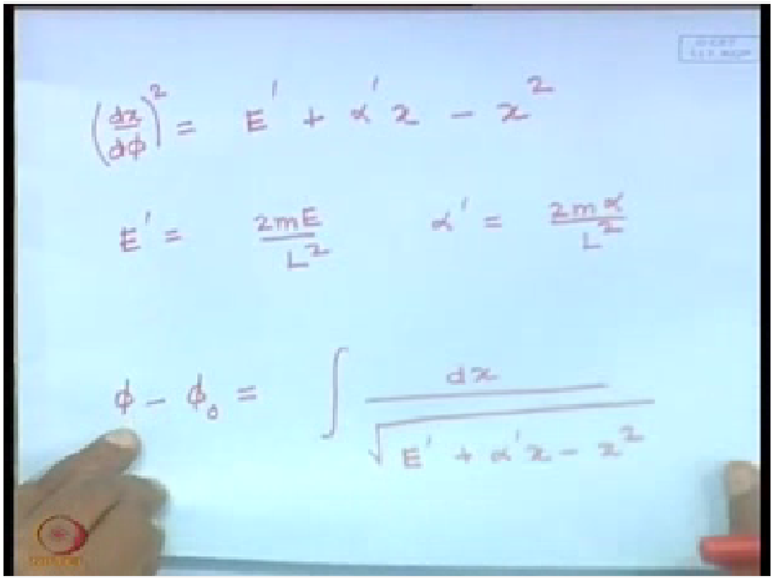

So I will get by multiplied with 2m/L square and I can take these 2 terms on the left-hand side. What I will get is dx d phi square = E prime, E prime is E/E*2m/L square -/+alpha x, alpha prime x-x square. So all that I have done is I have multiplied with 2m/L square and then taken this to the left-hand side, taken this to the left-hand side and I get this. Where E prime is 2m E/L square, alpha prime is 2m alpha/L square, right.

And this can be integrated straight away, it is not difficult at all and if you integrate this, what you get is phi - phi prime = the integral, phi - phi 0 where phi 0 is an arbitrary constant of integration, this is = dx/square root of E prime+alpha prime x-x square, right. So we have obtained a formal solution, all that you have to do is to do this integral and this can in principal be done for any kind of a potential interaction but the integral cannot be analytically done in most cases, it can be done, luckily it can be done for the gravitational interaction.

So we will just do this integral for the gravitational interaction which is what we have written down here. So the way to proceed, this integral is not very difficult at all.

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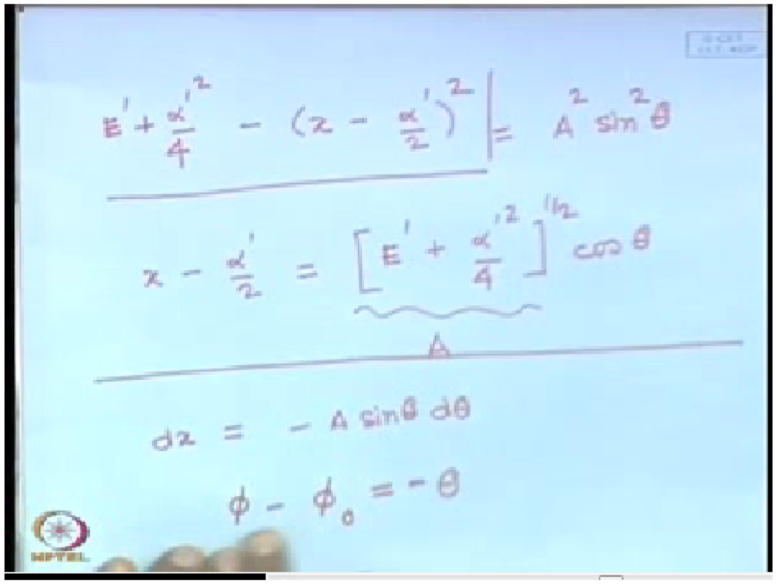

The way to proceed is as follows: what we do is we write E prime, we write this as E prime - x - alpha prime/2 square. So we want to write the term that occurs below the square, inside the square root and I am writing it as E prime - this, so I will have x square and I will have - x square which is there. I will have + alpha prime*x. So I have one extra term which is - alpha prime square/4, so I have to add that, + alpha prime square/4, okay.

So this is the term that occurs under the square root in the denominator and what we will do is, so this is a term, what we will do is, we will say that x - alpha prime/2 = E prime+alpha prime square/4 to the power of half cos theta. So with this substitution, this is quite straightforward now. The denominator, this term, you can see now, it becomes, I can take this term, the square of this, so this term will become, if I call this A, this will become, this is A square and this is A square cos square theta.

So this will become A square sin square theta, right and dx is -A sin theta d theta. So now we can go back to this integral, phi-phi 0 =, so dx will be -A sin theta d theta and this whole term in the denominator will become A sin theta, so A sin theta will cancel out and what we are left with is that phi-phi 0 = -theta, that is all, right. The integration is done and we can put back this now over here, so if you put it back over here, what you get.

And this phi 0 is an arbitrary constant, it essentially tells us what is the value of r when phi = 0, it tells us where the origin of phi is, so we will choose phi 0 to be 0 with this choice.

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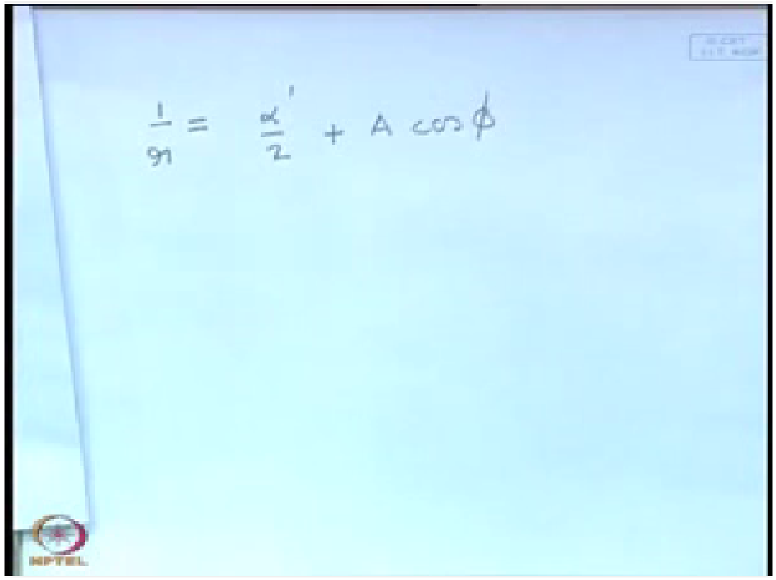

The equation it becomes what we have is, so we have the solution is over here basically, the solution is that 1/r = alpha prime/2+, I will write this as A+A cos phi, okay. So we have obtained the solution to the problem that we want to solve, it is not in terms of the energy or the other integrals of motion but we can do that. It is most convenient to write it in the following way.

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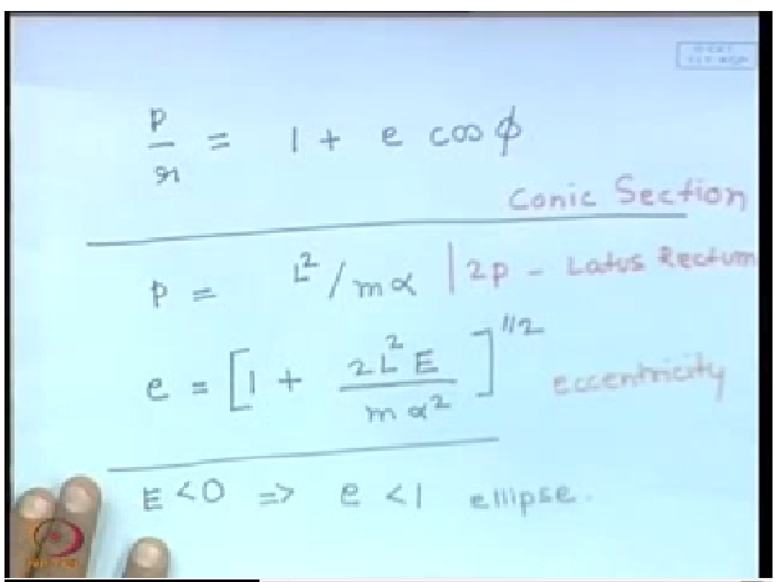

Let me write it in this, it is most convenient to write it in the following way, P/r, let me put it on a different sheet of paper. So it is most convenient to write it in this way, P/r is = 1+e cos phi, okay. So this is nothing, the same thing, I have just taken this factor over here and called it P and I have taken A and divided it by alpha prime/2 and I called it e. And if you go back and work out what the value of P will be, you have to replace alpha prime and e prime, etc.

The value of P is L square/m*alpha and the value of e is 1+2L square E/m alpha square to the power of half, okay. So this is nothing but straightforward algebra, you have to just work back and P we know, is we have seen, right. So you can identify what P is from this expression and what E is from this expression and write them in terms of e and L and alpha and the mass of the particle, the reduced mass and you can arrive at these expression.

This is just straightforward algebra, I am not going through it, okay. So this is the orbit of the reduced mass under the central force motion which is gravity, okay. So let us now spend a little time interpreting what we have just worked out. The first thing is that this equation, it describes what is called a conic section which I am sure all of us are familiar with. So this equation that we have worked out, the equation between r and phi, it defines what is called a conic section, okay. So this is a conic section.

And the conic section is defined in terms of 2 parameters, these 2 parameters here are P and e, okay. 2P is called the latus rectum of a conic section. 2P is called the latus rectum and e is called the eccentricity. So the orbit is defined in terms of these 2 things and there is phi 0 which we have already chosen a particular value such that, we shall come to that. So phi 0 has been said to 0, okay.

So this defines a conic section, once you tell me the value of p and e, you have told me precisely which conic section you have, okay and these are decided by the values of the angular momentum, the energy and the mass of the particle and the value of alpha, the reduced mass of the particle and the value of alpha, okay the mass m basically. Now, the first point is that, we know that there are 3 kinds of conic sections, there are ellipses, parabolas and hyperbolas, okay.

Whether it is an ellipse, a parabola, or a hyperbola, is decided by the value of the eccentricity e, okay. And now let us look at this expression for the eccentricity, if the energy < 0, then the eccentricity is going to be < 1 which is an ellipse, okay. So for eccentricities which are < 1, we have an ellipse. So the energy is negative. Let me connect up with what we had discussed in the last class.

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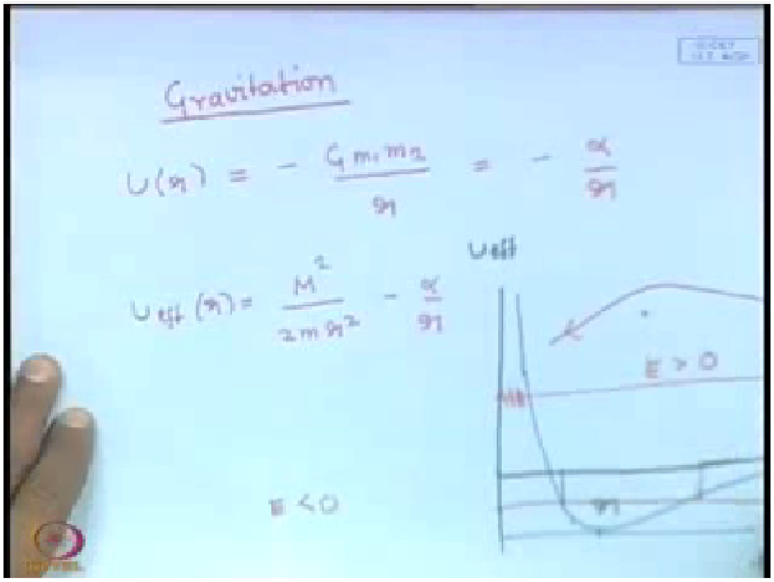

In the last class, we had seen that this problem is basically there is an effective potential which is the combination of the centrifugal potential which is positive and which dominates towards near the centre and there is the gravitational potential which is negative and which dominates faraway, okay. So the particle here is one, the motion in r, here is a 1-dimensional motion with an effective potential given by this, right.

And we see that for positive energies, the particle comes from infinity and comes to a finite distance and then goes back but for negative energies, there will be bound orbits and it will be, so if I have a negative energy like this, then the orbit is going to be restricted within these 2 values of r, okay and this is also what we see over here if the energy is negative, the orbit is going to be an ellipse which is a bound orbit. The value of r is going to vary between r1 or r2, okay.

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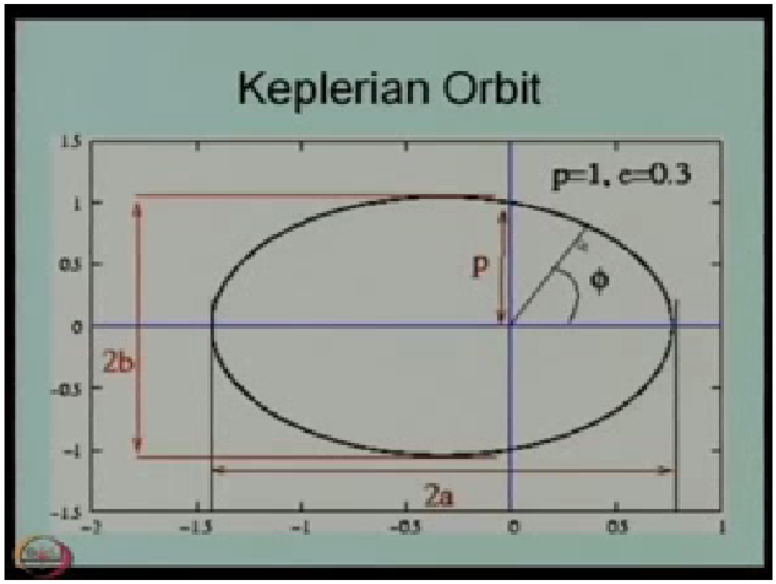

So this picture over here on the screen shows you one example of an ellipse, okay. So this is an example of an ellipse of one particular possible orbit. This has been drawn with p = 1 and the eccentricity = 0.3. The focus is over here and phi is the angle. Okay, the focus is over here and phi is the angle with respect to this direction, okay. What is the meaning of phi = 0.

Let us go back to our equation now. So phi = 0, if I had said phi equal to 0, then P/r is 1+e, that is the maximum value which the term on the right-hand side can have, which tells you that the value of r is minimum, okay. So phi = 0 corresponds to the position of nearest approach.

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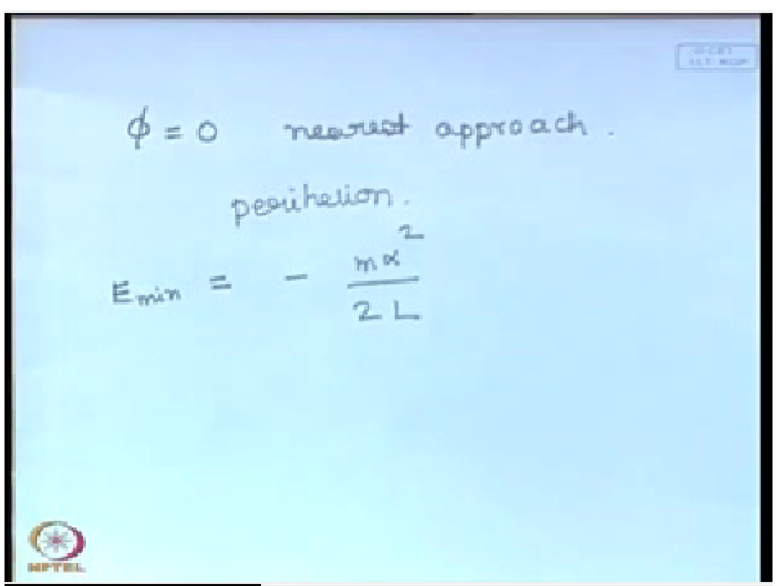

So the point where it approaches nearest to the centre of the motion, okay. So that is how the origin of phi has been chosen and this is called the perihelion and recollect that if I have 2 particles, if I start off with the 2-body system, then this focus, this is one of the focuses of the ellipse, this corresponds to the centre of mass of the 2 particles. So one of the focus corresponds to the center of mass and the ellipse as one of the centre of mass has one of focus, okay.

And phi is the angle with respect to the position where the ellipse is where the orbit is closest to the focus to the center of mass. Now let us discuss a few other interesting points, now as you increase the eccentricity, the orbit will get more and more elongated. On the other extreme, the eccentricity = 0, you will have a circle, right. You set the eccentricity to 0, you will have a circle, P/r is a constant, okay.

So the circle, when do you have a circular orbit. We have seen that you will get a circular orbit at the minima of this potential where there will only be 1 value of r, okay. Another way of arriving at that is by determining the minimum value of the energy that you can have, right. E will 0 and E cannot be lower than that. So the minimum value of the energy given the value of the angular momentum.

The minimum value of the energy is going to be m alpha square/2L-m alpha square/2L, okay. And you can also check that this also corresponds to the minima of that effective potential that is the minimum value of energy that you can have and eccentricity is 0. So for a given angular momentum, if the particle has the minimum possible energy, you have a circular orbit. If you give it more energy, the orbit is going to get more and more elliptical.

So for the fixed angular momentum, if you increase the energy of the particle, the orbit is going to get more and more elliptical, okay. Now this picture, let us go back to the picture of the ellipse. So this picture shows you the ellipse and for this ellipse, you also have the major axis, so this is the major axis, the distance between this extremity and this extremity and you also have the minor axis which is the distance between the 2 extremities along the y axis.

The latus rectum is the distance between the points where it intersects the y axis, that is going to be 2P that you can check easily, okay. But we are interested now in the major axis and the minor axis.

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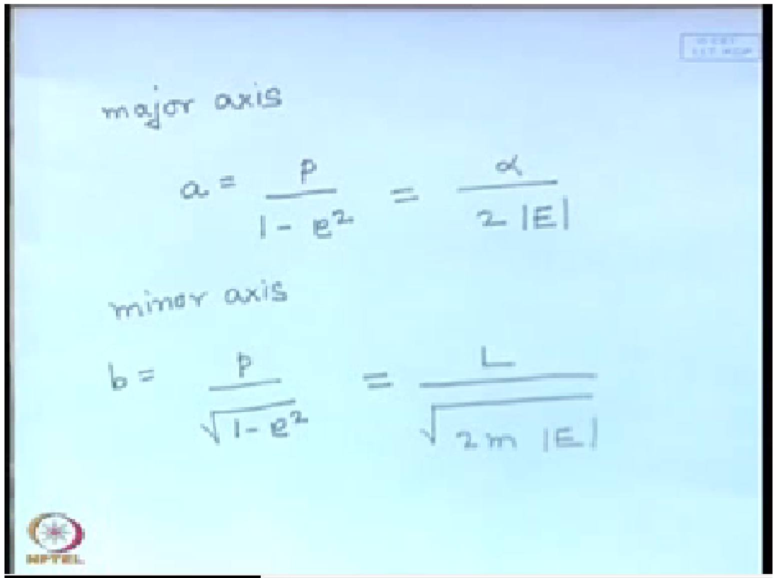

Now the major axis, we know that the major axis is P/1-e square. This is a straightforward calculation, you can just take the farthest distance and the nearest distance and just add them up, okay. So you will get the major axis and if you write P and e in terms of the energy's angular momentum, etc., then this comes out to be alpha/2 mod E. So very interestingly, the major axis does not depend on the angular momentum.

It is determined just by the energy, okay and the minor axis again you can calculate this by extremizing y, the distance from the x equal to 0 axis and then taking twice that and the minor axis you can check easily, comes out to be b - the square root of 1 - E square, okay and if you put in the values here, what you get is L/square root of 2m mod E. So the minor axis is what depends on the value of the angular momentum, okay.

So if you fix the angular momentum, then the minor axis is fix. Now if you keep on giving more energy for the least amount of energy that you can give, you will have a circular orbit with the value of the minor energy and if you keep on giving more energy, both the minor and the major axis are going to change, okay. So this tells us what the orbit is going to look like in such a central force problem.

Now if you want to calculate r and phi as a functions of time, then one has to use this equation and obtain phi in terms of time, okay and so it involves some integration again. Okay, we will not discuss this, so we are interested here in r as a function of phi. If you want to know exact r as a function of time and phi as a function of time, you have to use this, okay, this will give you phi, if you use this and then combine it with the things that we have worked out.

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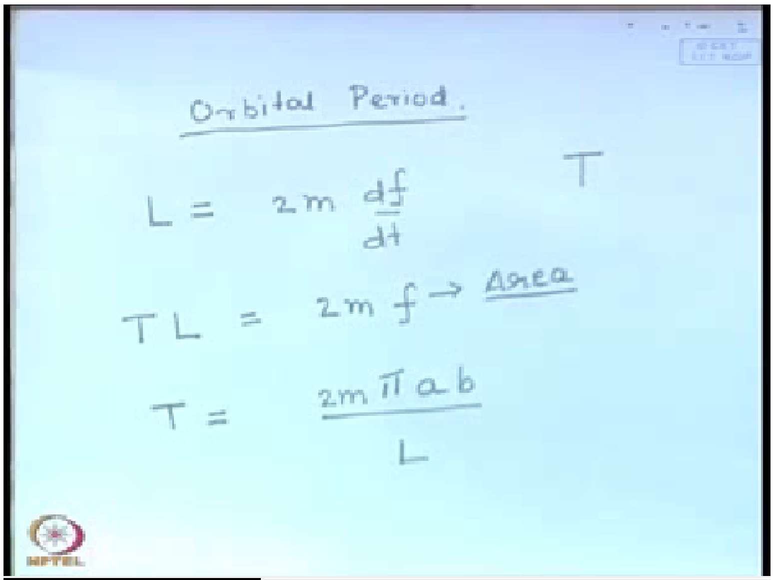

The next thing that we shall workout is the orbital period of the motion, okay. Now, we have seen that the sectorial velocity is conserved. It is related to the angular momentum and it is conserved. So what we saw was that the angular momentum L = 2 * the mass of the particle * df/dt where f is the area enclosed by the orbit.

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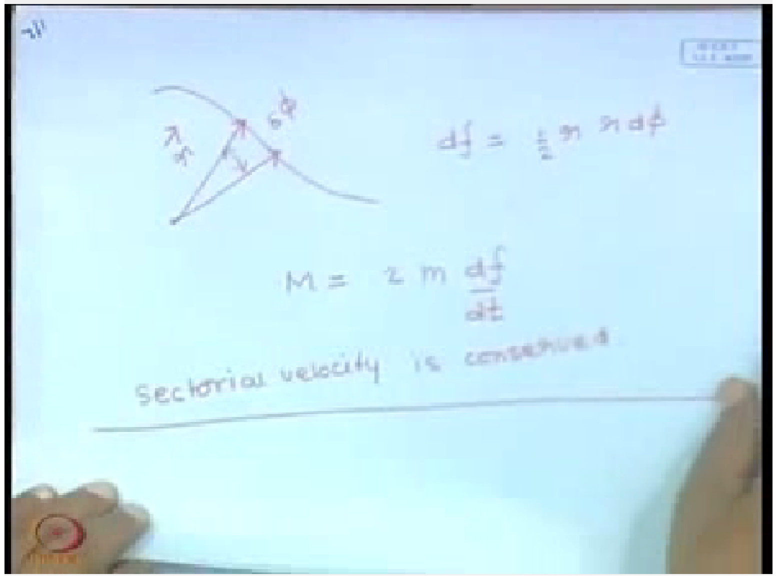

This we saw is true for any central force problem, need not be gravity, any central force problem whatever be the orbit, in this orbit, this is the center of the force, the area subtended by the orbit at the centre of the force, the rate of change of that area is constant, right this is what we worked out. So the sectorial velocity is constant, is conserved, it is related to the angular momentum. Okay, we saw this, this is true for any central force problem.

So now what we can do is, we could integrate this equation over 1 whole orbit. So let us say that the orbital time is T, orbital period. So we can integrate this over 1 whole orbit. If I integrate this over 1 whole orbit than what I will get is that T, the orbital period * L, the angular momentum = 2m * the area, this f is the area, right. This is the rate of change of area. So if I integrate over 1 whole orbit, I will get the area of the ellipse, right.

I will start from here and calculate the area that it subtends at the center. If I integrate over the entire orbit, I will get the area of the ellipse. So f is the area of the ellipse. So what we have from this is that the orbital period = 2m, we know what the area of an ellipse is, it is pi*ab, so it is 2m pi*a*b/L, okay. So we have worked out the orbital period for this motion, it is 2 pi m a b/L and we can substitute the values of a and b and we can determine what this is going to be.

So let us do that, let us substitute the values of a and b*this. So we will substitute these over here and work out what the orbital period is, right. So let us do that.

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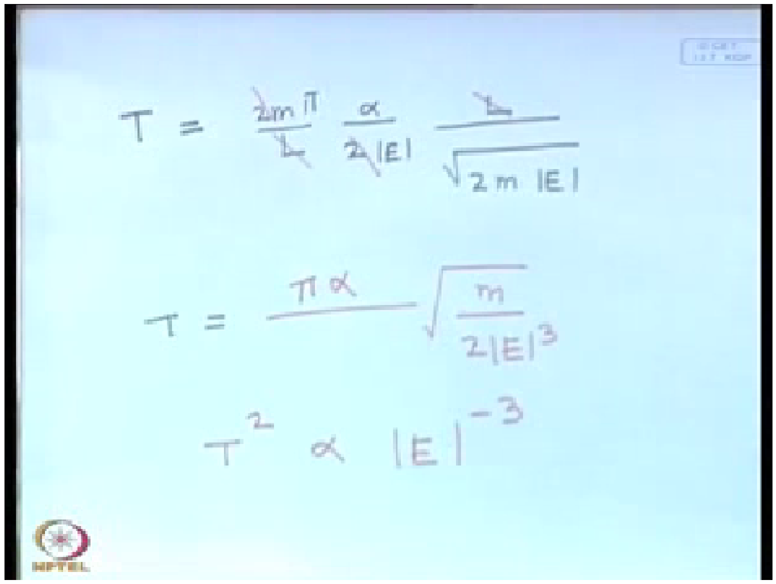

So what we get is that the orbital period =, so 2m pi/L and a, the semi major axis, is alpha/2 mod E and b is L/square root of 2m mod E and what we have is that the orbital period =, so L cancels out, 2 cancels out and I have pi and I have alpha, so I have pi alpha, okay * the square root of m/2 E cube, right. So this is what we get or the bottom-line of this is that T square is proportional to the energy won by the energy cube, okay.

So this tells us how the time period of the orbit, it does not depend on the angular momentum, it only depends on the energy of the particle and it scales T square scales as the energy of the orbit to the power of -3 or T scales as the energy to the power of -3/2, okay. So we have worked out certain features of the motion of objects under the influence of gravitation, orbits under the influence of gravitation and the planets.

The motion of our planets moving around the earth, the planets around the sun and the variety of other things, we can have binary stars, etc., etc., all of them are described to a large degree by such orbits. Now next in this class today, let me work out a generic feature, let me show you how we can use a generic feature to determine some properties, some very useful property of the orbit, okay. This principle that we shall use is called similarity, okay.

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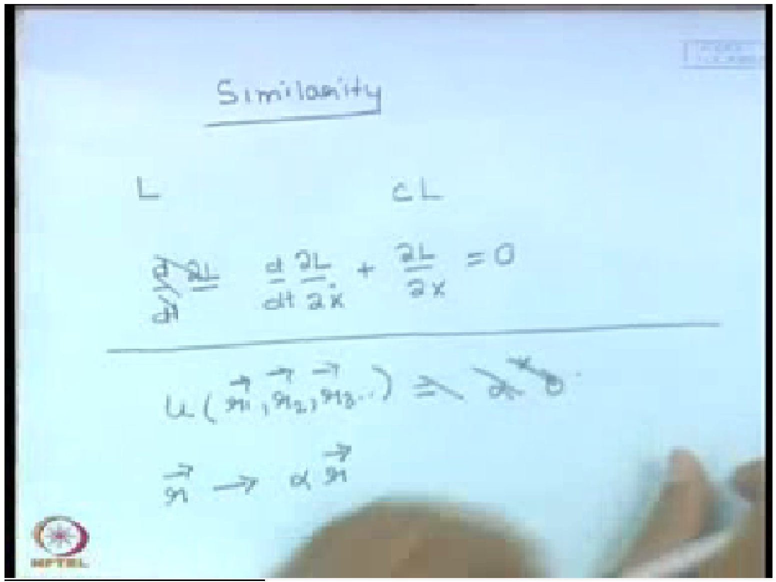

Let us start of by noting the fact that I have a system which has a Lagrangian L, okay. The system has a Lagrangian L and if I multiply the Lagrangian with the constant C*L, then the equation of motion, let me remind you what the equation of motion is, okayed. The equation of motion, it is not changed, okay. This equation of motion, if I multiply the Lagrangian with the constant, the equation of motion is unchanged, okay. The constant just goes out.

Obviously, it is a very trivial statement but we shall start from this. Now suppose my system, whatever system I have, has this interaction, the particles, the potential in this system which could be in principal of function of r1, let us say r1 vector, r2 vector, r3 vector, etc., so I have some potential in this, my system has got let us say whatever n particles and there is some interaction between them which depends on r1, r2, r3, rn, okay, which is there in the Lagrangian.

And if I scale, suppose my potential is homogenous in the sense that if I do a transformation, r goes to alpha r. So if my potential is, interaction potential is homogenous, what do we mean by that.

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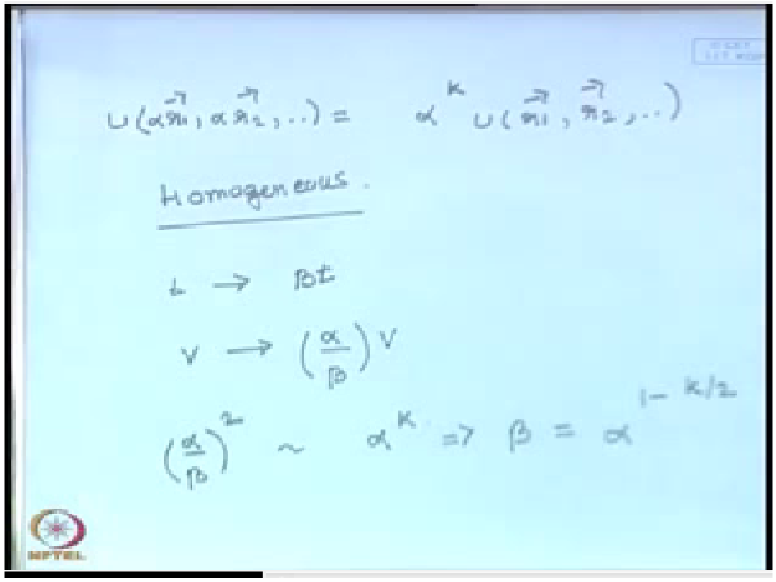

So if it is homogeneous of degree k, so what we mean by this is that if my potential has this property that U alpha r1 alpha r2 etc. = alpha to the power K U r1 r2, okay. So if I assume that my potential interaction is homogenous, okay. Now the Lagrangian, let me remind you has another term which is the kinetic energy term which is half m v square, okay. So if my potential term behaves like this.

Let us ask the question how do we need to scale the time so that the final thing is the original Lagrangian multiplied by just a number, okay. So the velocity under this transformation, the velocity will go to alpha / beta, right velocity will become this. Velocity is r/T, so it will become this, the r has changed to alpha r, T has changed to beta T, so the velocity will go to this and we would like, the whole Lagrangian, to become the original Lagrangian multiplied by a number.

So if that is true then the kinetic energy term, what is the kinetic energy term, half m v square. So we would like the v square term, the v square term will pick up alpha by beta to the power of 2. This should be the same as alpha to the power of K, okay. This should be the same as alpha to the power of K which implies that beta =, how much is beta, beta should be equal to alpha to the power of 1-K/2, right.

So if I scale my lengths in one way and my time in a different way which is given over here, then I will get back the Lagrangian, the Lagrangian will just be multiplied by a number and I will get back the same equation of motion, okay. So the equation of motion is going to be unchanged.

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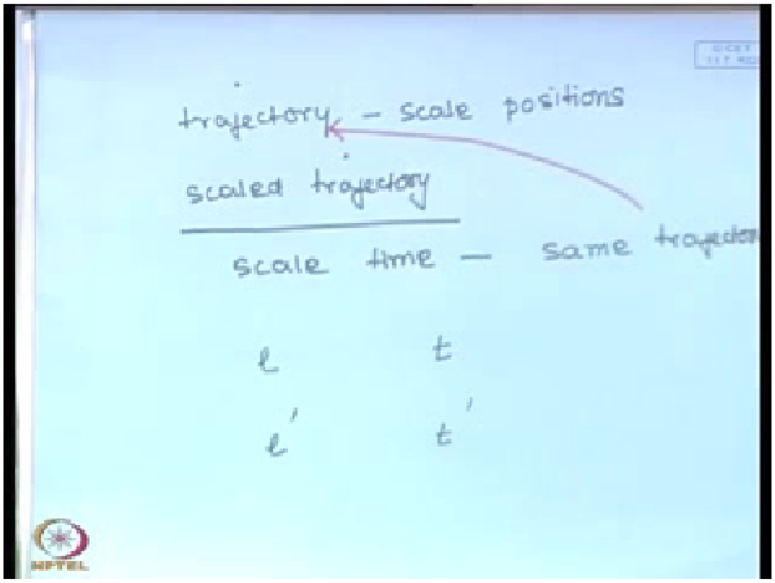

So now let us consider a system. For that system, I have some trajectory and I scale my positions/alpha, I will get a different trajectory, I will get the same trajectory which is scaled, right, I am looking in Cartesian, let us say. So I will get a scaled trajectory which is going to be a different trajectory, okay. So I have some motion, I multiplied that all the lengths/a number, I will get a different trajectory obviously.

But then I can make it the same trajectory which I started with by just rescaling the time that if I appropriately rescale the time then it essentially is going to be my same old trajectory which I had started out with, okay. So if I scale time, so if I scale the positions, so I will get a different scaled trajectory. If I scale the time in a different way, okay, then I will get back the same equation of motion, same time trajectory, okay.

So let us say that there is a trajectory which has a length scale l associated with it and the time t associated with that length scale, right, a particle let us say covers a distance L in a time t and now if I scale it, then how should if I scale it, so that it now covers a distance l prime, what should my t prime be, okay. So we can work this out, this is quite straightforward so what is the condition, we just went through this.

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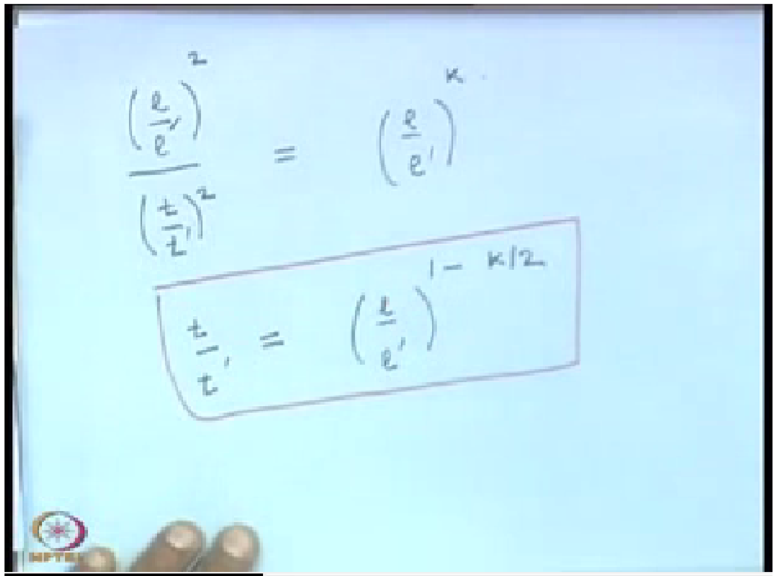

So the condition here is that l by, let us see, so l by right, l/l prime square/t/t prime square should be equal to l/l prime square to the power of k, right, that is the condition. This is the kinetic energy term, this is the potential energy term or t/t prime =. So if I scale my lengths in this way, from l to l prime, the time will also scale from t to t prime and this is related to l/l prime in this fashion, okay.

Similarly, not only does the time change but the energy and everything else also changes accordingly.

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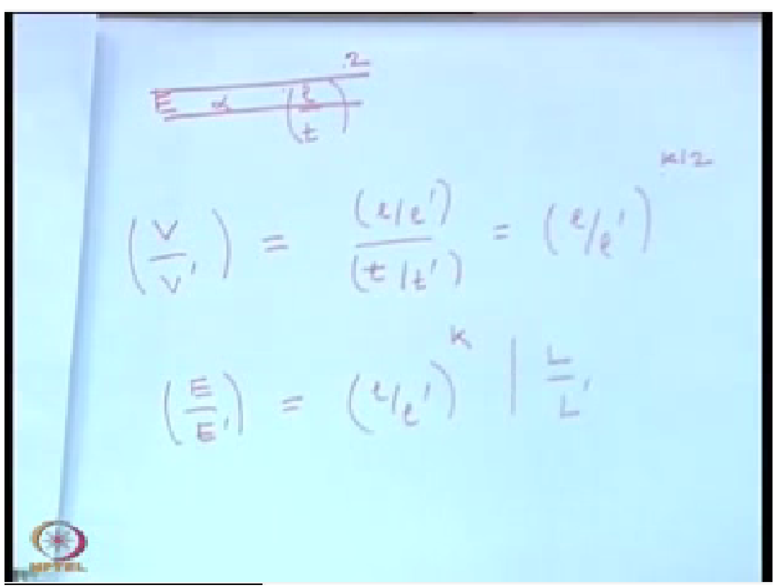

Let us work out how the velocity changes first. So the velocity v/v prime is going to be equal to, if I change my length scale, how will the velocities scale, so I will have a new trajectory, scale trajectory with the scale time, how will the velocity scale, so you can work that out by just noting that velocities l/t. So this is going to be l/l prime/t/t prime. So this is going to be equal to l/l prime to the power of, how much is it, so t/t prime is l/l prime to the power of 1-k/2.

So this is going to be to the power of k/2. And the energy E/E prime is going to scale as l/l prime to the power of k and one can also work out how the angular momentum l/l prime is going to change that it is l*V basically. Now what relevance does it have here.

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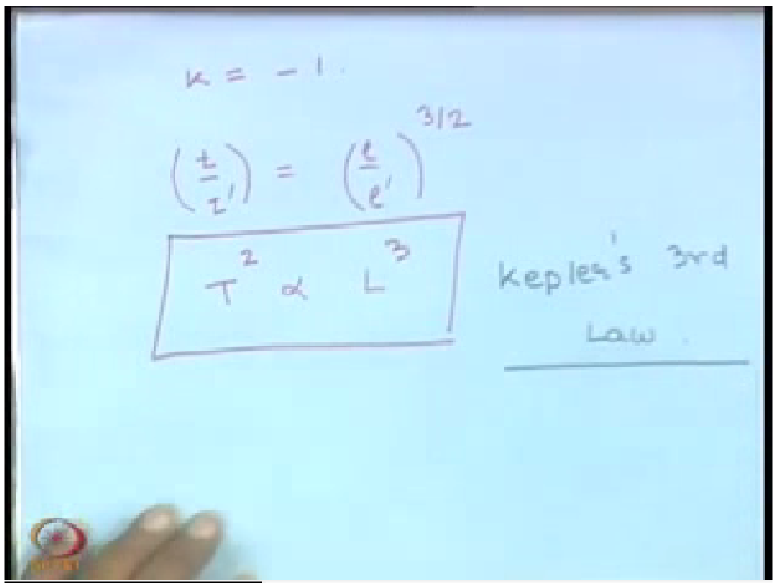

Now we can apply this to see how the time period for a gravitational, so let us take the gravitational interaction. For the gravitational interaction, so there are 2 bodies bound to each other by gravity, that is the orbit that we are considering, okay. So for gravity, k = - 1, okay and let us ask the question, how does the time period change with the size of the orbit, okay, with the length scale of the orbit.

So that is quite straightforward, we have just worked it out, how the time period is expected to change with the length scale of the orbit, it is given over here and k is -1, so what we see is that t/t prime = l/l prime to the power of 3/2 or the time period of the orbit is proportional to the length scale of the orbit to the power, okay. So the time period of any orbit, the square of that scales as the cube of the linear diameter, linear size of the orbit. Okay.

This is referred to as Kepler's third law and let me stop here for today. So in a nutshell, the last 2 classes, we have studied how the orbits in the influence of gravitational force, we have studied these orbits. Now we shall proceed to apply this to the orbits of different planets and the moon around us and various other astronomical situations.