Astrophysics & Cosmology
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Lecture - 28
The Expanding Universe and the Cosmological Metric

Welcome, let me start of today's class by recapitulating what we have been doing.

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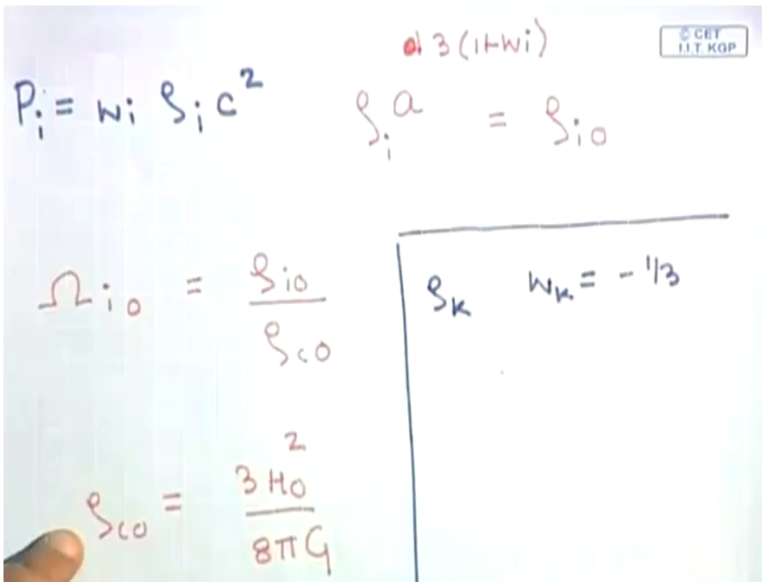

We have considered a universe which is filled with a variety of components each of which has a density rho i. so i here refers to the different components and we have allowed for the possibility that these components also exert pressure, so the pressure of the ith component is a constant w i*c square.

So here allowed for the possibility that the universe has different components which fill it and each such component is labelled by i different components are labelled by i and each of them has a density rho and a pressure P which are related through some wi and then we worked out with the expansion of the universe how does the density change that is the thing that we worked out in the last class.

And we found that with the expansion of the universe the density evolves as rho i so for the ith component the density evolves as rho i a to the power-3 (1+wi)=rho i0*a0 to the same power but we have assume that the scale factor at present has a value 1, so essentially the density of the ith component into a to the power-3*1+w is a constant rho i0 is the present density of the ith component okay.

So the scaling behaviour with the expansion of the universe depends on the value of the pressure how the pressure is related to the density and this is something we worked out in the last class, if there is no pressure then the sorry there should not be a- sign here it should be a+ sign there is no-sign here, so as the scale factor increases the density falls and if it is matter the usual dust with no pressure then the density into a cube is a constant.

If there is a pressure then the you can think of it as the material does work on the expansion in the expansion of the universe which modifies the behaviour you have to then look at the first law of thermodynamics and apply the conservation of energy from where we are led to this behaviour of the density as the universe expands, so these were the 2 things that we did in the right in the beginning of the last class.

The other thing is that we represented the present value of the density of any of these constituents using the density parameter the value of the density parameter at present, so the density of any component was parameterized like this and this is the ratio of the density of that particular component to the critical density of the universe at present okay.

So we model this was the basic idea that we had that are the universe can have different components each component is the present contribution from each component is parameterized by its density parameter, once you know the present contribution you can work out the contribution at any ratio any value of the scale factor using this okay, so this tells us how the density evolves with the expansion of the universe.

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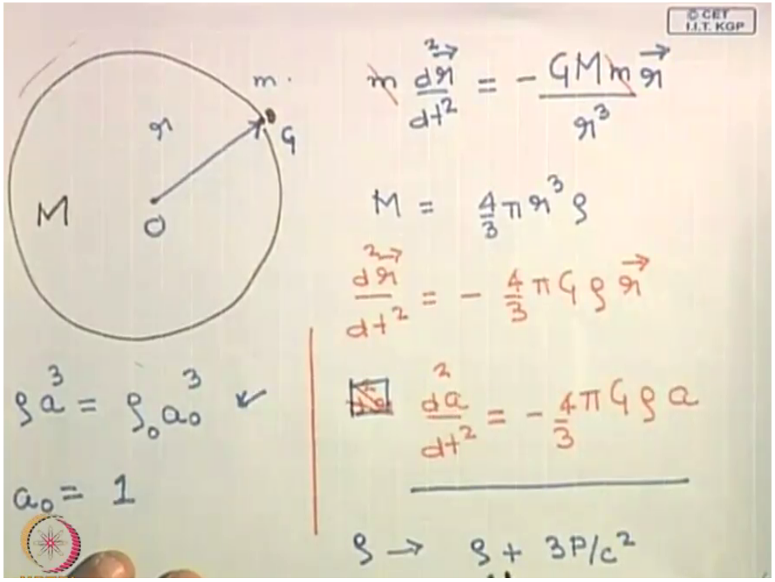

Then we use this to determine the dynamics of the universe let me remind you again, we determine the equation governing the dynamics of the universe by just considering the equation of motion of a any galaxy at a distance r the force acting on this is due to the mass inside which gives rise to the this equation for the scale factor, if the constituent of the universe has pressure this has to be modified rho has to be replaced by rho+3 P/c square.

So this is the equation that governs the acceleration or de-acceleration of the universe, this is the equation that governs the dynamics of the expanding universe, so let me write down the equation.

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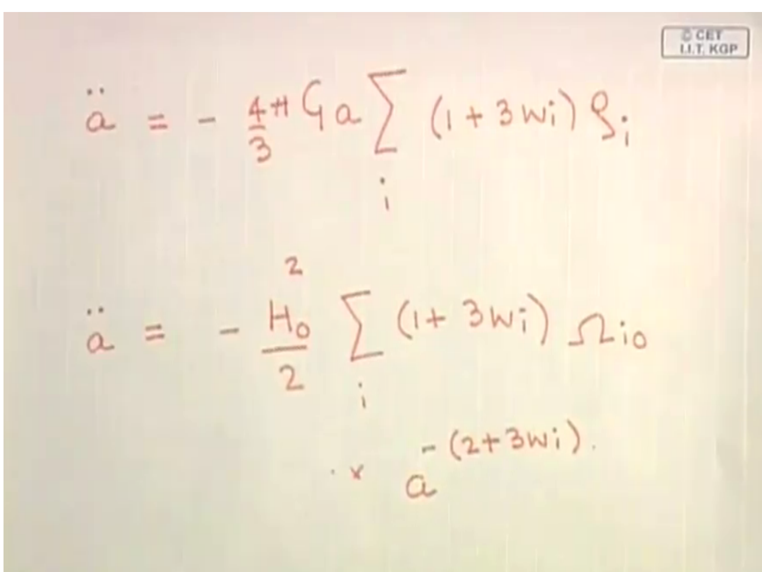

The equation is a double dot=-4/3 pi G*a into the contribution from all the components in the universe each component contributes (1+3wi) into the density right, because I told you the constituent has pressure if the particular component has pressure than the density has to be replaced by rho+3 P/c square this comes from Einstein's theory of general relativity okay, so this is the equation that governs the acceleration or the dynamics of the universe.

And this equation can also be written in terms of the density parameter at present for these different components, so all that you have to do is replace the density at this is the density at any arbitrary epoch, these are all functions of time, scale factor is a function of time, density is a function of time, you have to replace the density in terms of the present density of that component and the appropriate power of it.

Once you do this then the equation that you have is a double dot=-H0 square/2, where we have used the fact that the critical density the present value of the critical density is 3 H0 square/8 pi G so we have use this over here to replace 4 pi G/3 okay, so we have the ratio of the density to the critical density at present and you can use that you can use this you will get this relation okay, so this is essentially the equation that governs the dynamics of the scale factor.

Next we integrated this equation, we multiplied with a dot and integrated this equation or equivalently these equations are same there is no difference okay, we integrated this equation and let me again remind after we integrated this equation.

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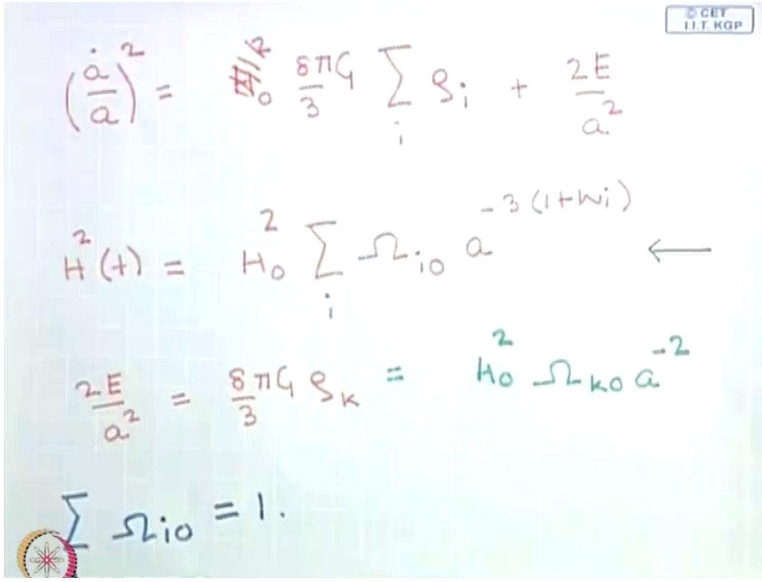

We obtain the following equation a dot/a square this is the Hubble parameter square=H0 square let me not write H0 here, so we integrated this equation and what we got was this is=8/3 pi G into some of the different densities +a constant of integration which was 2E/a square okay. So this is the integral our first integral of this equation multiplied with a dot use the fact that we know the time dependence of rho for each component.

And you can integrate this equation once you will be led to this we have done this exercise and this can also be written in the following way H square=this is the Hubble parameters square=H0 square again we have replace 8 pi G/3 in terms of H0 square and the critical density. Sum of all the components in the universe the corresponding density parameter omega i0*a to the power-3 (1+w i) in this process we have also included this, so what have we done we have also included this term 2E/a square as a fictitious constituent of the universe.

So 2E/a square we have represented this as a fictitious constituent of the universe, this term actually corresponds to the curvature of the universe spatial curvature. And we have we are here representing it as a fictitious density and as the universe expands this density should fall as 1/a square, so this fictitious density so we have introduced a fictitious density by hand and the fictitious density that we have introduced let me put it here, this fictitious density that we have introduced you can see that you want this to scale as a square as the universe expands.

So the w should be –1/3 okay. So we have introduced this constant of integration we have represented by a fictitious density called omega k which as w k=-1/3 okay, this is introduced as a fictitious density, it is not really a density it is a constant of integration which is Einstein's theory can be associated with spatial curvature okay.

So that has gone into this already, into this so this term has been put inside here, so this term can also be written as H0 square omega curvature 0 a-2 right, this term falls as a – 2, so this term also a contribution here in this equation is H0 square omega curvature 0 a-2 okay, this basically parametrized this constant of integration with this now this is the entire equation the first integral of the equation that governs the dynamics okay.

Now if you take this equation and look at it as present your led to the relation that H0 square=H0 square into this which essentially tells us that at present that this term must be 1 okay, so we have the sum of all the different omega i0 should be=1 okay and this is true because we have also accounted for this constant of integration inside this by introducing a fictitious omega curvature corresponding to this okay.

So this is the equation these are the 2 equation that we have discussed in the last class and this is the equation that we finally use in much of cosmology to study the expansion of the universe okay and the cosmological models that we shall be dealing with mainly has 3 components, 4 components in addition to curvature is one of the them, one of the possibilities, there are other 3 components let me introduce.

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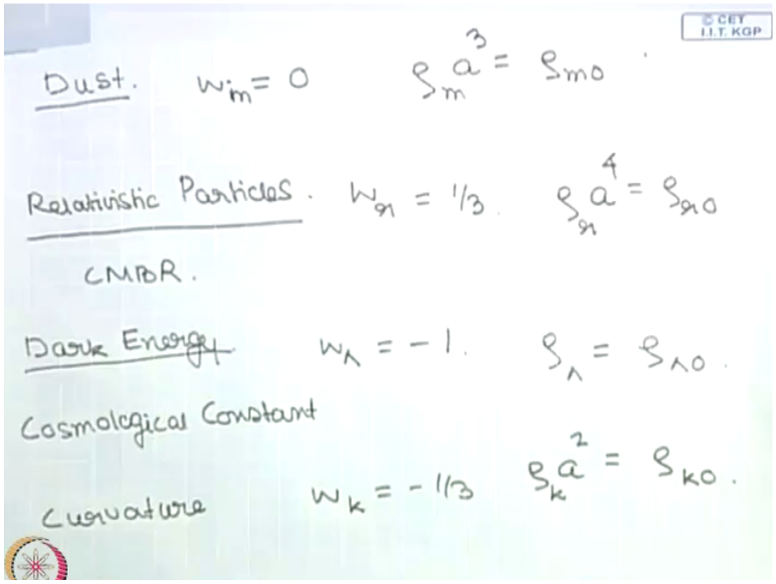

The first one was dust which has no pressure, so this is ordinary matter basically no pressure, so for this a cube then we had the relativistic particles for example the cosmic microwave background radiation photons and there are other relatives particles also possible which we shall learn about later, so let us write all of them as so for these particles omega r relativistic particles.

For example, CMBR photons are relativistic particles the isotropic distributed relativistic particles, this pressure is 1/3 rho c square and rho relativistic a to the power 4=rho relativistic 0, then we also talked about the dark energy and there is evidence that such thing exists, the dark energy of which the cosmological constant is an example is a possibility cosmological constant this is something for which the I have represented it by lambda in general okay.

So we are going to particularly focus on the cosmological constant there are other possibilities also which go under the generic name of dark energy okay, so we are going to talk about cosmological constant which is represented by lambda this is a constituent of the universe whose pressure –rho c square okay that is all, that is the basically by and large all that we know about, so for this rho lambda=rho lambda 0.

In addition, we have the curvature which is not a real component it is the way that we have introduced these constant of integration in this equation, so let us briefly just look at the solutions okay, so let us before we look at the solutions let us look at the behaviour of right hand side of this equation, in a situation where we have a combination of all these terms and we have the curvature also let me put the curvature here on the same paper.

For curvature this = -1/3 and rho curvature a square is a constant okay, now let us consider briefly the possibility that our universe has a combination of all 4 of these okay, so the equation that governs the expansion our universe is essentially this and it has a combination of all 4 of these okay, so let us look at and try to get a feel for what the general behaviour is going to be when I have a combination of all 4 of these.

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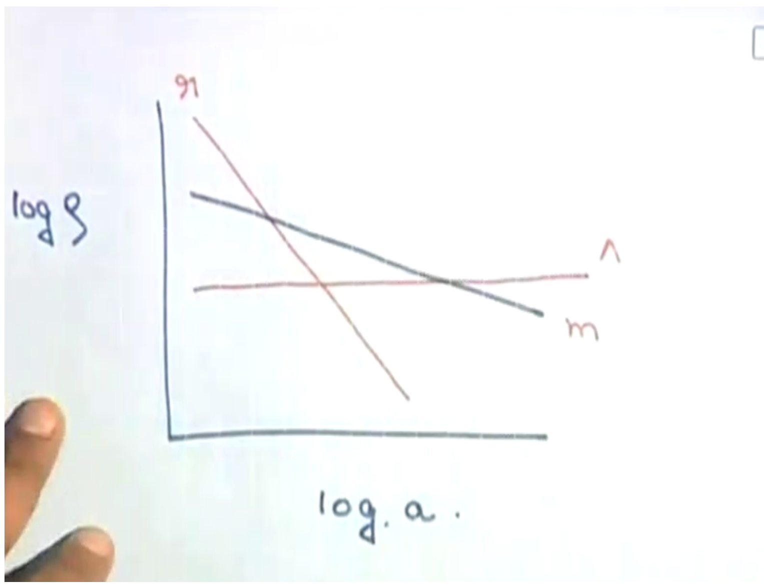

Let me make a plot of the density contribution to the density in each of these components as a function of the scale factor and let me make the plot in the log scale, so this is log rho, this is log a okay, so as the universe expands you are going to a larger and larger scale factor, the density is going to fall, for relativistic particles okay, for dust for ordinary matter the density will fall as a to the power 3 okay, for relativistic particles like CMBR the density will fall as a to the power 4.

So this is going to fall the steepest of all the components, so of all the components the contributions from the relativistic particle will fall as steepest okay this is relativistic particles r and this will have a slope on a log-log it will have a slope of-4 okay, next the matter as the universe expands the matter density will fall as a to the power –3, so that will be something like this possibly okay.

And let us look at the cosmological constant the cosmological constant will have a constant density, so that will be somewhere like this lambda this is matter and if you consider the curvature, if you have a combination of all 4 of these, the curvature will fall somewhere intermediate to the matter and the cosmological constant.

So you can see the lessons that we learn from this is that in the early universe when the scale factor is very small the universe is going to be dominated by relativistic particles radiation dominated it is going to be radiation dominated okay, and then you are going to have a transition to a matter dominated universe and if you had curvature then you would have a curvature dominated universe.

And finally if there is a cosmological constant you would have a cosmological constant dominated universe, if you do not have these things then the question does not arise okay.

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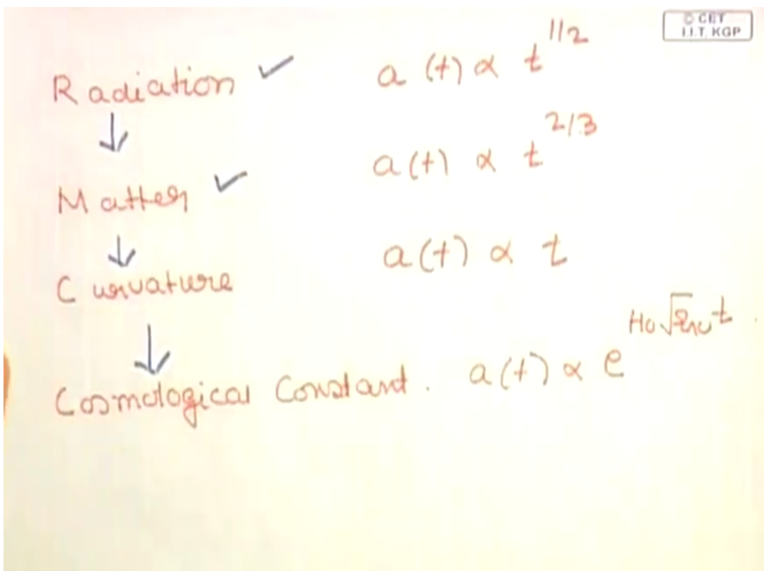

So in general you will have a radiation dominated universe to start with relativistic particles, then you will have a matter dominated universe, then if curvature is there you will have we know the radiation and matter both exist, if you have a curvature dominated universe, if you have a curvature component then the curvature will dominate and finally you will have a cosmological constant if this exists that it will be final thing that dominates the universe okay.

These 2 we know definitely for sure that the exist because we have measured the CMBR and we see matter around us okay, curvature we do not know, cosmological constant there is evidence that this is very small if it exists at all and there is evidence that there is a cosmological constant whatever okay, the behaviour chronologically is going to be like this the early universe will be dominated by radiation.

And the relative universe if there is cosmological constant it will dominate if not it will be continued to be matter dominator, simplest model you only have these 2 radiation and matter, so you have early universe radiation and later on matter okay, so if you have only 2 components radiation and matter then the evolution early in the universe the radiation dominates later on the matter dominates.

The transition between these 2 is determined by the value of omega matter and omega radiation relativistic particles okay, this is the broad behaviour let us now to get a picture of what happens, let us work out let me briefly tell you how do you determine the scale factor in a general situation how do you determine the scale factor as a function of time from this equation.

So if you look at this equation this is basically H as a function of a square, H is basically a function of a, the time dependence comes through a the scale factor in these models okay, so we have the dynamics of the equation we have integrated it once.

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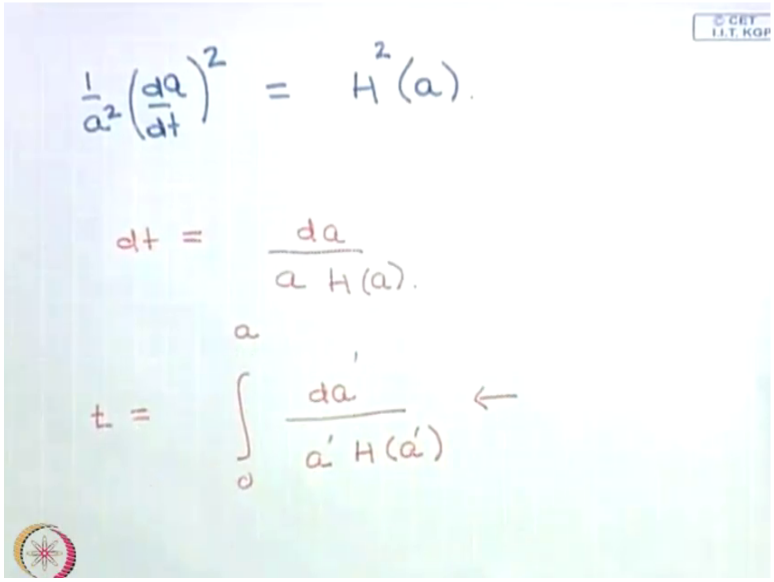

And the final equation that we obtain is 1/a square da/dt square=H square (a), where H square a is a known function depending on the different constituents on the universe, depending on the values of the different omega i's it is known function okay, so this is the equation that you have to integrate and this equation is easy to integrate.

So the integral of this equation is that dt=so I can take this on to the what I have to do is I have to take this on to the right hand side bring this in the denominator so this=da/a H(a), so if you want to solve for this scale factor at any instant of time, you have to essentially do this integral 0 to a da prime a prime H(a) prime and there will be a constant of integration which we set to 0, because we would like the big bang to occur when t=0 okay.

So to determine a as a function of time essentially what you have to do is you have to integrate this, this will give you t as a function in terms of a you have to invert this to get a as a function of t okay, this equation also gives you the age of the universe for any way a value of this scale factor and the present age of the universe.

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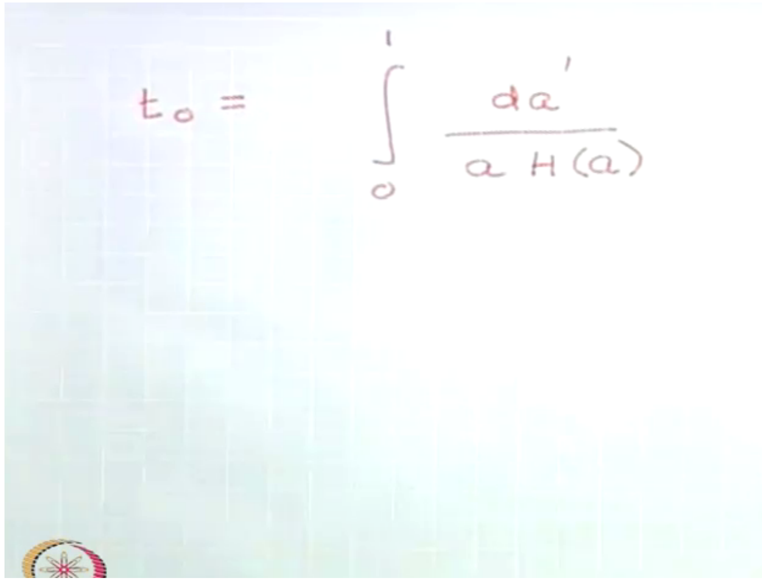

$$t_0 = \int_0^1 \frac{da'}{a\,H(a)}$$

Let me put it here can be determined from this equation t0=0 the present value of the scale factor is 1, so this can also be used to calculate the present age of the universe, you have to just do this integral from 0 to 1. So if in your model you know all the different density constituents what are their contribution at present you can determine everything about the model not only can you determine the expansion history of the universe how a behaves as a function of time.

You can also determine the present age of the universe okay let us quickly workout very simple models in general you can do this numerically put this in a computer and you some simple rule like Simpsons 1 3rd rule or something like that to do this integration numerically it is very straight forward okay, that will give you t as a function of a you can invert that, let us do a few situations which are simple to get an idea of what happens.

So we have learnt that early in the universe is going to be radiation dominated, so let us look at the behaviour in a situation where the matter cosmological constant curvature all can be neglected, so suppose we are looking at the behaviour of the universe somewhere in this region where the radiation is most dominant thing everything else makes a very insignificant contribution early in the universe okay.

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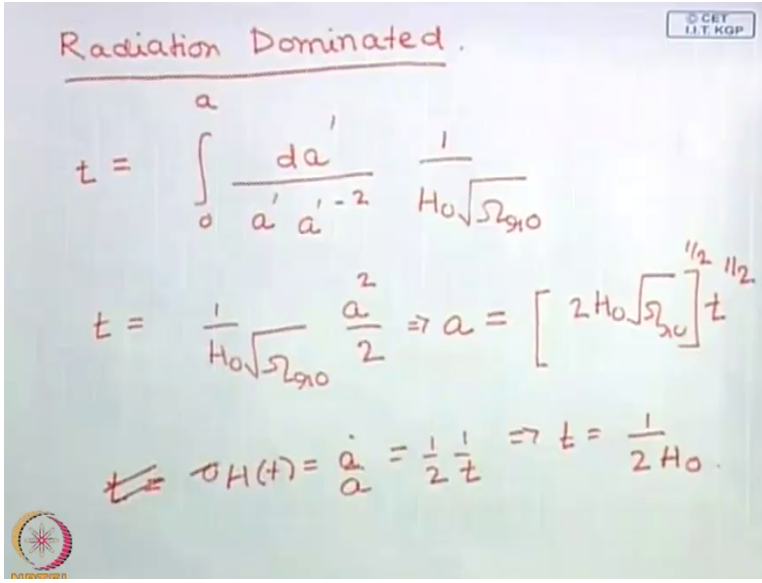

So in such a situation the equation that we have to solve this is radiation dominated universe and the early universe we know is going to be radiation dominated, because the density of radiation of relativistic particles will increase as a to the power-4 if I go back in time okay, so let us look at the behaviour of this equation under that condition.

So we have to basically do this integral where H(a) prime now let us put the form of H(a) prime for a radiation dominated universe I will have only the omega radiation 0 a to the power-4 okay, so the integral that I have to do is t=0 to a da prime a prime I have to put H(a) there so this is going to be H0 square root of omega r0 a to the power-4 so –2, so I have a prime to the power-2 and 1/H0 omega r0 okay.

So we know what this integral is, this integral is a to the power, if I multiplied this these 2 I will get a to the power-1 so which will be a prime on top if I integrate this I will get a prime square/2, I will get a square/2 at the limits I will get a square/2 okay, so this integral essentially gives me t=1/H0 square root of omega r0*a square/2 that is this integral okay or you see that the scale factor a= so the scale factor is 2/H0 is so I will have 2 over there 2 H0 omega r0 to the power 1/2 t to the power 1/2.

So the crucial point is that for this model the scale factor is proportional to t to the power 1/2 okay and the age of the universe at any instant of time is a dot/a sorry the Hubble parameter at any instant of time a dot/a so for t to the power 1/2 a dot/a is going to be half 1/t, or t=1/2 H0 okay, so this is the dynamics of the equation in a radiation dominated model, so in the radiation dominated model the universe expands as t to the power 1/2 okay.

Let me also remind you what it was for the matter dominated model in the matter dominated universe, we have already worked it out in a matter dominated universe the expansion law is t to the power 2/3, so you can do this check that you recover the same thing again for the matter dominated universe, I will not go through it here okay, so for a radiation dominated universe the expansion is t to the power 1/2.

For a curvature dominated universe there is no gravity so it is just a proportional to t, so we have already worked out what the expansion factor is for several of these models. So we have now work out what this here, so this is a (t) is proportional to t to the power of 1/2, a (t) proportional to t to the power 2/3, a (t) proportional to t in all of these models gravitation causes the universe to slow down the expansion to slow down, in this model there is no gravity.

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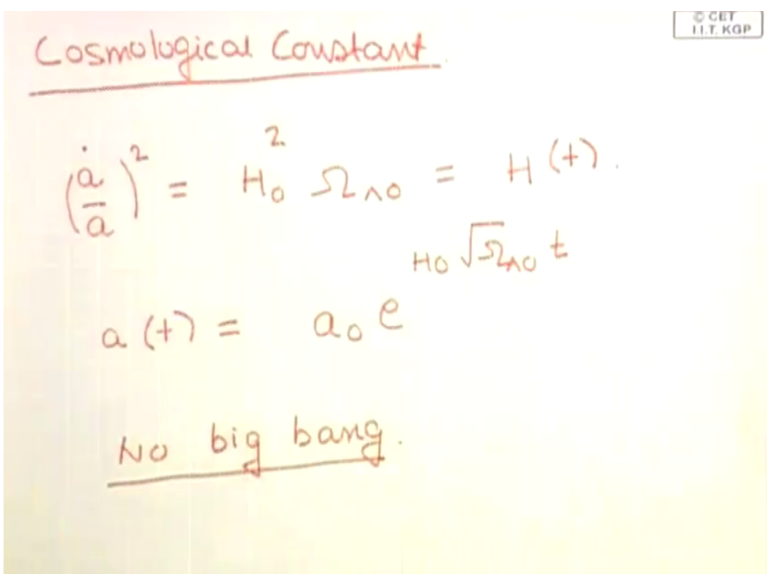

Let us look at a model with just the cosmological constant which is the only one that we have not discussed as yet, so in a model with a cosmological constant the integral that we have to do we have H0 H (a), so the cosmological constant here the equation that you have to solve is a dot/square=0 H0 square omega lambda 0, for a cosmological constant there is no dependence on the scale factor, this is the equation that you have to solve.

And the solution to this equation is quite obvious, this equation can be written as this has a solution that a (t)=t to the power some constant a0 if you wish t to the power H0 square root of lambda 0*t, it is an exponentially growing solution, so for all of these other situations where gravity tends to slow down the expansion of the universe we have power law solution for a freely expanding model it is proportional to t.

Whereas for the cosmological constant dominated model the expansion is an exponentially growing function of time, the scale factor is an exponentially growing function of time okay and the value of the Hubble parameter H remains constant it does not change with time, so and the universe has no singularity it has a singularity at t =-infinity which basically means that there is singularity there is no big bang in this model okay there is no big bang.

It is an ever expanding universe with an exponential okay, so in general okay so this is, so this is this summarizes the general behaviour in a model we have where we have the universe dominated by radiation, dominated by matter, dominated by the curvature and dominated by the cosmological constant, so if I have a universe where there is a combination of all of these then in this space of a universe it will expand as t to the power 1/2.

In the matter dominated phase it will expand as t to the power, let us take a simple model where there is radiation matter and cosmological constant, in this model you will initially have t to the power 1/2 and then you will have t to the power 2/3 expansion and finally you will go over to a phase where you will have an expansion which is exponential okay, which is an accelerated expansion and it is not accelerating.

The cosmological constant we have seen acts to instead of slowing down the expansion it acts to increase the universe the acceleration of the expansion of the universe which is acceleration so the universe will then go over to a phase of accelerated expansion okay, so this is kind of behaviour which you have in a model where you have these 3 components.

And in general given this equation all that you have to do the equation that we have just worked out is integrate this equation incorporating all the different components that we have okay, so I will give this to you as an assignment work out numerically the solution for a combination of matter, radiation and the cosmological constant, workout a (t).

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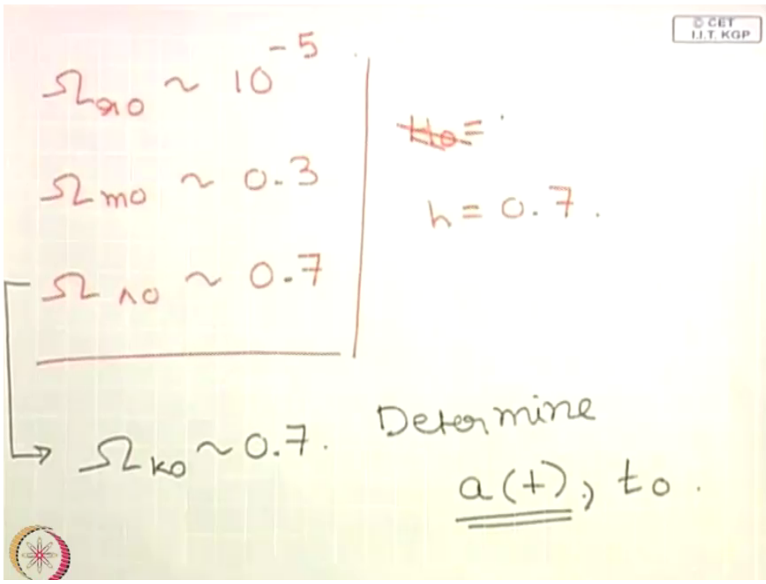

Take for example omega radiation 0 to be of the order of 10 to the power- 5, omega matter 0 to be of the order of 0.3 and omega lambda 0 to be of the order 0.7 okay, the total sum has to be 1 this is very close to all, work out the behaviour of this scale factor as a function of time for a cosmological model where you have these H0 rather h.

So this is the homework please work out the expansion history for a model with this, compare it with the model where instead of this you have omega curvature 0 of the order of 0.7, the total sum has to be 1 right, so consider 2 different models one where you have these 3, another where you have this 2 and this compare the behaviour of the expansion of the scale factor in both these models, so you have to determine okay.

So plot the graphs and compare these models, also determine the age of the universe which I have told you how you have to determine, assuming H0 h is 0.7 okay. Now let me switch over to a different topic, so I have told you how to work out the dynamics that I have told you about the equation that govern the dynamics of the universe and I have also told you how to work out solutions for these equations, we have worked out some simple examples.

Let us now switch topics and look at the cosmological metric okay, so before we going into the discussion of cosmological metric I have to tell you a few things, so I hope we all know that in Einstein's theory special theory of relativity you have this very important quantity call the interval between events okay.

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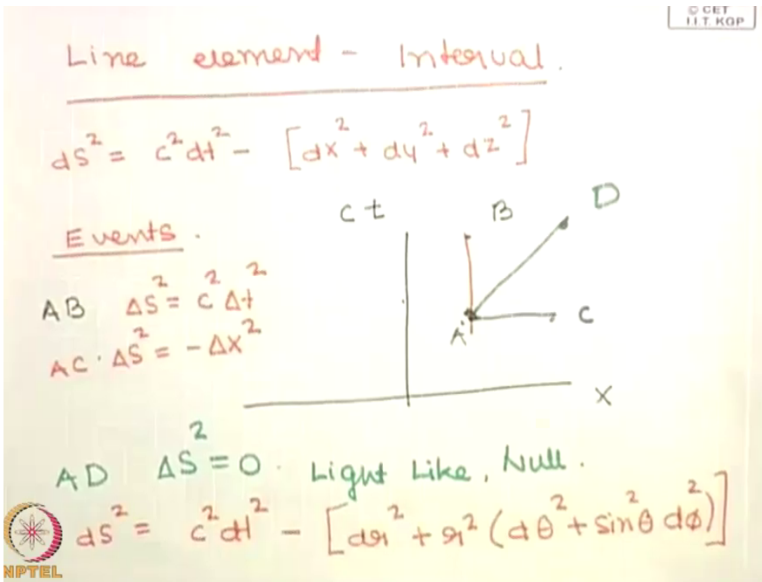

So let me write down the expressions for the line element or interval between events, let me write down the expression here I shall explain to you what it means briefly okay, this is how you calculate the line element or interval between 2 events, so in Einstein's theory of relativity we have to think of space time rather than space itself okay, so we are dealing with the 3-dimensional space with coordinates x y and z Cartesian co-ordinate system and there is time.

Now the things that we are dealing with are events, what is an event? We all know what an event is you are born, the clock ticks one second that is an event okay, so we can represent events on what is called a space time diagram, in a space time diagram we have the position of the event along the x-axis and we have the time at which the event took place along the y axis and here we will only refer to one of the spatial coordinates x, y and z are not being shown.

Let us say that y and z do not change okay, instead of using time we shall measure time in units of speed, the speed of light okay, so this is an event A it is labeled by the place of the coordinate of the place where it occurred and the time instant at which it occurred that is an event A, there is another event let us say B which in this diagram is vertically above A, then let us calculate the interval for the event AB.

The way you calculate the interval is you evaluate this expression for these 2 events, now you see for these 2 events they are not separated in space in position they only separated in time, so for these 2 events the interval delta S square=c square delta t square okay, such events are said to be time like, this is the time like these are 2 time like, these events are time like okay, and the delta S square is essentially c square into the time interval between these 2 events.

So these 2 events imagine a person sitting at one place with the and you are sitting at one place, so this is at t=0, this t=1 time will just evolve for that person in a particular co-ordinate system okay. Similarly, let me take another event here and call it C, these 2 events AC delta S square is basically-is negative-delta x square just apply this formula.

So for these 2 events you see that the interval this is called the interval, interval is negative it is the-the length squared such 2 events are said to be space like and the value of interval is essentially-the distance between these 2 events square, now the utility of the event is apparent now when you consider something that travels at the speed of light, so imagine a photon being emitted from here and it travels at the speed of light.

So after a time interval this photon has reached here, this is the event D, so the photon is emitted here and this is where it is let us say detected by a detector AD, let us look at the separation AD, for the separation AD you see it is travelling along the x-axis delta x and it is also evolving in time so there is a delta t, now since it is moving at the speed of light delta x=c delta t.

So for this these 2 events which is basically the propagation of light from here to here delta S square=0 okay, these are said to be light like or null, so along the propagation of light the interval is 0 and the interval plays a very important role in special theory of relativity, its value does not change event if I change go to moving frame of reference etc. etc. now this is briefly what is meant by an interval.

And the same thing can also be written in spherical polar co-ordinate system, so let me write it here instead of using a Cartesian co-ordinate system I can also use a spherical polar coordinates system where this can be written as c square dt square–dr square+r square d theta square+sin square theta d phi square okay, so I have written the same thing in polar spherical coordinates that is all okay, which I am sure is familiar this polar coordinates.

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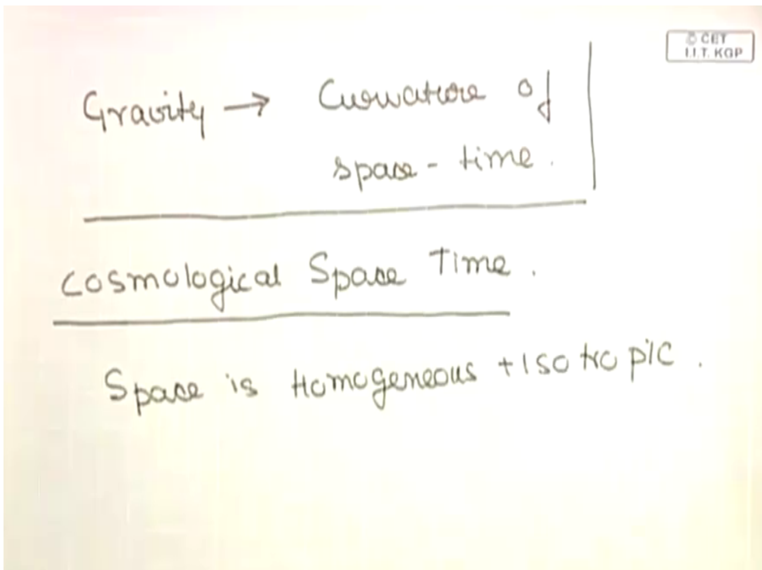

Now here we are looking not only at relativity but we are also interested in Einstein's general theory of relativity, so in the general theory of relativity the gravity manifest itself as curvature of space-time, so this is the general theory of relativity, by the curvature of space-time how the way it manifests itself is essentially it modifies the way in which we calculate the interval corresponding to 2 events.

It modifies the way in which we calculate the interval between 2 events that is how the curvature of space-time manifests itself in this okay, that is the first point that we have to understand we have to accept let us say that we will not bother to understand it here we have to accept it that the curvature of space-time manifest itself as that the way you have to calculate distances becomes different.

And it is intuitively quite clear and that if something is curved the way you have to calculate distances is different from the way if I it were you would calculate distances if it were flat okay that is the main point, so the question is how do we calculate distances in this cosmological space time, so what is this space time that we are interested in we are interested in a cosmological space time and the cosmological space time is one that satisfies the cosmological principle.

The cosmological principal is that space is homogeneous and isotropic okay, so if you have a space-time where this space is homogenous and isotropic that is the cosmological principle not only is the density of matter isotropic and homogenous, but the gravity rate nuisance also is homogenous and isotropic everything he is homogenous and isotropic so the space itself is highly homogenous and isotropic everything is homogenous and isotropic.

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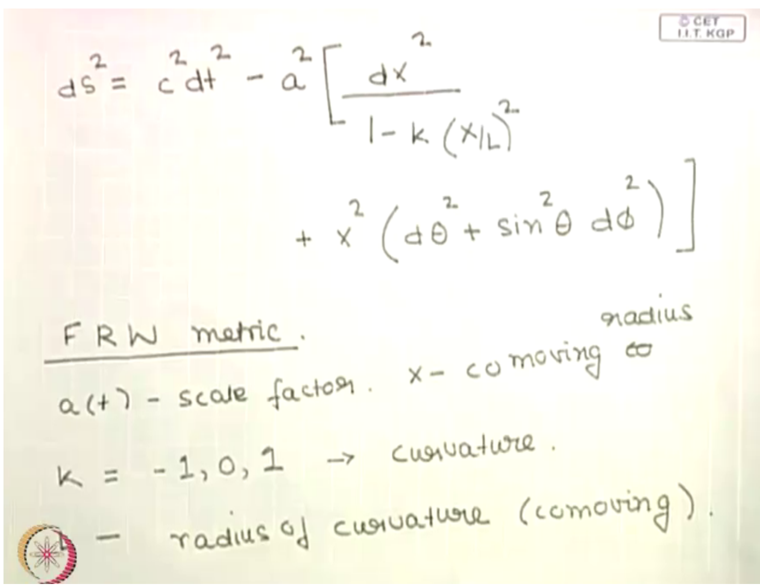

So the space itself is an homogenous and isotropic in such a space the interval is calculated let me write down the formula, the way you would calculate the interval, the interval is calculated according to this formula, so in such a space-time you calculate the interval using this formula which is very okay this is the first thing this is called the FRW metric Friedman Robertson Walker metric okay.

So the metric is how you what you used to calculate the distance between 2 coordinates points and this is referred to as FRW metric, this thing that we wrote down before is the flat Minkowski metric okay this gets modified to this that is the first thing in for the homogenous and isotropic universe okay, now you see one should not get very intimidated looking at this, it is not very difficult to understand.

The let us first see what are the differences with what we are familiar from what we are familiar with, so you should compare it with this, this is the space time this is the usual space time interval written in spherical polar coordinates, the difference essentially lies in 2 things there are 2 things which are different one is this a(t) which is the familiar scale factor we have already encountered, that is the one difference so a(t) here is already familiar to us it is a scale factor.

So x here is the comoving coordinate, so we are working in a comoving coordinate that is quite clear comoving radius x is the comoving radius, theta and phi are the usual angular coordinates right, so it is just like this here you have the radius r, here you have the radius x it is comoving because the whole thing is multiplied by a factor of a square, a*x tells me the length right, so this is the comoving coordinates system.

The whole length in this coordinates system gets multiplied with a function a square okay, if a is 1 then this looks exactly like this there is one more difference where the difference being I have this K, K is a constant which can have values -1, 0 and 1, this tells me whether there is if there is curvature and 0 is no curvature so if you set K=0 you recover back exactly this, if you make the scale factor 1 okay.

So this is the spatial curvature not the curvature of space-time, the space time is curved already okay, this is the spatial curvature is the space curved forgot about this part, see this is just the space, is the space curved that is there in this factor K, L is the radius of the curvature of the space and it is comoving because we are working in the comoving length scale, so this is the comoving radius of the curvature okay.

So let me bring today's lecture to a close over here, in the first part of today's lecture we discussed we recapitulate the dynamics of the expanding universe and then I told you the behaviour in general and also told you about the behaviour in for 2 particular new cases which we are not considered earlier where we have a cosmological constant and the other where we have only radiation.

In the later part of today's lecture, I introduced the cosmological metric, so this tells you how to calculate distances, time intervals in this expanding cosmological space-time and I introduced the terms that you encountered in the metric in this line element, in the next lecture we shall look at this in more detail and understand what this really tells us.