Astrophysics & Cosmology
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Lecture - 27
Dynamics of the Expanding Universe (Contd.)

Welcome in the last class I told you that in addition to galaxies or universe is also filled with another kind of component which is the cosmic microwave background radiation and at the end of the last class we are trying to figure out what happens to the cosmic microwave background radiation due to the expansion of the universe.

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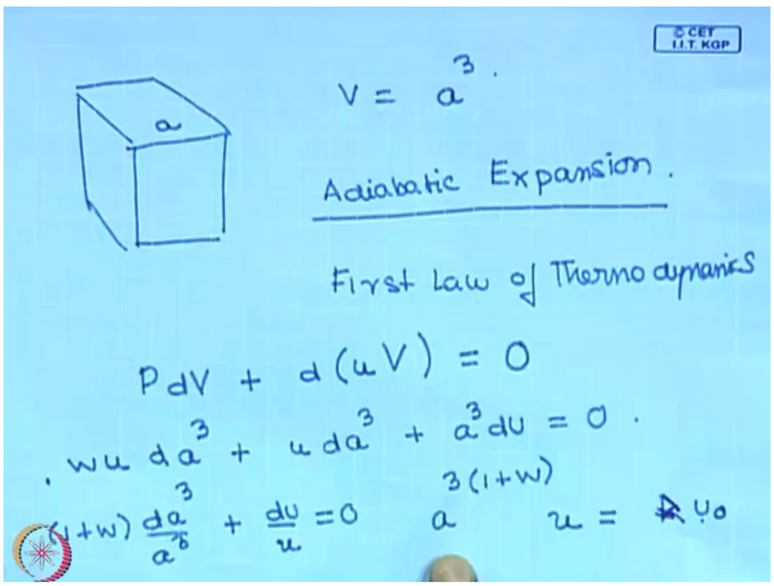

So let us assume that discussion, so we were looking at comoving volume sorry a volume of unit comoving length so the volume is a cube and we are assuming that the expansion of the universe is adiabatic entropy is not generated in this process, so we then applied the first law of thermodynamics do the constant to the cosmic microwave background in this volume, so and the entropy of the cosmic microwave background inside this volume should remain unchanged due to the expansion.

So it should satisfy P dV+ the change in internal energy should be 0, now we will do a somewhat general derivation now, so we will assume that okay that the pressure so the medium material which is there inside we will assume that the components that we are discussing the substance that we are talking about.

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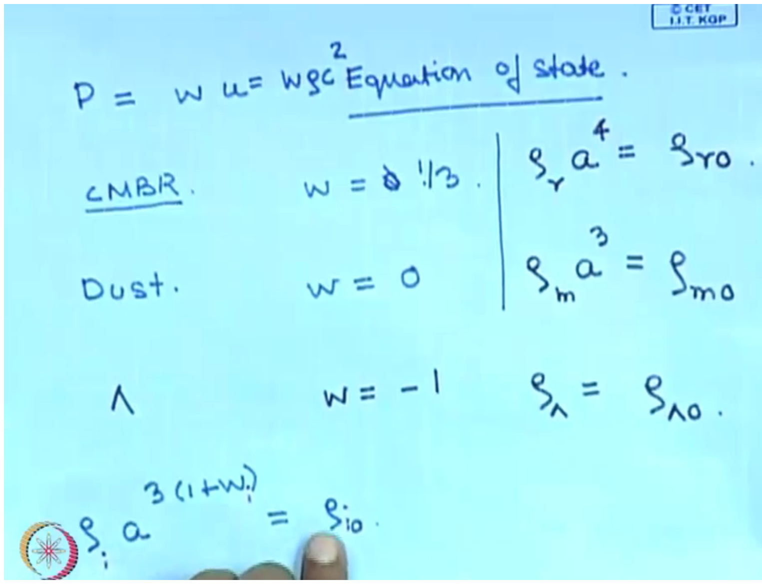

The pressure can be written as w times the internal energy density this is the equation of state, and for the CMBR cosmic microwave background radiation we have discussed this already for photon black body spectrum for photon isotropic distribution of photons the pressure is 1/3 the energy density, we have discussed this when we were discussing we have learnt about this when we were discussing the radiative transport and black body spectrum right.

So w has a value which is=0 sorry w has a value which is=1/3, we were we have also right in the beginning of our discussion of cosmology we have also talked about galaxies and galaxies do not exert any pressure they do not their random motion are very small that is what that is an observational fact okay, so they sit where they are more or less whereas photons move at the speed of light and they carry momentum with them.

So there if you have a gas of photon there is always a momentum flux through any surface, a gas kind of galaxies distributed in space they will sit and where they are and just expand with the universe okay, so galaxies and stuff like this like that are referred to as dust, this is general relativistic parlance okay in general relativity such things which do not have pressure are referred to as dust and galaxies are an example and w is 0.

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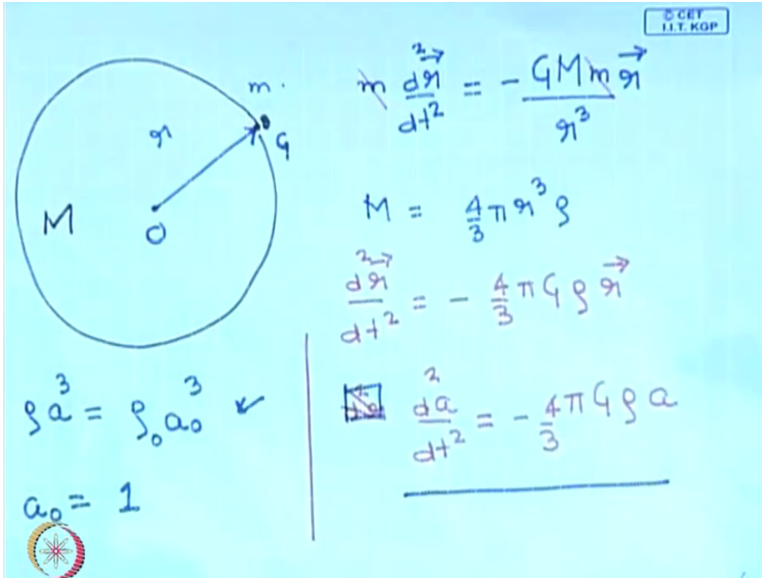

So our discussion is going to be for a very general class of its going to encompass both of these and for galaxies we have already seen from the conservation of mass, that the density into a cube should be a constant as the universe expands okay, but for something else from other things which have pressure it is not the conservation of mass which is important but the first law of thermodynamics which is essentially the conservation of energy okay.

So if there is expansion the material can do work and that the energy will come from the internal energy that has to be accounted for that is the basic consideration okay, so let us now do this calculation, so the pressure can be written as w into the energy density and this is going to be d a cube+ I have the energy density here into a cube the volume is a cube, so I can write this as u d a cube+a cube dU=0.

And I can now combine these 2 terms and divide throughout by a cube, so if I combined these 2 terms and divide throughout by a cube what I will get is (1+w) d a cube/a cube+ and I divide throughout by u also the internal energy density, so dU/u=0 okay, so now we can do this integration it is quite straight forward this integral is log a cube to the power 1+w, this integral is log u.

So what this tells us is that the a cube*(1+w)*u energy density is constant which I can write as u0 because we have assumed that the scale factor has value 1 at present, so we are led to the relation that in general if there is pressure it is okay.

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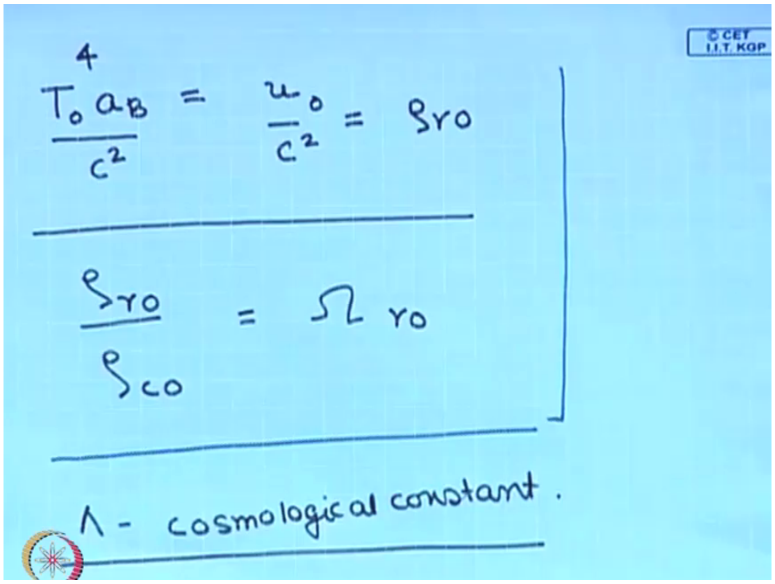

First of all it is more convenient to work in terms of the mass density in terms of energy density instead of energy density that 2 being related like this okay, so the mass density is the energy density/c square rho okay, so then we are led to the relation that the rho a to the power 3(1+w) = rho 0 okay, so for dust we see that rho a cube is constant rho 0, so for dust rho the dust is ordinary matter, so rho matter a cube=the rho m0.

And for any kind of relativistic isotropically distributed relativistic particles the pressure is 1/3 the energy density, so for any kind of for even or just for the CMBR let us say does not matter this is 1/3, if it is 1/3 then we have rho a to the power 4 is a constant, so for the cosmic microwave background radiation it is rho gamma a to the power 4 = rho gamma 0 okay.

And if I had some other constituent of the universe which has a different value of this equation of state w parameter w, I would get a different relation between the density and the scale factor that is the first thing that we have to appreciate okay, let me write at this point introduce another component in our discussion this component goes by the name of cosmological constant or lambda.

Now this lambda is also called the because lambda is actually I mean you may think of it as being a cosmological constant, this is the quantity that was originally introduced by Einstein into general relativity and the way Einstein introduced it, it is a geometrical thing that appears in the Einstein's equation, well as far as we are concerned we shall think of it as a kind of matter for which the pressure =-the energy density.

So for our discussion we shall think of it as a kind of material whose pressure =-the energy density and the Einstein cosmological constant has exactly the same behaviour okay, so for such a material such as a substance the value of w is -1 and you can see that for this rho lambda a to the power 0, so rho lambda=rho lambda 0 the density is not affected by the expansion of the universe okay it is a cosmological constant right.

It is just that we are thinking of it as a material which fills the universe with a certain density and the pressure which is negative exactly-the density it has a negative pressure, normal substances have positive pressure okay, negative pressure means what is the let us just interpret what is the significance of negative pressure and positive pressure, if something has positive pressure if I make this volume expands then the substance which is there inside does work P dV.

So if the universe if this volume expands due to the expansion of the universe the material inside does work and because the material inside does work it has to lose internal energy, so the internal energy density goes down due to 2 reasons, one is due to the expansion of the universe, other is due to the one is because the volume is larger and other is it does work, we are looking at the internal energy density okay.

The density the total internal energy goes down because of the work done, the density goes down also become the volume is increased, if it were just due to the increase in volume then the density would fall as a cube density into a cube would be constant, but it also loses some internal energy because it does work, so it rho*a it falls as a to the power-4 with the expansion of the universe.

Whereas if something has negative pressure then if it expands you are actually doing work on that material you are putting in work into that material, so because you are putting in walk the internal energy goes up and if the pressure is exactly = -the density, then the internal energy goes up in such a way that the density inside remains fixed, the density is getting diluted because of the increase in volume.

But you are also doing work with a just exactly compensate for that and the density finally remains a constant okay, now something with negative pressure is not altogether unfamiliar think of an elastic solid if this there an elastic solid, for an elastic solid if I have to increase its volume I have to do work, if I have to increase the volume from the equilibrium position I have to do work it is like a spring right solid elastic solid I have to do work.

And that work gets stored up as the internal energy of that as the potential energy of that spring, so a solid is an example of something that you can think of as having negative pressure okay, here we have the cosmological constant as the material which has negative pressure, its density reminds invariant if I expand the universe as if under adiabatic expansion okay, so this is the first change that we have to look we have to consider.

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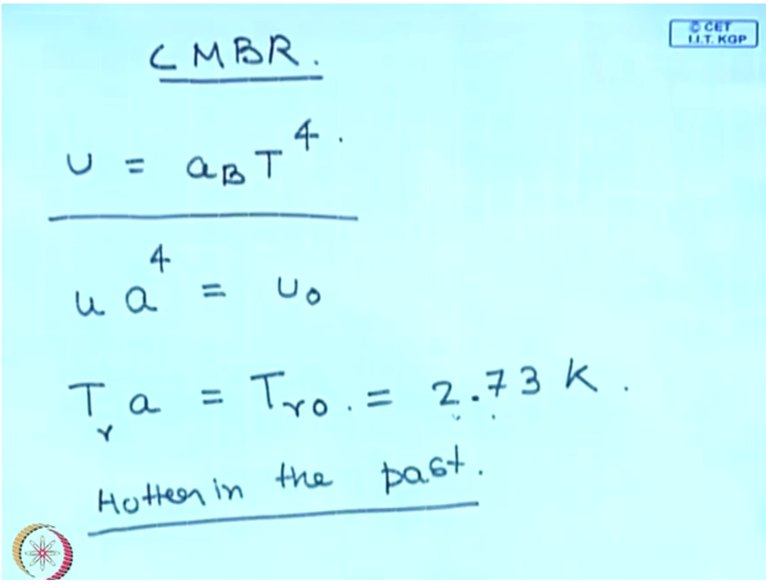

If you wish to look at the CMBR or anything else that has pressure, let us go back to the CMBR quickly, so the CMBR for the CMBR as the universe expands the energy density for CMBR, let me just remind you once more the energy density=aB T to the power 4 and under expansion we also know this is the Stefan Boltzmann constant should not be mistaken for the scale factor.

Further under expansion we know that a*u the internal energy density a to the power 4 is a constant present value of the energy density, from this we can and say that the CMBR temperature so this also T to the power 4 and there is a to the power 4, so we can say that the temperature of the CMBR into the scale factor is a constant and the present value is 2.73 kelvin.

So when the universe was half the size the scale factor was half the temperature was doubled okay and when the scale factor was 10 times smaller the temperature was also 10 time larger, so the temperature of the CMBR was the same cosmic microwave background radiation which has a temperature 2.73 kelvin was hotter in the past and at the big bang it has an infinite temperature, if we just extrapolate it okay, so it is hotter in the past.

So the cosmic microwave background radiation essentially establishes a model called the hot big bang model okay, it is the hot big bang model and the fact that it was the past was hot is extremely important in our understanding of the universe, so not only does the energy density in the cosmic microwave background radiation increase in the past it continues to be a black body spectrum that we shall see later whose temperature is inversely proportional to the scale factor.

Now let us go back to our general discussion, in general we have seen for any arbitrary pressure medium with arbitrary pressure this is how the density will evolve as the universe expands okay, now let us again go back and work out the expansion of the universe what will happens to the expansion of the universe in such a situation, we have looked at the expansion of the universe assuming dust where rho*a cube is a constant.

What will happen to the expansion of the universe when we consider constituents which have pressure, so one change we have already seen the relation between rho and the scale factor is modified, there is another change that has also to be incorporated that changed unfortunately cannot be understood in terms of Newtonian gravity cannot be explained, it cannot be handled in terms of Newtonian gravity.

The change that the other change that one has to account for is the fact that in the correct theory of gravity in general relativity it is not only the mass density which gives rise to gravity, the energy the entire energy momentum tensor stress tensor actually contributes to the gravity, so not only does the density inside contribute to gravity also the pressure inside contributes to gravity okay.

So that has to be incorporated by hand into this theory, it is not there in the Newtonian theory okay, so we shall incorporated by hand the change that has to be made is extremely simple, the only change that has to be made is that we have to now not only take into account the most density inside but the contribution from pressure also, so the pressure of the material inside the sphere also contributes to the gravitational attraction that is the only difference okay.

If you do the full relativistic theory and that modifies this equation, the modification that we have to introduce is that the density now gets replaced by rho+3 P/c square, that is the only change that one has to do, so you have to replace the density the mass density so it is exactly the same thing only the matter is substance inside will contribute to the gravitational acceleration.

The only difference is that it is not only the mass density inside but also the pressure that will contribute to the gravitational slowing down and if we replace the density with the density+3 P/c square okay that is the only difference, so we now have the 2 equations that we have to solve let me write it here.

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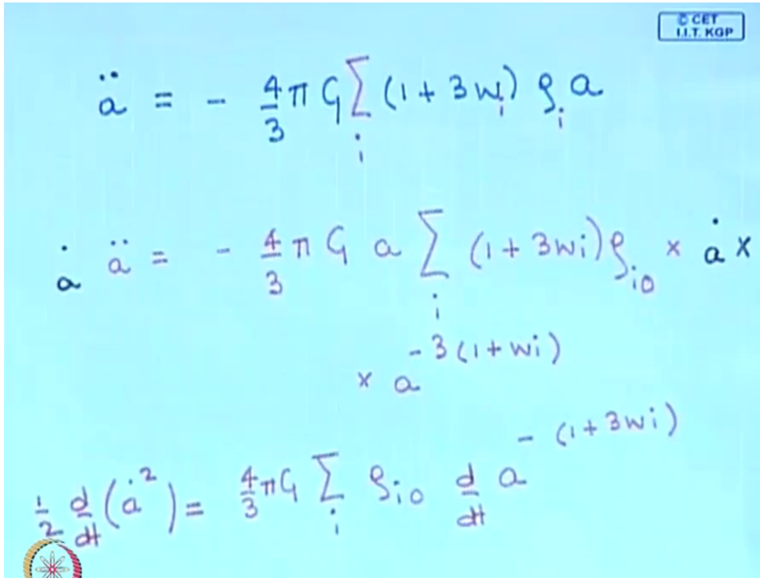

So the first equation is a double dot is =-4/3 pi G and we have to replace 4/3 pi G we have to replace rho/rho+3 P/c square and P we have seen can be written as w times the energy density or it can also be written as w times rho c square, so basically what will happen is that this will be replaced by 3 times w*rho, so the so compared with the earlier equation we have –4 pi G rho a we have now have 4/3 pi G rho a*(1+3w) 3w is basically 3P/c square okay that is the only modification that has to be made.

Now we shall also you see in the entire universe in the universe that we have dealing with not only do we have galaxies over here, we also have the cosmic microwave background radiation possibly cosmological constant and maybe many other things that we do not aware of like the dark matter itself, so we have to account for a variety of components in the universe, so let us allow for different components in the universe.

So what we shall do is we shall put sum here, sum over all the components each component will have a different value of w and they will have their own density, so the equation now becomes this, so the gravity is due to the superposition of all these different components of the universe and we have also seen that for each component rho*a to the power 3*1+w is a constant, so we can now replace that over here.

And we can write it as a double dot =-4/3 pi G into the scale factor sum over i, i refers to the different components of the universe (1+3wi) rho i0, so I am writing the density of any component as rho i0, so for the ith component I am writing is as it has the present density of that component divided by a to the power 3 (1+wi), so I will have a factor a-3 (1+wi) into this, the procedure to solve this equation.

So this equation is an equation involving only the scale factor the only thing is that here we have sum of different powers of the scale factor, now the way to solve this equation is that we again do the same thing we multiply this equation by a dot on both sides, so I will multiply this with a dot let me put the a dot here okay, so I have multiplied both the right and the left hand side of this equation with a dot.

So having done that now we can see that the left hand side of this equation again is 1/2 d/dt a dot square=that is the left hand side of the equation, the right hand side of the equation now becomes 4/3 pi G sum over i rho i0 d/dt a to the power –(1+3wi), so the right hand side of this equation can be written like this, we can easily check that if you differentiate this you will get the factor of –(1+3wi) over here and you will get a dot.

And what will remain will be -i will have 2 here+3 wi which is what we have here also because we have –3+wi and there is a here outside okay, so it is quite clear that this is the same as this okay, now I can integrate this equation in exactly the same way as we have done earlier.

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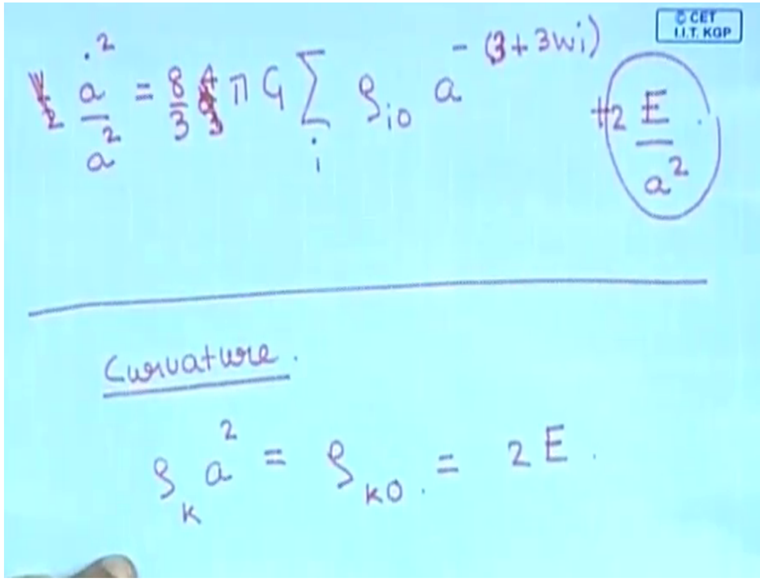

And we then have 1/2 a dot square=4/3 pi G we have sum over i rho i0 a to the power –(1+3w i) +a constant of integration which again I will call E, it is exactly the same thing only that you have different some of different powers of a here okay, it is exactly the same except that we have, now we will do 2 things here, we will divide this equation throughout by a square.

So if I divide this equation throughout by a square I will get this, this will become-3 and I will have by a square and let me also multiplied throughout by 2, so I will have 8 here and I will have 2 I will have 8 I will have 8/3 pi G okay, now we will do 2 things, first is we will identify this term, we will identify this entire time 2E/a square, as a different kind of constituent of the universe, we will not write it as a constant of integration.

We will identify this as a different kind of constituent of the universe we will identified with something called curvature and later on when we do in to talk about the curvature of space and time a little bit, you will see that this term is related to the curvature of space okay, so we sell identify this with another kind of component called the curvature, so rho curvature and I have a 1/a square here, so rho curvature into a square=rho curvature 0.

So at any instant of time T rho curvature is rho curvature 0/a square, this is which is why I have a square so I have basically done that okay, so this=2E I am just calling 2E rho curvature 0, so at any epoch this will be rho curvature 0/a square okay, so at any epoch this will be rho curvature

0/a square I am just giving it a name that is all, I am not saying that it is actually a constituent or anything I am just calling it this okay.

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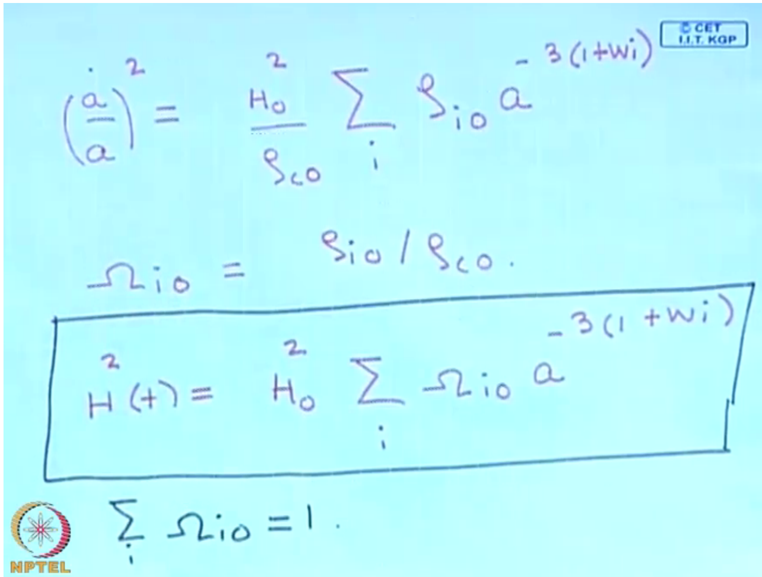

So with this identification I can write this equation as a dot/a square=now the other thing is we know that 8 pi G/3 is H0 square/rho critical 0, the value of the critical density at present is 3 H0 square/8 pi G, so this I can replace by, where in addition to all the physical constituents of the universe we also have fictitious constituents called the curvature which has gone into this equation.

Curvature, so if I identify curvature as a constituent of the universe let me just see, so what should be the value of w for curvature, for curvature we want rho*a square should be constant, if rho*a square should be a constant than 1+w should be 2/3 if 1+w has to be 2/3 w has to be-1/3, so this can be thought of as fictitious component for which w curvature is =-1/3.

If I had a fictitious component for which the pressure is-1/3 the energy density it would behave just like this curvature okay, so in my equation I have to accounted for this constant of integration by a square by introducing one extra time in the density which has w =-1/3 and the entire equation can then be written this way, and I can divide throughout by rho critical 0.

So the ratio of the present density in any component to the critical density is what I called the density parameter of that component, so this is, so with this identification the entire equation take this simple form that H square t is=H0 square sum over all components omega of the different components into a to the power-3(1+wi) for the individual components, so this is the equation that you have to solve to determine the expansion of the universe.

Further, you also have the condition that at present if you make t=t0 this will become H0 square this also become H0 square, the scale factor has value 1, so you have in addition to this the condition that sum of all the omega i's=1, so this is a very general equation that governs the expansion of the universe in a situation where each component can be represented through this through such an equation of state okay.

It is not necessary that each component of the universe can be represent through such an equation of state, if it is so that for each component my universe has several components all of which has an equation of state where the pressure is proportional to the density a combination of such things, then the expansion of the entire expansion of the universe is governed by the single equation like this.

Where the relative contribution from each component at present is on quantified by the corresponding density parameter omega i0, so in order to specify your cosmological model fully you have to specify the value of the Hubble parameter and the value of the different omega i0 the different density parameters okay, once you have specified the Hubble parameter and the value of the different density present value of the different density parameters.

Then you have specified the entire cosmological model and this so this is the then you can determine the entire evolution history of the universe, so the main problem then is how to determine the relative contribution of the different constituents of the universe okay and this is the problem of basically determining cosmological parameters.

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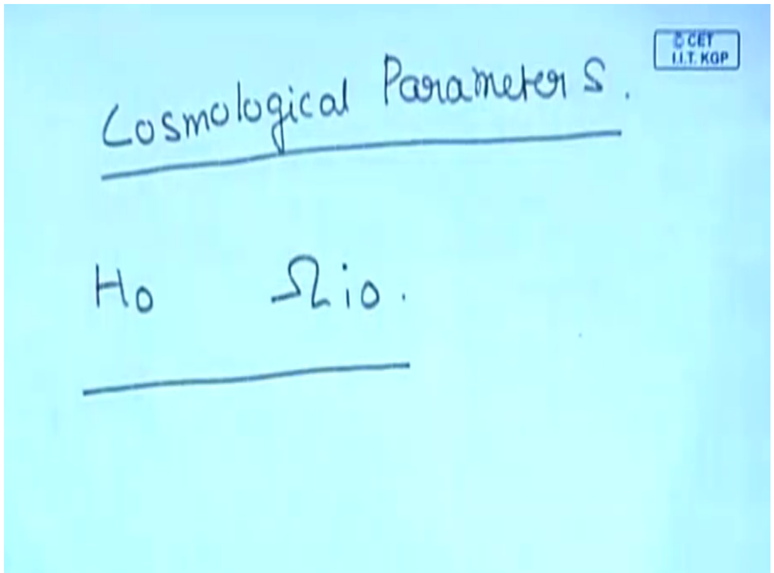

So in this model there are several cosmological parameters, so the entire cosmology has been parameterized by certain parameters, one is H0 and other is the set of density parameters omega i0 okay, if you can determine this you have determined the dynamics of the universe that we live in, and this has been one of the very important problems in cosmology to determine the cosmological parameters okay.

So our expansion is essentially parameterized in this way, the expansion of the universe is parameterized to these parameters we may not understand what the microscopic physics of each of these constituent is, for example I have told you that there is dark matter we do not know what it is, but we will parameterize its contribution through a corresponding omega for the dark matter okay.

And you may you are free to postulate if observations require it a variety of other such components, we may not understand what it is but we will parameterized its constitution to the expansion through a corresponding omega okay, so the problem has been reduced simplified in the sense that we parametrize the expansion history of the universe through such parameters.

And one is now trying to determine and one now has a reasonable idea of what are the values of these parameters for our present universe that is something that we shall discuss later on in this course okay, so once you have determined this then you can use this equation that we just wrote down to solve for the scale factor as a function of time okay, in the simplest cosmological model, let me just mention the simplest cosmological model that we have.

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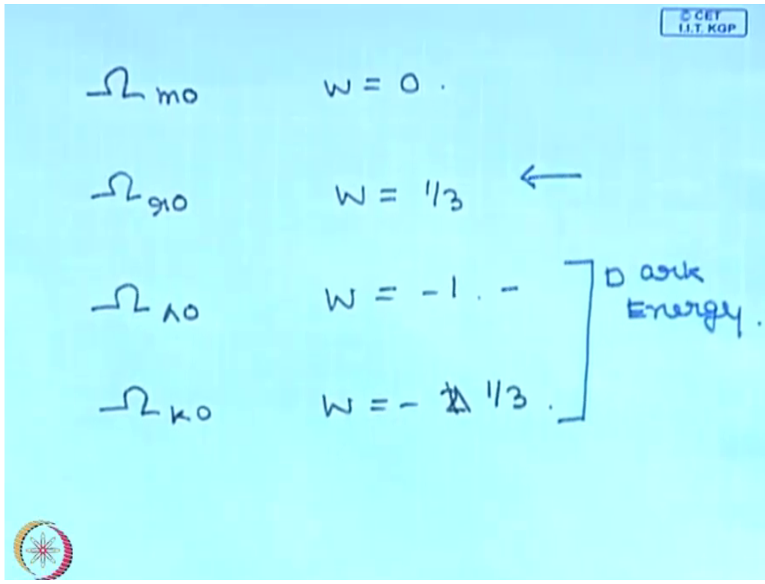

There are you will require for components let me just mention this in the simplest cosmological model you will require 4 components one is the matter this includes the galaxies the baryonic matter that we see and dark matter if it does not have a pressure cold dark matter, so dark matter without pressure is called Cold dark matter okay, so it involves the contribution from all of these, so this is matter with no pressure w=0.

And then you have a variety of relativistic species which includes the cosmic microwave background radiation for which the pressure is 1/3 the energy density and you have the possibility of a cosmological logical constant lambda 0 for which w is -1, and you also could have a curvature for which w =-2,-1/3 not – 2,-1/3.

And I should find out that these 2 may not be, the second one is definitely not matter it is just parameterizing that constant of the integration which is related to the spatial curvature of the universe, so it is not matter but we have parameterized like this, the cosmological constant is possibly not matter or it may be some kind of matter it is exact nature is not known or so this is often called something called dark energy also this was often given the name dark energy also.

So there is evidence that our universe also has something which has negative pressure and we do not know what it is the cosmological constant is one such possibility it is not matter but there are other candidates also which can produce negative pressure and there is observational evidence that our universe does have a component with negative pressure okay, we shall discuss what the observational evidence is later on.

So you are required at least you may your model these are 4 very simple things which you can put into your model and then maybe variety of other things also which you may if required introduce, but we shall talk about the cosmological model in terms of these parameters in our discussion okay and you can estimate this straight away if you know the Hubble parameter for the at least contribution from the CMBR you can estimate.

I have asked you to do this so you know the energy density of the CMBR at present convert that into mass density divided by the present value of the critical density you will get the contribution from the same here okay, so this is the homework you should do, please determine the value of the density parameter for the cosmic microwave background radiation what is the value?

The model that we have discussed till now did not have these 2 did not have radiation either they did it have cosmological constant it had only matter and it had this constant of integration E=0 the constant of integration is 0, omega curvature is 0, omega matter was 1, that is the model you have solved t to the power 2/3 okay that is the model that we have solved, where omega curvature is 0, omega matter is 1 the sum of all these should be 1 okay.

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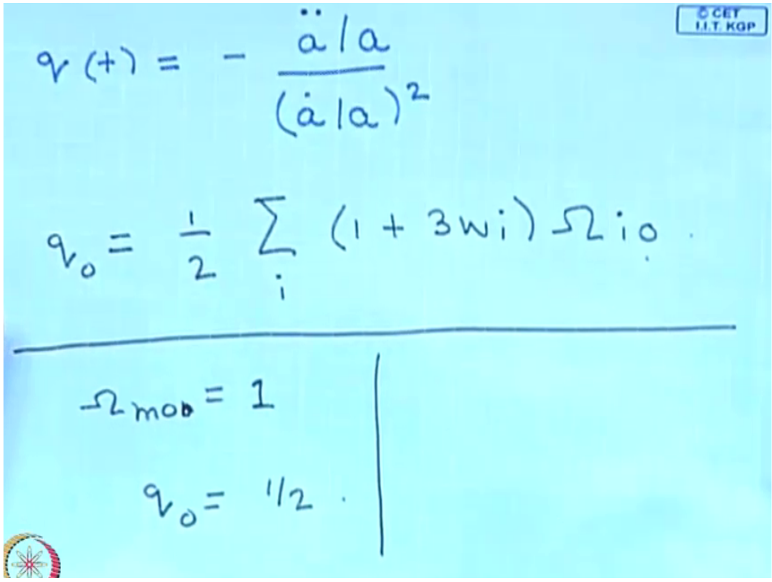

Let me also finally introduce something called the de-acceleration parameter, the Hubble parameter we have seen tells us the rate of expansion of the universe, it is positive with the universe so that is a dot/a, it is positive if the universe is expanding it will be negative if the universe were contracting it would be 0 if the universe was static, it has information about the first derivative of the scale factor which is the velocity.

Now the dynamics of the universe is there in the acceleration and if we know gravity is attractive so if the universe is filled for example with ordinary usual kind of matter we expect the expansion to slow down, so the double derivative the second derivative of a would be negative and I have in the definition of the de-acceleration parameter we have a-sign so this would be something positive if the universe were slowing down this is what we expect gravity.

So if you consider an observer over here a galaxy over here the galaxy is moving away but we expect its motion to slow down because of the gravitational attraction of the matter inside, it is this which is quantified by the de-acceleration parameter it is positive and it indicates this slowing down, it is dimensionless because it has 1/T square in the numerator 1/T square in the denominator.

Now let us calculate what this is for the cosmological models that we have been discussing, so the cosmological models that we have been discussing the equation can so this where this is the first equation that we had and this equation can be written in the following way it can be written as, so this is the equation that we have forgot about the a dot, we can write it in the following way.

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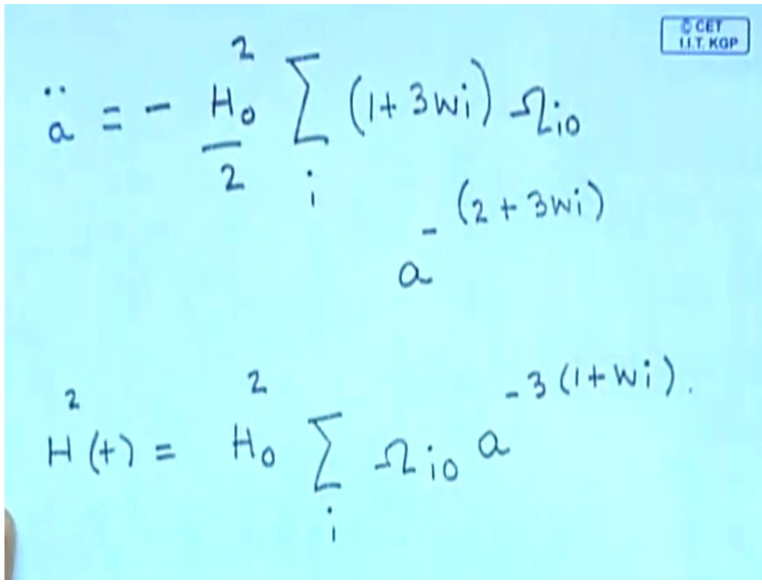

We can write it as a double dot =-H0 square/2, so H0 square/2 is 4/3 pi G into the critical density, so this into a sum over i (1+3wi) then we have the density of the ith component which we write in terms of the present density, so that will be omega i0 because this will give you the critical density which if I multiply I will get the density corresponding to that component at present into a to the power-I have a factor of a over here and I have –3+wi so – 3*1+wi.

So I will have a to the power –(2+3wi) and this is the equation for the acceleration of the universe, we also have the equation for the Hubble parameter this equation we have just walked out H0 square sum over i*omega i0 a to the power –(1+wi), so let us use this equation these 2 equations to calculate the present value of the de-acceleration parameter.

So the de-acceleration parameter we have seen is the ratio of a double dot/a*H square with a negative sign and let us use this to calculate the present value of the de-acceleration parameter, so we will calculate use this to calculate q0, so at present the scale factor is 1, so I do not have to bother about this I have – here and I also have a-here so that is gone, H0 square will cancel out, with this H0 square I will be left with half and sum of omega i's present is 1.

So this is gone from the picture, so what I have left is that this=1/2 sum over i (1+3wi) omega i0 that is the present value of the de-acceleration parameter, so the value of the Hubble parameter at present is basically H0 square which cancelled out with the H0 square in the numerator okay, so if I have a model which has only matter nothing else, let us consider different possibilities.

If I had a model which has only matter if omega matter 0=1 which means that the other components are all 0 they do not contribute to the density, then what we have is q0=half for matter w is 0, so this=1/2, on the contrary let us now consider a cosmological model where in addition to matter we also have some dark energy or cosmological constant.

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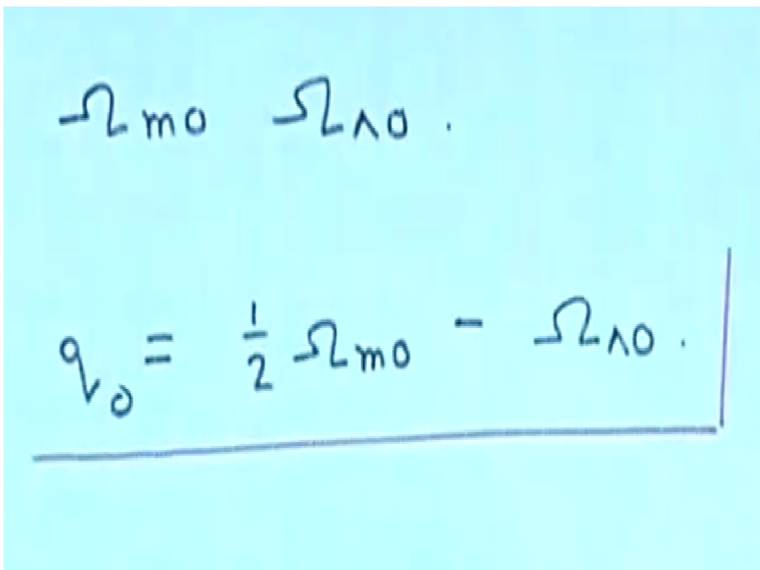

So we have 2 things, for such a model we see that q0 turns out to be half for matter w is 0, so ordinary matter so I have 1/2 omega matter 0, for the cosmological constant which has negative pressure and the value of w is -1 this over all factor is -2 which cancels out with the 1/2, so I have, so we see that there is something very strange that happens if you have a cosmological constant in the universe.

The cosmological constant causes the de-acceleration can cause basically the effect of the cosmological constant is that it will caused the de-acceleration parameter to have a negative value, if this exceeds this it will have a negative value okay, the de-acceleration parameter having a negative value implies that instead of the motion the deaccelerating instead of slowing down it is actually speeding away, so the gravity the sign of gravity gets reversed okay.

So if you allow for the possibility of a cosmological constant, then the nature of the gravitational force due to the cosmological constant is exactly opposite that of the usual matter it causes an acceleration okay, so anything which has an equation of state where w is more negative than 1/3 will cause acceleration will act with an opposite sign in de-acceleration parameter okay.

If w =-1/3 it does not contribute to the de-acceleration or acceleration which is quite consistent with what we have been doing, because we have introduced a fictitious component with w =-1/3 which is the constant of integration, we do not want it to affect the acceleration it is just the constant of integration, so it is quite consistent that if you have w =-1/3 it does not affect the acceleration at all okay.

So let me summarize what we have done today in today's class we considered the possibility that the constituent of the universe, the universe have can have many constituents and these constituents can also have pressure and the we assumed that the pressure is related to the density through an equation of state.

So first thing that we saw is that if you have adiabatic expansion then we worked out the relation between the density and the scale factor as the universe expands this follows from the first law of thermodynamics the total energy has to be conserved, energy in a volume has to be conserved, then we incorporated this material this constituent which has pressure into the expansion of the universe.

And the Newton is laws gets modified if there is substantial pressure in the constituent in the gravity submitted substance okay, the pressure has to be comparable to the energy density only then the pressure is important and this possibility we have to allow for this possibility in cosmology, so we have to use the current equation which are obtained by just replacing density by density+3 P/c square.

And we worked out what the equation governing the scale factor is allowing for the gravitational attraction of the pressure also, we worked out the most general such equation so this was the starting equation that we had the equation for the second derivative of the scale factor and we integrated this equation to obtain this that is basically what we did okay.

This equation has the entire tells us everything about the expansion of the universe in a situation where the equation of state pressure is proportional to density holds, let me finish today's discussion here, in the next lecture we shall look at a few consequences of this equation for different kinds of constituents of our universe.