Astrophysics & Cosmology
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Lecture - 20
Stellar Physics – V

Welcome we shall just start today's class by discussing the Saha ionization equation.

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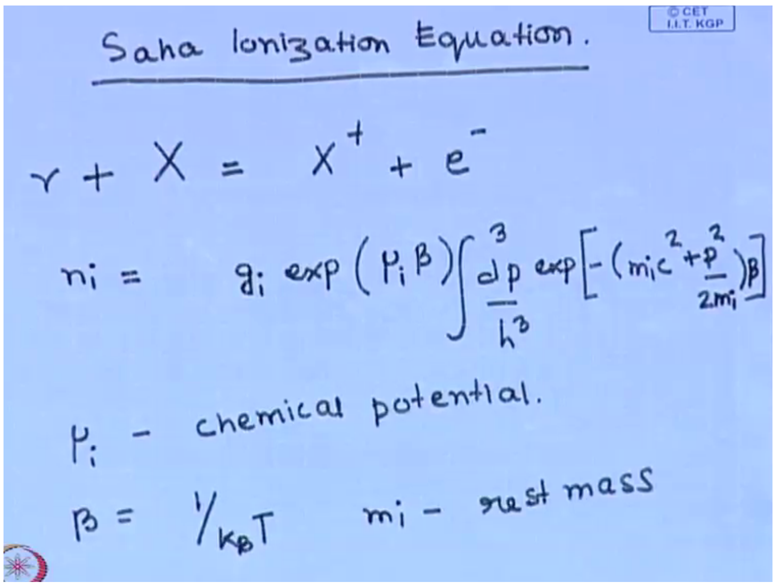

So we consider an atom let us call it X and we consider a reaction where this atom interacts with the photon which has sufficient energy to ionize the atom and as a consequence the atom goes to an ionized state + an electron, the same thing could also be applied in principle to study the transition from a singly ionized state to a doubly ionized state etc. so it is not only for the first ionization but for subsequent transitions to higher ionized states also.

In general, we are going to consider a situation like this and the question that we are interested in is that if I have the whole thing in equilibrium at a temperature T with certain density rho then our number density n, then what is the fraction of the atoms that is ionized and this is governed by Saha’s ionization equation which was worked out first by Meghnad N Saha okay. So we know that at sufficiently high temperatures the in thermal equilibrium the number density of any species ni i here could refer to the atom X or the ion X+ or the electron not the photon.

So in thermal equilibrium the number density is given by the Maxwell Boltzmann distribution and the Maxwell Boltzmann distribution tell us that the number of particles with any specified value of momentum. The number density of particles with any specified value of momentum is given by gi*exponential (mu i*beta)*exponential [- (mi c square+P square/2 mi)*beta].

So the Maxwell Boltzmann distribution tells us that the number density of particles with momentum in the range d cube p/h cube is given by this expression here gi is the degeneracy of any of these states it is a spin degeneracy gi. Mu i is the chemical potential corresponding so we are looking at the ith at species gi is the degeneracy of the ith species, mu i is the chemical potential of the ith species, beta is the same for all of these species same for the photon.

These atoms the ion and electron, it is 1/the Boltzmann factor into the temperature, mi is the rest mass of the ith species and i could refer to either the atom, the ion or the electron okay. So let me remind you what we are in what we are interested in, so we have this atom X which is in thermal equilibrium at a temperature T with its ion and an electron.

And this is a possible reaction to so if at a temperature T all these things will be in thermal equilibrium and provided the temperature is sufficiently high the number density of any of these species is given by Maxwell Boltzmann distribution and the number density is given by this expression, it has a temperature and the chemical potential of that species also this spin degeneracy, to find the total number density I have to integrate overall momentum states okay.

This is basically the energy of the particle in to beta e to the power-the energy into beta that is the Boltzmann factor okay, to find the total number density of particular species I have to integrate overall momentum values okay, so this will give me the number density of that particular species right.

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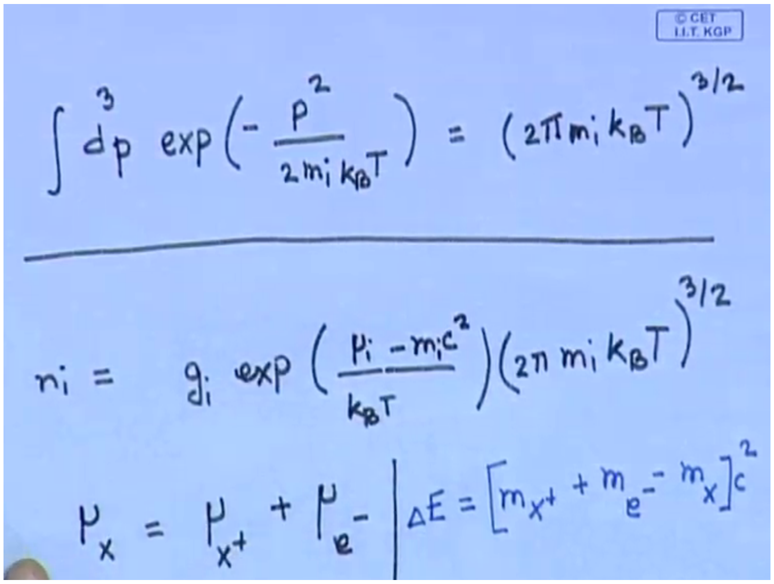

And we also know that this integral d cube p, e to the power – mi c square into beta is constant that comes out that come outside the integral, so we have to essentially do the integral d cube p*exponential (-P square/2 mi KB T) where I have written out the value of beta and this is a 3-dimensional there are basically 3 Gaussians in this 3 different Gaussians each of them being integrated from-infinity to +infinity.

So each of these integrals gives as a factor 2 pi into this number over here, so mi KB T the whole thing to the power half is one of the integral so here I have 3, a product of 3 integrals one for the X component, one for the Y component, one for the Z component, so we have to the power 3/2. So we can just straight away substitute that here and we get the number density of any particle any of these species to be gi exponential mu i/KB T mu i-mi c square where I have taken this factor outside.

And then we have this factor over here which is 2 pi mi KB T to the power 3/2 okay. So this allows us to write down the number density of each of these species provided we know the chemical potential and we know the mass of those of that species and we know the temperature of the entire system which we are looking at okay, now in thermal equilibrium the chemical potential of all the species on the left hand side of the situation is equal to the chemical potential of all the species on the right hand side of this equation.

We know that in thermal equilibrium this should be true, further photons can be created and destroyed at will there is no conservation associated with the photons, the photon number is not conserved so the chemical potential for the photons is 0 which essentially tell us that the chemical potential for the atom should be equal to the sum of the chemical potential for the ion + the chemical potential for the electron okay.

The second input and that is required is the fact that the energy is conserved in such a reaction, so here if I take the rest mass of this atom and take the sum of the rest masses of this ion and the electron the sum of the rest masses of the ion and the electron is going to be more than the rest mass of the atom because the atom is obtained by binding these 2 together, so there is a loss of energy in that process.

And the difference between these is the energy that is required to ionize the atom, so if this is the energy required to ionize the atom this is = the mass of the ion + the mass of the electron-the mass of the atom into c square, so this difference is the energy that is required to ionize the atom that is ionization potential delta E okay, so with these 2 inputs and we have this we can now take the ratio.

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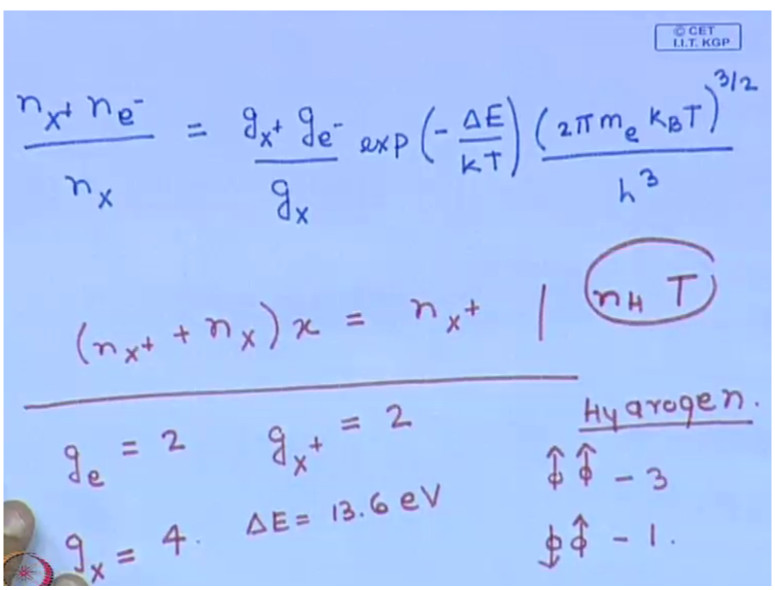

We can now consider the ratio of the number density of particles in the ion into the number of density of number density of particles of the electron divided by the number density of particles in the neutral okay, so let us consider this when we consider this ratio we will have 1 factor which is the ratio of the corresponding degeneracy factors, so we will have gx+g electron/gx.

Then we will have the ratio of exponential mu/K T for this species in to that of this species divided by that of this species, so since the chemical potential is in the exponent we will have the exponent of this + this – this which is 0, so the chemical potential term exactly cancels out from the whole thing okay, now we have another term which is the mass term the rest mass term, so we will have the – the exactly this when I consider this ratio.

So I will have another term which is exponential this that is the ionization potential energy ionization potential and we have a ratio of this number for the product of this number for the ion and the electron divided by that for the neutral atom, the ion and the neutral atom the masses are approximately the same so in the ratio it cancels out and what we are left with is, this should be a factor of h cube here which I have forgotten to write down.

The h cube comes from the phase space counting right, you have to count per unit volume in phase space which is h cube, so we have this factor 2 pi*m mass of the electron KB T to the power 3/2/h cube this is this Saha ionization equation it plays a very important role in astrophysics in a variety of situations okay, the ionization fraction let me complete this, the ionization fraction is the fraction of the total number density that is ionized.

The total number density of the particles is, so the total number density nx+ that is the neutral number density + the ionized number density this into ionization fraction X gives me the fraction that is ionized okay, so we can use this equation to solve for X okay, it is quite straight forward let me just leave it here for that or we can okay, we can so you can use this equation and solve for X the ionization fraction okay.

And this equation has a variety of applications for example if you are dealing with hydrogen let me just look at hydrogen and the interior of the star we are here interested in a star, so the interior of the star is to initially largely made up of hydrogen so if you are interested in looking at the state of hydrogen at a certain temperature then the spin degeneracy of an electron we know it is to the spin its spin of particle and it has 2 possible spin states.

The hydrogen ion is a proton which again is a spin half article so that 2 has spin degeneracy of 2 this is for hydrogen okay, and the hydrogen atom the ground state of the hydrogen atom there are 2 possible states in one state let me just look at that in one state the proton spin and the electron spin are aligned, in the other state the proton spin and the electron spin are an anti-aligned okay.

So that is now this has a total spin of 0, so it has only one state this has a total spin of one it is a triplet so there are 3 possible states, so totally there are 4 possible states for the ground state of neutral hydrogen okay, so for the ground state of neutral hydrogen this is for hydrogen gx = 4 okay, so you can put in these numbers here, the ionization potential for hydrogen is 13.6 electron volts this we know from Atomic Physics.

So you can put that in here the mass of the electron is known and then if you assume that the hydrogen has certain number density, the number density is going to be here right there is one number density here, one number density here so it is a number density square divided by just a number density, so there will be 1 factor of number density hanging around, so if you assume that you have a certain number density of hydrogen atoms.

And then you can calculate what fraction of it is going to be ionized, so you need to know the number density n of hydrogen atoms of hydrogen basically could be atoms ionized nH and you also need to know the temperature ones you put these 2 things into this equation it will tell you what fraction of the gas is ionized, what fraction is neutral okay, and this is a very important equation it has a variety of applications.

And we shall be discussing another application of this equation in cosmology, but here we have been looking at the star and the reason why we have been discussing the Saha ionization equation is that we have been looking at the equation that governs the structure of a star okay, and the Saha ionization equation tell us whether the gas inside a star at any location is going to be what fraction of it is going to be ionized and what fraction of it is going to be neutral okay.

This is important because we need to know the number of particles if it is ionized then the number of particles increases and for the same temperature and density you will have a larger pressure right we have seen this.

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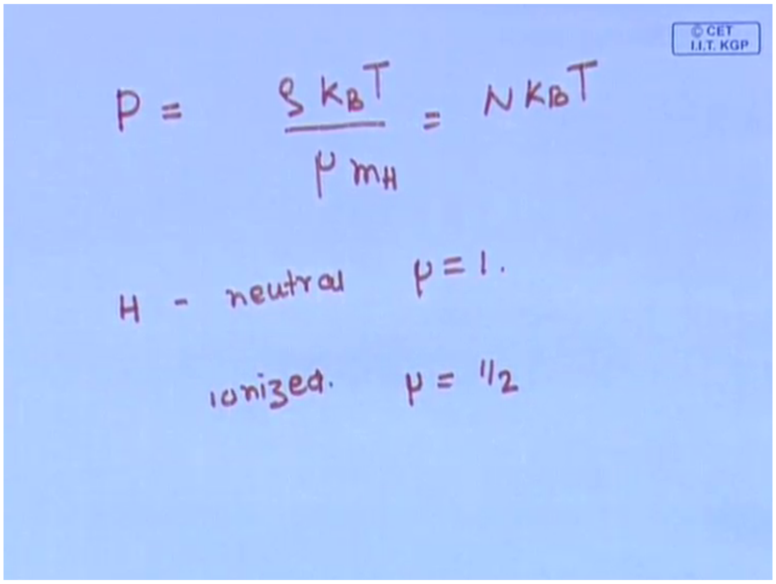

So the pressure at any point is rho KB T/mu into the mass of an hydrogen atom and mu is the mean atomic weight, so for hydrogen we have already discussed this for hydrogen if it is neutral mu is 1, whereas if it is ionized then the mean atomic weight is half, because electron has weight approximately 0, the proton has the same rate as hydrogen atom approximately, so but the number of particles has doubled okay.

The number of particles has doubled because we have the electrons and the protons both contributing to the pressure right, this is actually N KB T this is the formula which we have written like this in terms of the mass density right, the pressure is N KB T, N is the number of particles, pressure sorry by volume okay, so the number of particle doubles and so you the pressure is different okay.

So when you want to solve for the state of the interior of a star one this is an important quantity that has to be it also is important at when you want to determine the opacity neutral atom and we will have a certain opacity the ionized atom electron and ionized atom will have a different opacity, the electron is after all a charged particle, so it can interact directly with radiation. Whereas atom will interact only at certain frequencies corresponding to its spectrum spectral lights okay.

So this is something the Saha ionization equation is something that one has to solve simultaneously, so let me know discuss how one goes about solving the equations that govern stellar structure okay.

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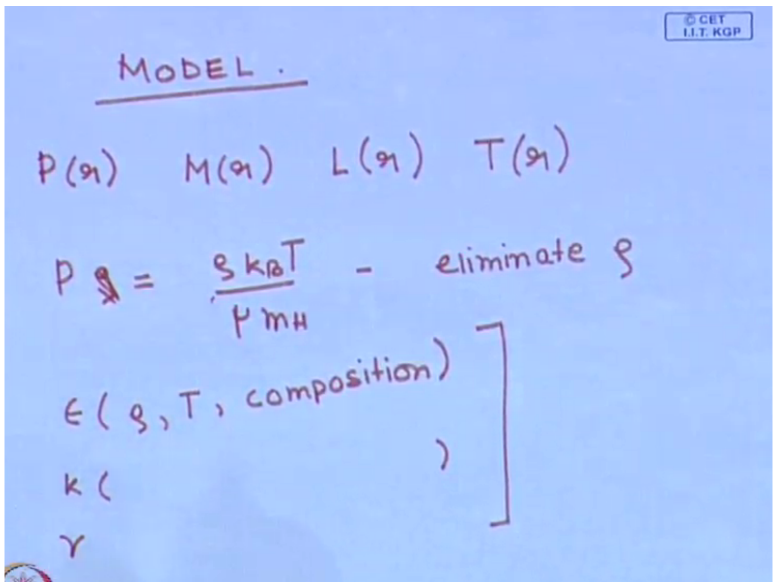

So how does one actually model the stellar structure, so we are now in a position to do this qualitatively at least discussed the whole thing, so we have already studied the 4 basic equations that govern the structure of a star, we had 4 equations one equation for the pressure as a function of r, the second equation for the mass as a function of r, the third equation is for the luminosity as the function of r.

And the 4th equation is for the temperature as a function of r, we had 4 equations, the 4th equation we have seen is different depending on the energy transport mechanism if the energy is transported through radiation you have a certain equation, if it is transported through convection we have a different equation and we have discussed the condition for which equation should be applied okay.

So we have these 4 equation and for these 4 quantities and the corresponding equations, so we have to simultaneously solve for the 4 equations governing these 4 quantities, in addition we have to provide the equation of state externally this has to be given this has to solved, because the pressure gradient involved the density, so we use this to eliminate the density from all of this right.

All of these equations depend on the density and we can use this relation to eliminate the density and write it in terms of a pressure and temperature okay, we also have to externally specify the energy generation rate epsilon which we have seen depends on the density the temperature we have already discussed this and it also depends on the composition, this equation has to be how much this is what functional form it has to be externally specified.

The opacity has to be again specify externally specified which again is a function of all of these 4 things, the opacity comes in the temperature the energy generation function comes into the equation for the luminosity, the opacity comes in here when energy is transferred through radiation transport and we also have to specify the ratio of the specific heat capacities externally.

So these things have to be either determined theoretically or experimentally from the study say this has to determine from measurements of nuclear reaction or atomic cross sections, this has to be determined by the measurements of optical cross sections all of these have to be externally given okay, so all of these have to be given and once you have this you have a set of closed equations which you can now attempt to solve.

Now in addition to this it is also necessary to give boundary conditions if you want to solve these equations, so you have 4 differential equations you have to also give provide a suitable boundary conditions if you wish to solve these equations okay, so the boundary conditions for a star would be something like this.

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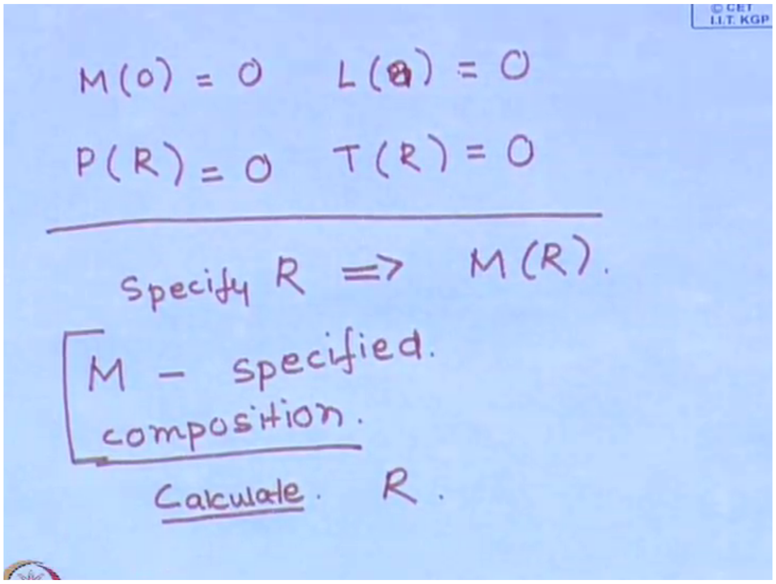

So the pressure okay first of all at the center of the star the mass M at the center of the star, now let us see the pressure okay would be 0, at the center there would be no mass that is obvious the mass contained within the center is 0, the luminosity at the center sorry at the center is 0 at the origin, the pressure at the surface of the star should vanish if it is has to be in equilibrium, the temperature at the surface also has to vanish.

So one could put in these boundary conditions and you have the equation 4 differential equations for these in addition to these conditions, now what one could do is one could then tell that okay, so I know the mass at the origin is 0, I know all of these boundary conditions what one could have to do now is that one would have to specify the radius of the star because which are the points where you want to set these 2 to 0.

One would have to specify the radius of the star and by solving this equation these equations you would get the mass of the star but that is not typically how one would like to formulate the problem, so the way I have set it up till now you would have to specify R the radius of the star and then solve for this which would give you the mass of the star which is the M at R but typically we would not like to specify the problem like this okay.

So the problem is reformulated in a slightly different way the conditions that are set up in a slightly different way let me mentioned that, so the conditions are set up so that you can specify the mass of the star that is how you would like to solve it, so the problem is reformulated the boundary conditions are reformulated, so that you can specify the mass of the star and the composition.

So this is how we would like to specify that problem given the mass and the composition we would then like to calculate the radius of the star, the size of the star and various other things the temperature profile, the density profile, where the energy is transported radiatively, where it is transported through conversation everything else we would like to solve given this okay, so the problem is reformulated so that these 2 things can be specified okay.

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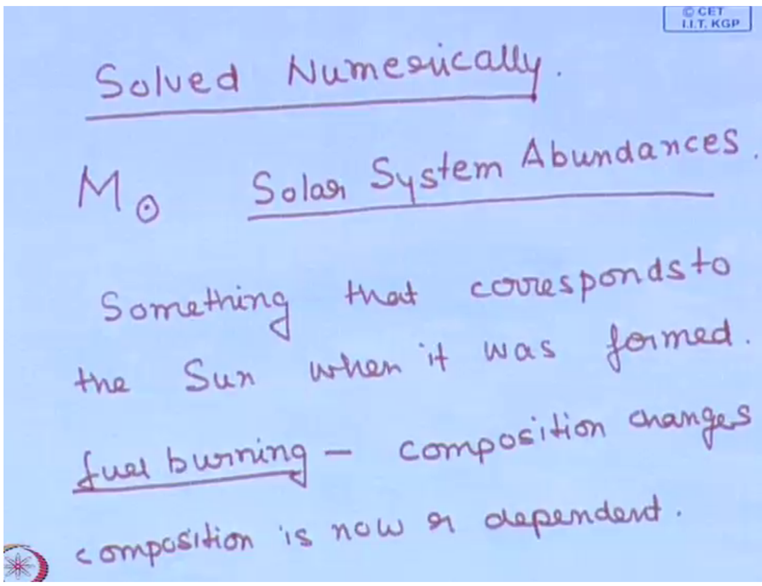

And the whole thing has to be solved numerically, so the whole thing then has to put in a computer and the set of equations have to be solved okay, the whole thing has to be solved numerically it is not tractable analytically this exercise it is a complicated set of equations with various conditions okay, so the whole thing is put in a computer and it is solved numerically. Now, so suppose you take a star of the mass of the sun.

And you set the solar system abundances and you set the so the composition you set it at the solar system abundances and you solve these equations what you would get as a solution would be something that corresponds to the star to the sun when it was formed right, because you have started off with the initial composition of the sun and you set up these equations then you solve it. Then people actually are able to reproduce something that people believe is what the sun look like when it was formed, now as the sun burns the fuel the composition changes right.

The whole thing evolves so what happens is there is fuel burning and the composition changes and one has to take this into account and again solve the equation, so initially the composition was the same throughout the solar abundance. Now the composition becomes so the composition now is the function of r okay and one has to incorporate that so basically there is evolution affect which one has to take into account right.

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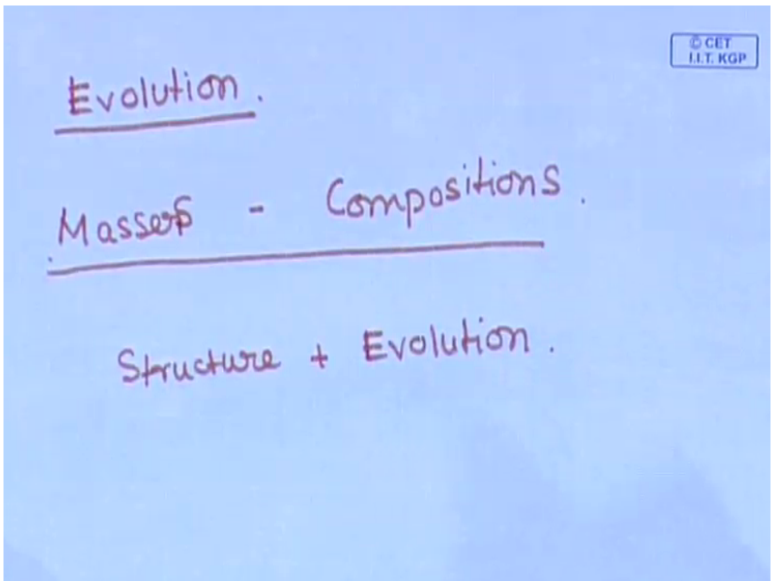

So there is evolution has to be incorporated and then you are finally lead to a model which is you expect to match with the sun as it looks now okay, further one can also carry out this exercise for different masses okay, other masses and other composition and look because one could very the mass and the composition and study the structure of stars in general and the study what both things structure and evolution of stars in general of a wider range of stars.

So the whole thing is a very intensive computational exercise which is not I mean it is not feasible to discuss more details of it here we can discuss the results okay, so there are now currently existing solar models which have been obtained by this process okay.

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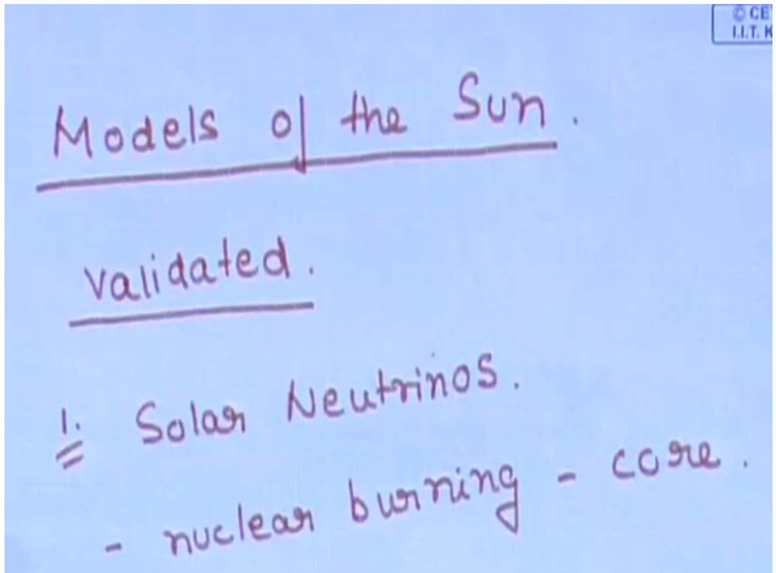

So there exist models of the Sun which have been built up like this, these equations solved with the equations for the energy generation through nuclear reaction, so when you solve these equations you know in which part of the sun reaction is taking place you can predict simultaneously solve for all these the kind of reaction, the reaction rates etc. so you have now models which are reasonably believed to be correct.

These models have been validated mainly against 2 kinds of very precise observations, one are there Solar Neutrino observations, the solar neutrino observations probe the fuel nuclear burning which takes place in the core of the sun okay, so they probe a very small region at the center of the sun neutrino burn this nuclear the solar neutrino observations and these are very precise through these observations put very precise limits in give very precise inputs on the model.

And the models now are consistent with these solar neutrino observations okay, there are another kind of observation which I will just mentioned here again very important in field of astronomy.

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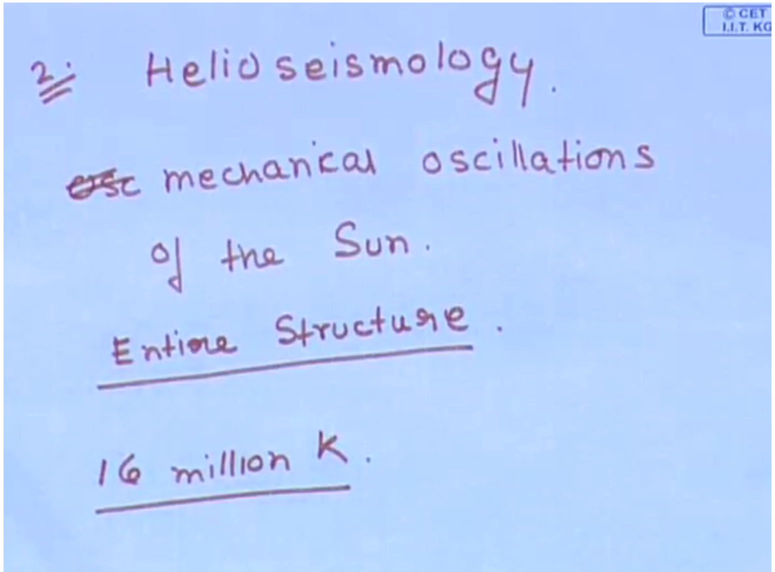

These are Helio is a subject actually Helioseismology, so here what is studied is the oscillations or mechanical oscillations of the sun, the quantity that we studied are mechanical oscillations of the sun, so it is like something like monitoring earthquakes on the earth surface, so you monitor the vibrations on the sun and it can be done by looking at the intensity on the radiation that you received from the sun okay.

And this is relatively easier model then you can so this is relatively easier than modeling the earthquakes in the earth, because the sun is made up of simple gas it is a sphere of gas whereas composition of the earth is much more complicated there are all kinds of rocks etc. so the whereas the sun is easier, so it is easier to model oscillations of the of the sun and mechanical of oscillations of the sun.

And these observations the monitoring of the observation monitoring of such oscillations probe the interior of the sun, so they essentially probe the entire structure of the sun helioseismology and models of the sun are now consistent with this helioseismological observations also okay. And these models predict that I will not go into the details.

I have already told you many of the important parameters of the sun in earlier classes we have discussed in great detail, so they tell us the temperature of the sun center of the sun is around 16 million kelvin at the core and it falls off as you go outside okay and a wide range of other parameters in the interior of the sun are predicted by current solar models okay.

The subject of modeling stars has also been applied to different masses and different compositions that goes into the realm of different subject which is the study of the space, the study and classification of stars which is something and that we shall take up in the next class, so today we are going to stop here.