Astrophysics & Cosmology
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Lecture - 02
Kepler's Law

I hope you had a look at the night sky last night and you could identify Orion and Ursa Major and also get feel for those things that we talked about in the last class, particularly right ascension and declination. In the last class, we saw that the motion of the sun and the planets and the moon on the sky can be interpreted in terms of the mood going around the earth and all the planets going around the sun.

Now the study of these orbits, the motion of these planets and moon is what led to the birth of what we now know as mechanics. So the mechanics that we lean in school, high school and in college can essentially be traced back to the study of the orbits of planets and moon around the ear, trying to understand that. One can explain, one can mathematically quantify this whole thing using the Newton's laws of motion.

So the Newton's laws of the motion you may say were came about from the study of these orbits, okay trying to understand these orbits. So, there are 2 things that are required, Newton's laws of motion, which we all know.

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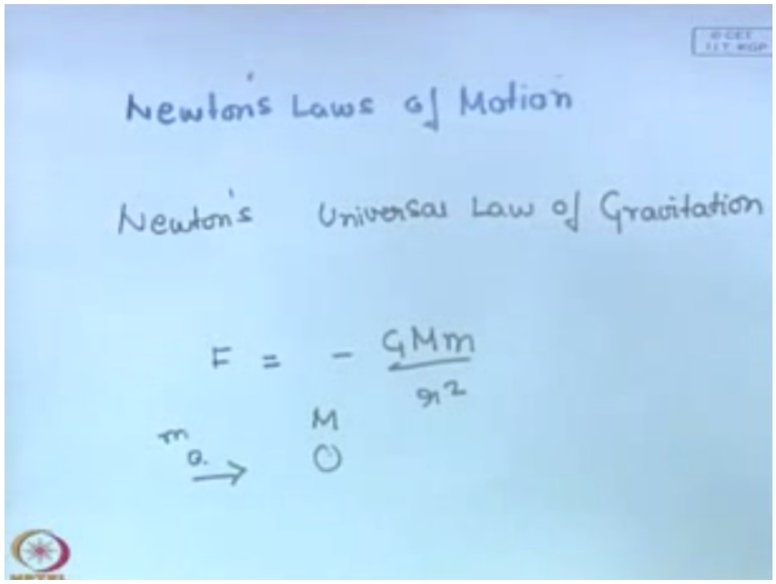

So, that is the first input and the second thing that is required is the Newton's universal law of gravity. Both of these were proposed by Newton. And the second one is a very remarkable thing, it tells us that any 2 objects in the universe exert a force F, which is equal to, so if there are 2 object, this object will experience a force, due to the gravitational interaction between these 2 objects on the force is going to be directed, it is an attractive force and it is going to be –GMm/R square.

Where M and m are the masses of these 2 objects and such a force acts between any 2 objects in the universe and it was a great realization, it was a great jump, the realization that the force that pulls an apple that fall from the tree down to the earth, the force that holds us to the earth surface is also the same force that is responsible for the moon going around a earth and the earth going around the sun.

So, this is a universal law, is not specific to certain place or object, it is universal in nature, it holds for any object. Okay, so using these 2, one can explain these orbits, one can quantify these orbits. Before we go further, let me also briefly discuss another point, we now know that there are 4 fundamental interactions in nature. We have the strong interaction, the weak interaction.

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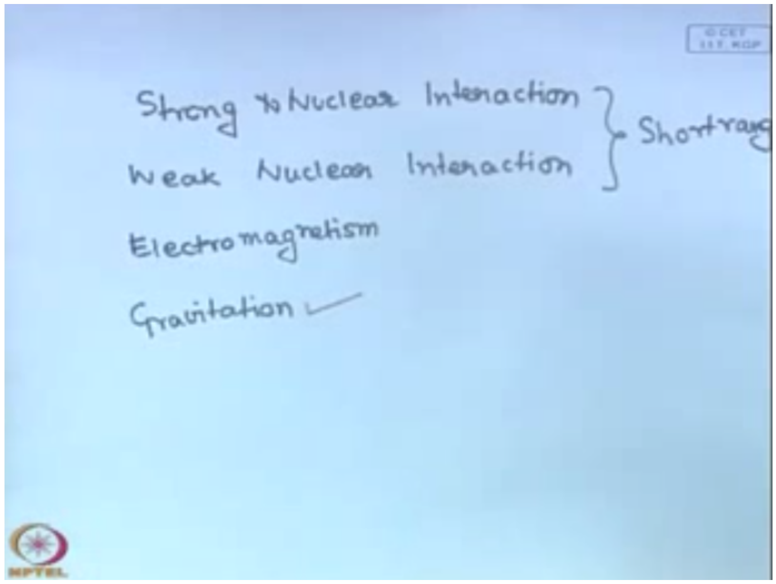

So we have 2 nuclear interactions, one is the strong interaction, strong nuclear interaction and then we have the weak nuclear interaction and we have electromagnetism and we have gravitation. Now, the first 2 interactions are short-range and in astronomy, we are usually dealing with motion of the very objects at large distances from one another. So these 2 are not very relevant if you want to study the orbits of say planets or on larger scales, motions are larger scales.

So, there are only 2 possibilities that the forces there could be electromagnetic in nature or they could be gravitational in nature. We also know that the universe on the average, if you take parts of the different objects in the universe there are on the average neutral, the do not have any excess charge. There are limits one can place on this and electromagnetic interaction acts only between charged particles.

So, this again, is not a very important force, if you are looking, trying to explain the orbit of planets around the sun or the mood around the earth, or various other motions that you observe in the universe, on large scales in the universe. So, it is gravity, gravitation alone, which is important, okay much of cosmology, much of the dynamics which is relevant on astronomical, for astrophysics or for cosmology is gravitational timings.

Other 3 also play important roles, but if you want to study large-scale motions, they are governed by gravitation. This is an important point to bear in mind and we shall now study the motion of objects under the influence of gravitational interaction. So we are going to start off by writing down the Lagrangian. And the systems that we are considering is as follows there are 2 objects.

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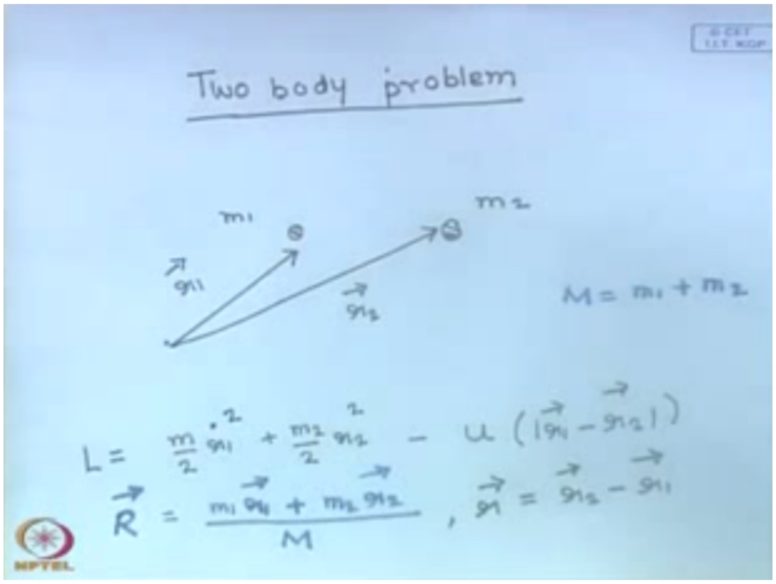

What we shall study is called the 2-body problem. So there are 2 objects of different masses, m1 and m2 and we chose some origin for the coordinate system and r1, r2 are respectively the vectors, the displacements with the respect to this origin, so they are the displacement corresponding to these 2 positions, so there are the positions of these 2 particles, m1 and m2 with respect to some arbitrarily chosen origin.

Now, the interaction between these 2 particles, it depends only on the Newton's laws tell us that it depends on the mass of this object, it depends on the mass of this object and it also depends on the distance, so it is a function of only the distance between these 2 okay. So we can write down the Lagrangian for this. Now, those are few who are not familiar with the Lagrangian, there is no reason to worry.

We could have equally well proceeded by writing down the Newton’s equations of motion, okay, but this is an easier approach if you know the Lagrangian formalism which I presume most many of the students here know, then it is an easier approach, okay so we shall adopt the Lagrangian approach, but you can arrive at the same result from just the Newton's law of motion. So let us write down the Lagrangian for this.

So the Lagrangian L will have one term corresponding to the kinetic energy of the first partial, so that is good to be m/2, r1.squre, this is the vector velocity square, basically, the velocity of this particle square, it is a kinetic energy of this particle and then we have the kinetic energy of the second particle +m2/2, r2 square and we have to subtract out the minus the potential interaction potential between these 2 particles.

And here, we shall do a very general analysis to start with, so minus, we will write that there is some potential which is given by u and u is the function of just the distance between these 2 particles, so u is the function of mod r1-r2, so r1-r2 is the separation between these 2 particles, the potential, the interaction depends only on the distance. So gravitational potential, we know depends only on the distance between the objects.

So here there is some potential, some interaction potential, which depends only on the distance between the objects, so this is the 2 body problem and we would like to analyze this problem, we would like to solve this the motion of these 2 particles. So there are 6 degrees of freedom, 3 for this particle and 3 for this particle and there are the 6 corresponding movement of these 6 degrees of freedom, so the degrees of freedom are the 6 different coordinates.

Now, the problem is, if you write down the equations of motion, you will get 6 equations of motion which will be coupled. The problem is considerably simplified, if you shift to a different coordinate system and the different coordinate system that makes the problem easier is as follows. One coordinate are, let me write down the coordinates, so there is one coordinate, R, which is the centre of mass of the system.

And you can calculate the centre of mass it is not very difficult, the center of mass is m1, r1, the weighted sum of the distances, r2/M, where M is the total mass. So, we are changing coordinates, instead of choosing the positions of the individual particles, it is equivalent, if you specify the center of mass and if you specify the relative displacement between these particles, so we shall use r to denote that and r is r2-r1.

We shall use these coordinates instead of the coordinates for the individual particles. These coordinate are convenient because the potential interaction depends only on the separation between these 2 particles, so now we use the displacement between these 2 particles as a coordinate. So, this also has 6 degrees of freedom, so you can equally well describe this system using these 2 vectors, instead of r1 and r2, okay.

We have to just transform everything from these coordinates to these coordinates. So to do that, let us just do the algebra, it is not quite straight forward business. So, let us go through the algebra, a little bit of algebra.

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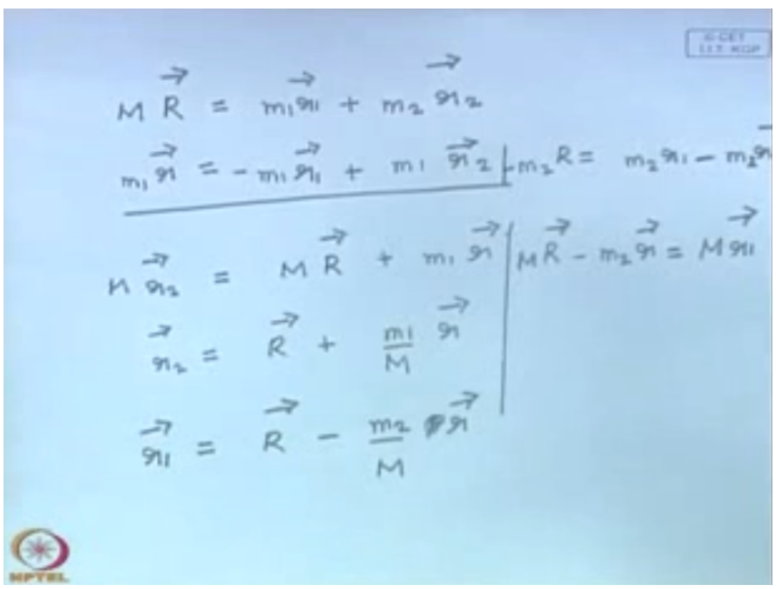

So, we have M*R. So we are just writing this equation again, so M*R = m1, r1+ms, r2 and we also that the displacement, the separation between these 2 are is r2-r1, so I can write r2 here and-r1 here. No what we will do is, we will multiply this with, so we want to let us say first eliminate m1, so we multiply this with m1, if you want to eliminate r1, so we will multiply this with m1 and add these 2 equations.

So, if you do that, what we get is this cancels out and what you have is M, r2 = M, R+m1 r or r2 = R+m1/M to r. Okay, so we have worked out what r2 is, now we have to just repeat the same exercise using eliminating r2 from this, we have now eliminated r1, so to eliminate r2, let me just write down the equation here, so if you want to eliminate r2, what you have to do is, you have to multiply this equation with m2 instead of m1.

So what we have to do is m2, r = m2 r1 and we will minus, this will have a - sign so, m2 r2 and we will multiply this whole thing with a minus sign, so that this cancels out so we will have a minus sign here, this will have a + sign, this will have a -. So, we have these 2 equations and when we add these 2 equations, proceeding in exactly the same way what we get is M R-m2 r =, this term will now cancel out, so we have M r1, or what we see is that r = R,-m2/M r.

So that is what we get. So we have these 2 expressions for r1 and r2, in terms of R which is the position of the centre of the mass, and r which is the displacement between the 2 particles. Okay, so what we have to do next is we have to substitute this in the expression for the Lagrangian. So let us do that. So the expression for the Lagrangian is shown over here, okay, so we have to substitute these expressions for r1. and r2., so a dot is just time derivate of this and this, time derivate of this and this.

The square of that we have to substitute here, okay so we have to substitute these 2 expressions here. Now let us just do this exercise quickly, so when I square this and multiply it with m1, I will have also another term, the square of this multiplied with m2, so when I add these 2 terms, I am going to square this essentially, I am going to square this and I am going to square this. So, there will be one term which is, let me write down the resulting Lagrangian.

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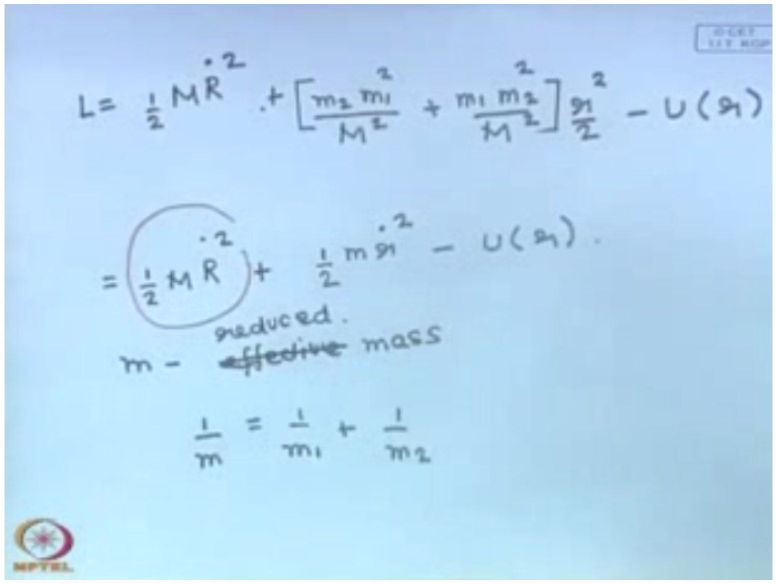

The resulting Lagrangian is L, there will be one term which is half M R square, that comes from the square of this and the square of the derivative of this, when I put them here. Then, there will be one term which is the product of this term and the product of this term, now note that there is an M1 here for r2 and r2 is multiplied by m2, similarly, there is m1 in r1 and that is multiplied m2 for r1 and it is multiplied by m1, so the cross terms, when I square the derivative, time derivative of this, the cross term will essentially cancel out.

And the term, the square of this, + the square of this is going to give rise to another term, so let me write that, so what we have is one more term which is +, so we have m2, m1 square/M+m1, m2 square/M, this*r square/2 and we have -u, which is just a function of r. And here, we can take, m1*m2 common outside, there will be m2 square here sorry. So, one term comes from the square of this, one term come from the square of this.

And here I can take m1, m2 common, if I take m1, m2 common, what remains is m1+m2, which cancels out with one of the m over here, so we get finally is that we can write this as half M r. square + half m r. square-u r. Where this m is called the effective mass. And it is 1/m = 1/m1+1/m2. So, we have gone through the algebra and what we see is that in terms of these new coordinates, the 2 body problem, can be written as just a free particle motion of the center of mass, it turns out that you have just a free particle motion of the center of mass.

And you have this motion of the displacement between the 2 particles, which is governed by this Lagrangian. Where the mass that now comes in is not the effective, it is called the reduced mass. The reduced mass is 1/m is m which is defined such that 1/m = 1/m1+1/m2. You just do the algebra, if you just simplify this, you will find that this is the mass that you get over here.

The center of motion, the motion of free particle is not very interesting, if it is at rest, it will remain at rest, if it moves, it will continue to move with a same uniform velocity. So that is not of much interest, so we can drop this term and the entire 2 body problem now reduces to a one body problem.

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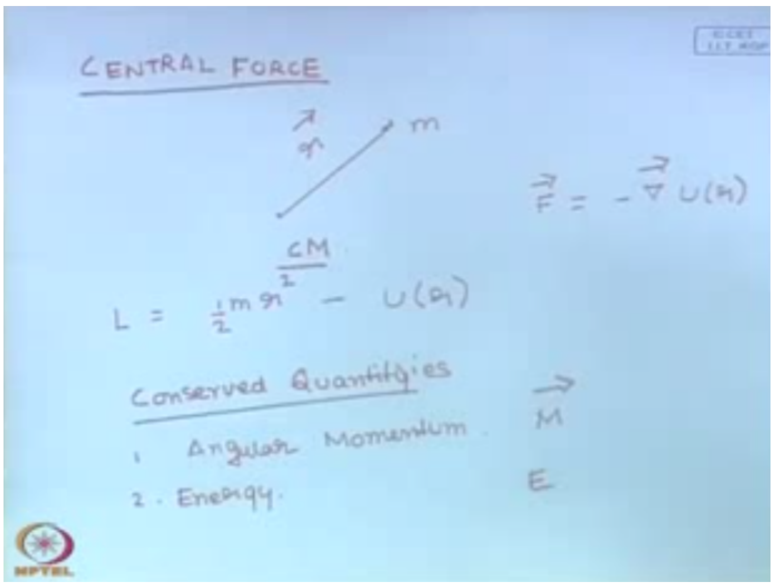

The one body problem is that there is a particle of mass m at a distance r, so there is a particle m and this particle m is under the influence of a force which acts towards the origin, where the origin now, the potential depends only on r and the force is the gradient of this potential, we know that, if the force on this particle is - gradient of u and since is spherically symmetric function or radial function, the force which is the gradient of this u - gradient of this u is going to be radial, it is a radial force.

This is called, so the whole thing, is a central force, so this is called the central force problem, the force acts towards this center, okay this is called the central force, so a 2 body problem, where they interact with some kind of interactions, which depends only on the separation, can be reduced to a central force problem, where there is one particle which experiences a force which acts towards the center, either towards at a way, and that force is represented by a potential which depends only on the distance r from the center.

So bear in mind, the relation between the system that we started out with, the system that we started out with had 2 particles, now we have just one particle of mass m, which is the reduced mass of those 2 particles, the distance r, the displacement vector r, the position of that particle is essentially r2-r1, so it is this vector, the displacement vector between these 2 particles.

And this displacement vector, r2-r1 is what we are now studying. And the center is now fixed, I cannot choose it arbitrarily, it is a center of mass of this, the origin is the center of mass, so this single particle which has the reduced mass, experiences a force towards the center of mass. Now, this again, at the end of the day if you wish you can convert back to the motion of these 2 particles.

This is the Lagrangian that we have to deal with, this is the central force Lagrangian and we know it is a well known fact that if you have a Lagrangian that is in variant under rotation, a Lagrangian that is spherically symmetric in variant under rotation, then there is a corresponding conserved quantity that conserved quantity is the energy, so is the angular momentum, so we know that for this system, there are conserved quantities.

One of them is the angular momentum and the angular momentum is conserved because the whole thing is spherically symmetric, the potential depends only on the distance from the center, it does not depend on the direction, if the whole thing is rotationally in variant and if we have a system that is isotropic or rotationally in variant the angular momentum is conserved, corresponding to this symmetry, there is a conserved quantity, the angular momentum.

There is another conserved quantity for this system and that is the energy, why is the energy conserved, well the energy is conserved in this case, or in any case, because if the system does not have any explicit time dependence, so if the interaction, whatever force I have is independent of time does not have any explicit time dependence, then I have a conserved quantity corresponding to that symmetry and that is the energy, so here we have 2 conserved quantities, angular momentum, it is a vector M.

So there are 3 components of this and there is one energy 4, so totally there are 4 conserved quantities. So the 2 body particle, 2 particles interacting with each other, through an interaction, which is dependant only on the distance, has been reduced to a one body problem, where that single body moves under the influence of an external force and that external force is a central force.

It has this isotropic and once we have an isotropic force, which is time independent, there are these conserved quantities, the angular momentum and the energy.

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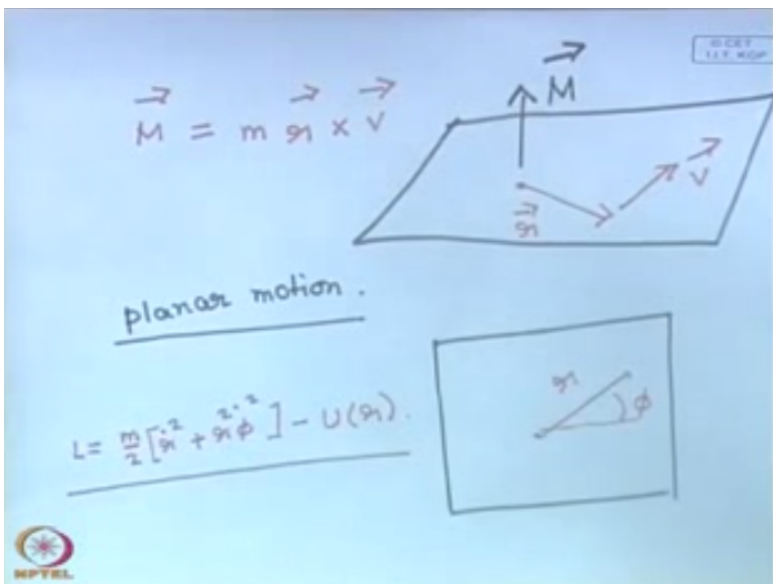

Now the angular momentum, we know is r, mass of the particle * r cross v, so if this the center of mass, if this is r and if this is v, the angular momentum is perpendicular to both of these, that is the cross product and it is a conserved quantity and since it is a conserved quantity, we know that the direction of angular momentum has to remain fixed, so r and v have to continue to lie in the same plane.

From the conservation of angular momentum, we can say that the motion is going to be restricted in this plane, the plane defined by r and v at any instant of time and the motion is going to be planar, so this implies that the motion is planar, so the conservation of angular momentum implies that the motion of this particle which has the reduced mass it is going to be a planar motion.

So, motion under a central force is always the planar motion, we now know this and that is the consequence of the conservation of the angular momentum. So, we can choose a plane aligned with this plane of the motion, and in this plane, we now have 2 coordinates, one of these, so this is a center of mass, one of these is r and the other coordinate is phi. 2 coordinates suffice in the plane to describe, completely describe the position of the particle.

And the Lagrangian can be written in terms of these 2 coordinates now and the Lagrangian is m x 2, so we have one term corresponding to the velocity, r velocity which is r. square, then another term corresponding to the phi motion, which is r square phi. square and we have the potential u which is the function of r. So this is the Lagrangian that we have to deal with. Okay, so, this is considerable simplification, we had 6 degrees of freedom.

Then it got reduced to 3 and the center of mass moves like a free particle and now we have got only 2, the conservation of angular momentum tells us that, the direction of vector. Now this Lagrangian notice, does not have any explicit phi dependence. It does not appear anywhere in this, so the equation of motion if you do not have any particular coordinate, that is called the cyclic coordinate and the momentum conjugate to that is conserved, so if you write down the equation of motion.

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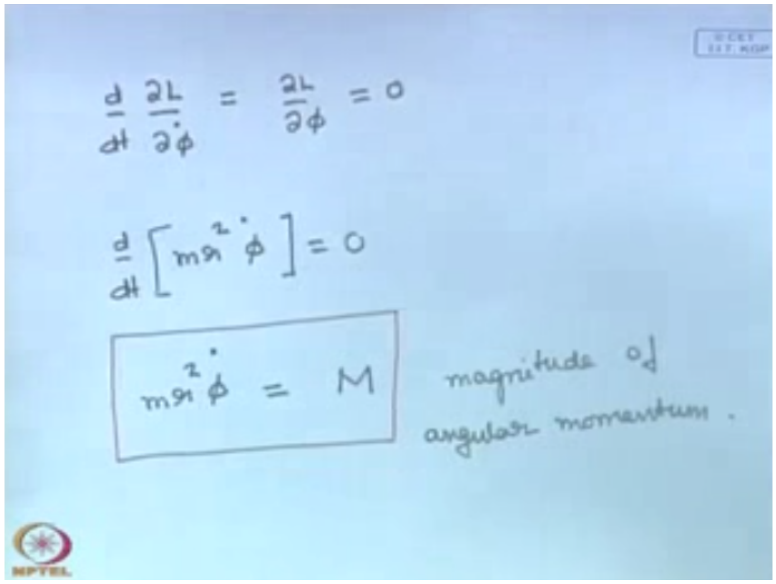

And del L, del phi, we know here is 0. So it tells us that the momentum conjugate to phi is conserved and let us calculate that so that del L, del phi. if you calculate that, it comes out to be m, r square phi., right what is this tells us is that this is 0 or this quantity mr square phi. is conserved and this conserved quantity is very easy to interpret, it is the magnitude of the angular momentum.

The direction of the angular momentum vector tells us we have planar motion and the equation for phi it tells us that the magnitude of the angular momentum is conserved. So, this is M, the magnitude of the angular momentum. Now, this is very important thing, let me spend a little time just discussing this. The fact that the magnitude of the angular momentum is conserved.

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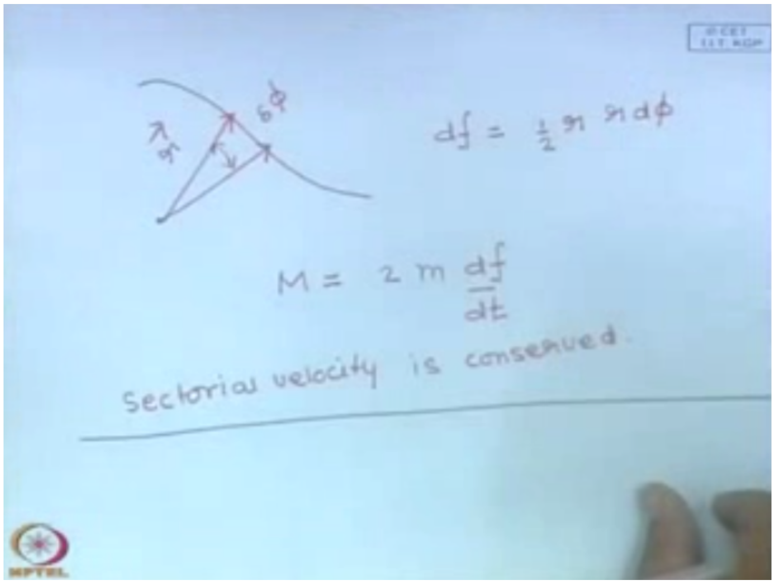

So let us consider, this is the center of mass and this is the trajectory of the particle and let us say that this is the position of the particle at some time, and this is the position of the particle at some later time and it has the angle between these 2 radial vectors has changed by an amount delta phi. So let us ask the question, what is the area subtended by these 2 positions, with respect to the center of mass with respect to the origin.

And this area, if the displacement is d phi, this area d f, we can call it df = half times r times this displacement approximately which is r d phi. Now let us go back to the conservation of the angular momentum, we see that this can be written as M = 2* the mass of the particle *df/dt. So for any central force, the sectorial velocity, this is called the sectorial velocity, the sector covered for unit times, sectorial velocity is conserved.

Or the orbit covers equal area per unit time which is Kepler's second law, Kepler observed these planetary orbits and he proposed different laws, phenomenological laws. So one of these laws is just an outcome of the fact that this we have a central motion and the law is that in a unit time, the orbit will cover equal area, in every unit time, it will cover equal area. The area covered in some time interval is going to remain fixed, the sectorial velocity is constant.

So this is called the sectorial velocity, df, dt, that is covered per unit time. That is the conserved quantity for any central force, need not be gravity, okay any force that is central tax towards a fixed center. So, this is a very general feature of the central force problem. Now, we can use the conservation of angular momentum, so this is the conservation of angular momentum, we have one more conservation law, so let me write down that.

The other conservation law is the conservation of energy.

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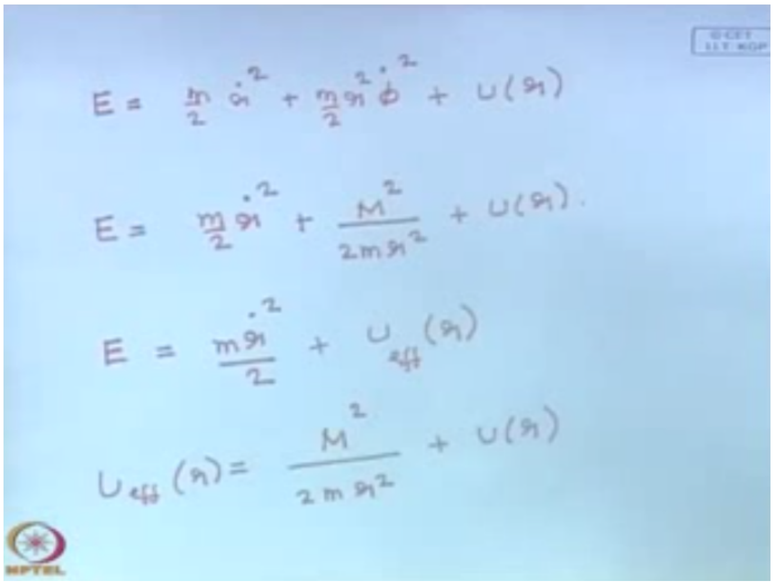

So this tells us that E = half m r. square+ no we have the, so this is the kinetic energy due to the radial motion, we have the tangential motion also which is m/2 r square, phi. square + the potential emergency u (r) that is the energy that is conserved. So this is other conserved quantity, the energy of the particle. Now, what we can do is, we can eliminate phi. from the expression for the energy using this magnitude of the angular momentum, the conserved magnitude of the angular momentum.

So, phi. is the angular momentum divided by the mass r square. Using this, we find that the energy =, this + r square, we have r square in denominator and when you square this, so when I square this, I will get r2 to the power of 4 in the denominator, I will get m square which will cancel out with one m over here as the half will remain and phi. will be replaced by M, so I have one term which is M square/2 m r square+u of r.

Or we can write the energy, as mr. square/2, now notice that these 2 terms, are just functions of r and we can think of this as some kind of an effective potential, write, we can write this as +u effective r, where u effective r = M square/2 m r square, that is the effective potential, so this whole thing has been reduced to a 1-dimensional problem where the energy is conserved and we have a new potential the effective potential.

This effective potential is the actual potential + an extra term. Do you what this extra term, yeah, that is the centrifugal term, so when you take the gradient of this, this will give you the external force and this is the centrifugal force, the angular momentum is conserved, so you will have the centrifugal force, so this term is what will give you the centrifugal force. So this is the centrifugal potential and it goes as 1/r square, so the centrifugal force goes as 1/r cube.

Now, let us next restrict our attention to the gravitational interaction, so we are now going to restrict our attention to the gravitational interaction, so for a gravitational interaction, so let us now get down to business.

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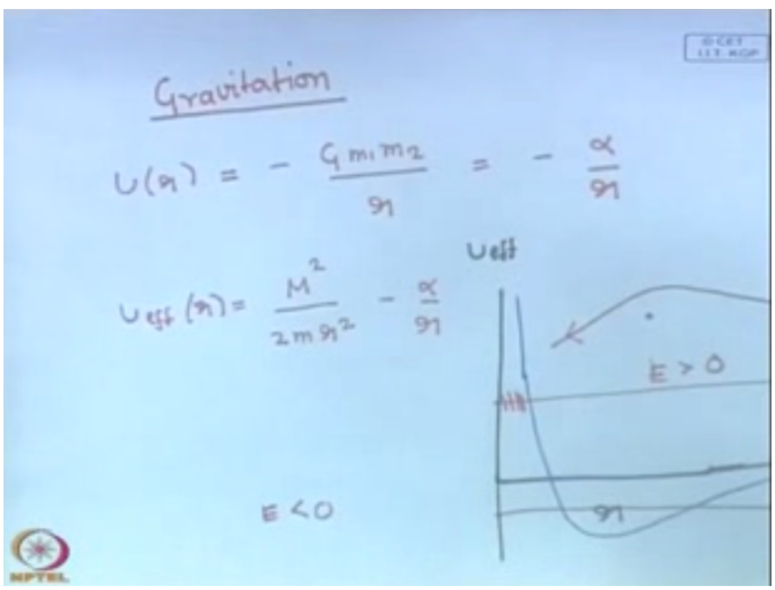

So we have, for gravitational interaction, the potential u(r) can be written as what is the form of the potential for gravitational interaction it is, -G m1 m2/r, right we know this, it is 1/r, the force is 1/r square, which we can write as - alpha/r. Okay, alpha is constant G m1, m2 and the effective potential is M square/2 m r square-alpha r. So let us plot what the effective potential looks like as a function of r.

This is very useful and instructive plot, so let us make this plot. Now, at small r, r square is smaller than r, so at small r, it is the first term, that is going to dominating, so as r tends to 0, the potential blows up, tends to infinity, okay. There will be some point, where these 2 terms are going to balance, and then beyond that it become negative and as r goes to infinity, this will tend to, both the terms will tend to 0 but its behavior is going to be dominated by this term.

So the nature of this potential is going to be dominated by this term at small r and is going to be dominated by this term at large R and there will be some transition region between these 2, so the graph looks like this and here it falls of as 1/r. So there are few general features that we have to note. First thing is, if the particle has finite angular momentum, it will never reach the origin, only particles with 0 angular momentum, can reach the origin.

If you have 0 angular momentum, this term is not there, I just have-1/r, it is always negative and the particle can reach the center of mass, otherwise, there is no change of reaching there, so the motion is going to be restricted outside. The second point is that the energy is conserved and it = the kinetic energy + this effective potential, so if I have positive energy, it will never reach here, so this part is out, the particle will come from infinity, it will reach a finite distance and then it will turn back again.

So, there will be a position of nearest approach, after which it will go past the particle, okay. So the trajectory for this energy, this is the center of mass, it is an attractive force, so the particle will come, it will get attracted and then it will go off like this, there will be a position of nearest approach, it never reaches the center. Whereas, for negative energies, $E < 0$, we will have bound orbits.

So bound orbits will be there only for negative energies and the orbit is going to be restricted between 2 values, the r1 and r2, the distance from the center of mass from the origin is going to vary between 2 values, r1 and r2, so there will a point of nearest approach and the point, where the particle is furthest.

You will have circular orbits, when do you circular orbits, at the minima of the potentials, at the minima of the potential there will be one value of energy for a given angular momentum, there will be one value of energy where these 2 curves will touch that is the minima and you will have circular orbits over there. Okay, and the circular orbits are stable in the sense that if you disturb it slightly, it will just very close to the circular orbit, it will not diverge.

So these are the broad features of this problem, so we are interested in bound orbits so the bound orbits are there for negative values of energy alone, so we will have bound orbits for negative values of energy and we have to solve this problem, now the procedure is quite straight forward, let me very briefly go through this, so how do you solve this problem, the way you proceed, you want to solve for the orbit, so what you do is you take this, start from here and this can be written as follows.

Let me not proceed in this way, let me proceed as follows: So here, we have r as a function of time, now when we want to solve for the orbit, what we would like to get as r as a function of phi, how down r, we want to solve for the orbit, how does r vary with phi. Once you have r as a function of phi, it is very easy to eliminate phi and get, it is a function of t using this. So what we will do is we will eliminate time from this whole thing first. So we will proceed as follows:

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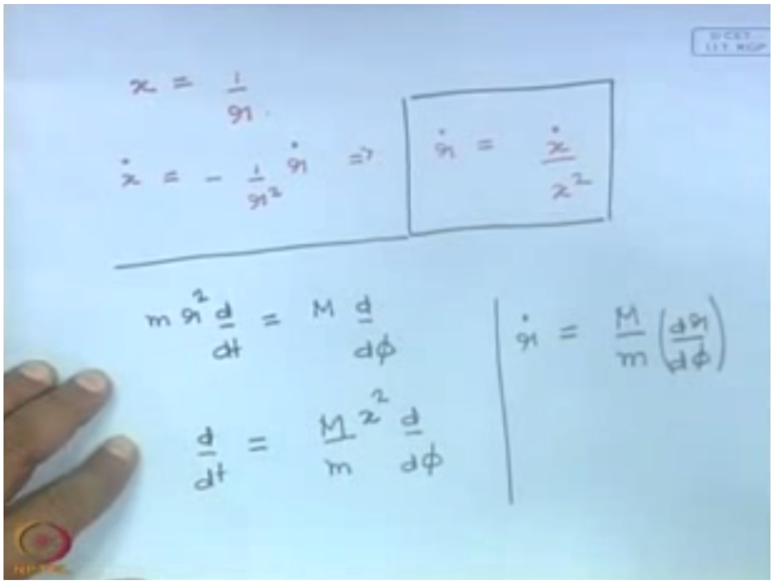

So we will time and the other thing simplification that we will do, is we will use u, let us say, x = 1/r as my variable, so we will do 2 simplifications, first we will use x which is 1/r as my variable, so let me do this, so x. =, right so this is x square, 1/r is x square, 1/r square is x square and if I take it onto that side I have r. is x./x square. Right, we can replace r. in this expression with x 1/r, that makes life a little convenient that is all, nothing profound in this, some algebraic steps.

And the second thing is we will replace time derivates with phi with phi as my variable, so from this we get m r square d/dt = M d/d phi, okay that is the conservation of angular momentum, right d phi dt = m r square d/d phi dt = the angular momentum, so we can write it in this way. What it tells us is that d/dt =, so I have, M, then I have m here and I will have 1/r square, 1/r square can be written as x square, so x square d/d phi.

So combining both of these, what we get is r. = M/m, the angular momentum divided by the reduced mass, x square cancels out, dr/d phi. So, what we have to do is, we have to replace this r. in terms of dr, d phi in the expression for the energy and then we have to integrate it. So, I will stop here for today and we shall resume on this in the next class.