Astrophysics & Cosmology
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Lecture - 18
Stellar Physics III

Welcome let us start off today's class by considering 2 problems.

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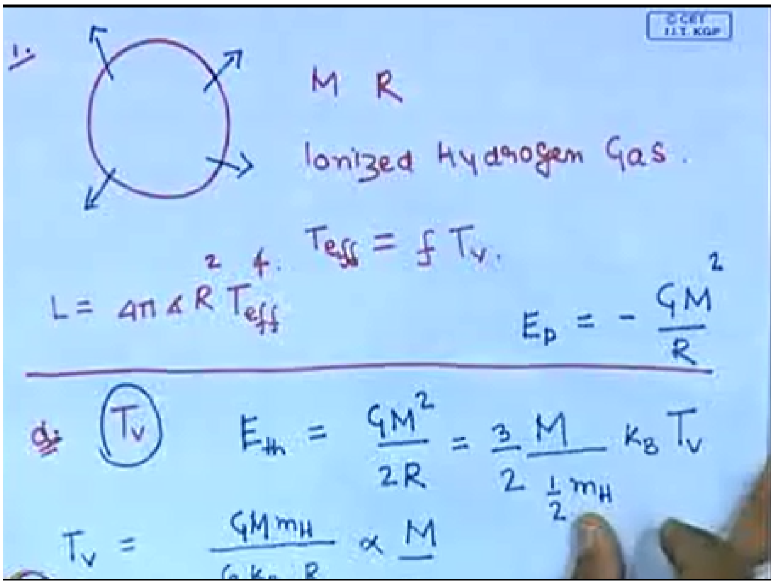

In the first problem, we have sphere like this you can think of it as the sun or some other sphere of gas and the sphere has mass M and radius R and it is in gravitational equilibrium. So if it is in equilibrium the sphere satisfies virial theorem. So this sphere is made up to hydrogen gas we shall assume that it is made up of ionized hydrogen gas and it is in virial equilibrium.

Now this sphere in virial equilibrium it will have some virial temperature we shall come to us it has some virial temperature and this sphere we will assume that it also emits radiation from the surface and the temperature T effective which is a fraction F of the virial temperature. So the luminosity of this object is 4 pi sigma R square T effective to the power 4. So the first part of this problem. This is the information given.

So the first part of the problem is to estimate the virial temperature of this object A so let us call it A. What is the virial temperature of this object? So the virial temperature Tv and how does this scale with the mass of the object and the radius of the object. Now we know let us ask let us address this question. So we know that if an object is in virial equilibrium under its own self gravitational force, then the average kinetic energy = the total energy in magnitude and it is half the potential energy.

And here we will also assume that the potential energy is – G M square/ R. So we want to find out how does the virial temperature depend on the mass and the radius of this object. So we know that in virial equilibrium the kinetic energy here is essentially the thermal energy Eth. So this is going to be = half of this value. So half of this value means it is going to be G M square/2 R.

And this is an ideal gas ionized hydrogen. So we know for an ideal gas the thermal energy is 3/2 into the number of particles. The number of particles is going to be the total mass of the object M/ the mass of the hydrogen atom MH and there is also the mean atomic mass of this that you have to take into account or another way of thinking of it is that every hydrogen atom essentially contributes 2 particles not 1.

Because it is ionized or you may say that the mean atomic mass is half. So there will be a factor of half here that is the number of particles. It is the mass of this object/half the mass of the hydrogen atom. So NK the Boltzmann factor KB * T virial that is how the virial temperature is defined. The internal energy is written in terms of the virial temperature in this way. It is basically 3/2 NKT. N is the number of particles KB is the Boltzmann constant *T.

So this straight away gives us the dependence of the virial temperature on the mass and the radius. So the virial temperature is equal to so let see how much is it equal. So the virial temperature = one factor of M cancels out. It is = G M * the mass of the hydrogen atom. These factors of 2 will cancel out and if I bring this on to the left hand side I will get a factor of 6.

So this divided by 6 * Boltzmann factor KB * R. I hope we have got it correct let us just check. So that is the virial temperature it is essentially proportional to the mass of the object and inversely proportional to the radius of the object. So that is the first part. The second part we would like to calculate the luminosity of this object.

And I have told you that this object we are assuming that this object emits from its surface as if it were a black body of temperature of this effective temperature which is a fraction some fraction some unknown fraction F of its virial temperature. In principal if it is optically thick and the temperature is the same throughout then it will be exactly equal to the virial temperature, but it may not be exactly optically thick.

So let us just use this and see how the luminosity depends on the mass of the object and the radius for such an object.

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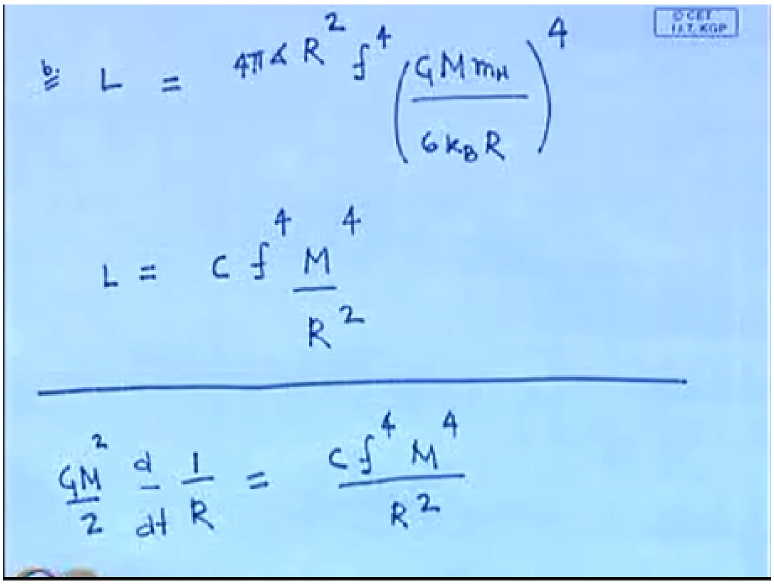

So the luminosity L=4 pi. So let me just keep the factor of 4 pi as it is. 4 pi and then we have this Stefan-Boltzmann constant sigma and then let me write those factors down 4 pi sigma. And then we have R square then we have this T effective to the power 4 which we can write in terms of the virial temperature. So the luminosity can be written as some constant C whose value you can work out by taking all the coefficients over here all the constant * F to the power 4.

And what is the dependence on the mass we are interested in the dependence on the mass and the radius. So the dependence on the mass is M to the power 4/R square. Okay so that is the second part of the problem. The luminosity is proportional to the mass to the power 4/R square. The thermal energy is proportional to the mass square/the radius. The virial temperature mass/R and the luminosity mass to the power 4/R square.

Now this object when it radiates it loses energy. So the total energy and the thermal energy are equal in magnitude and the thermal energy will increase if it losses energy we have already discuss this. So we can now equate the rate of loss in energy to the luminosity. So let us do that and what we get is D/DT the rate at which the thermal energy is lost and the thermal energy is G M square/2 R. Now when the energy is lose it is only R which is going to change the mass remains fixed.

So it is essentially 1/R and this is =C F to the power 4 M to the power 4/R square. So I can absorb these coefficients G M square/R * this constant C and redefine another so I can have another constant which is 2 C/G basically. 2 C/G.

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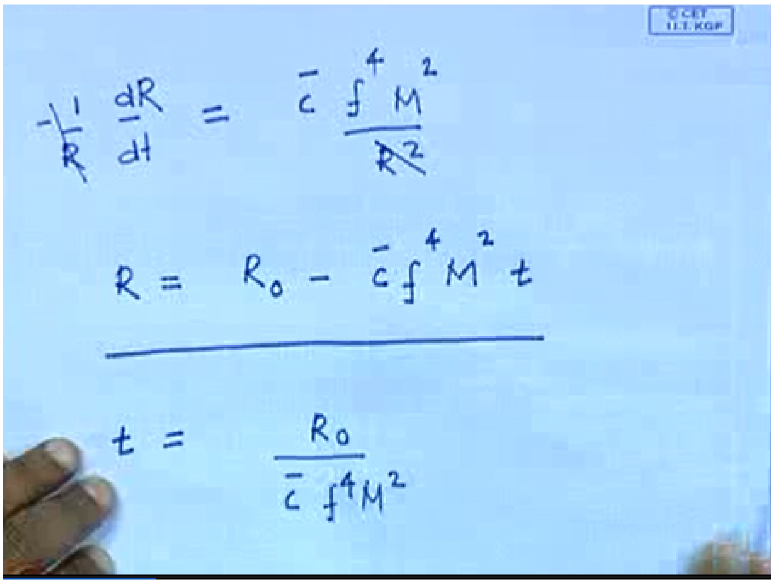

And what I will get is that D/DT of R -1/R square =so this was C let us call that give it some other name let us call it C bar this is = C bar and we now have M square over here F to the power 4 M square/ R square. So this 1/R square cancels out and the problem is that when this so the question is as follows when this losses energy its thermal energy is going to increase, its potential energy is going to get lower and lower.

So this thing essentially contracts and we want to find the rates at which it contracts. We want to find the solution to R as a function of time by solving this equation. So which now we can do straight away it is quite simple. So R =R0 which is a starting value of the radius –C bar F to the power 4 M square *T. So that is how the radius changes with time where you can work out exactly what the numerical value corresponding to C bar is by just backtracking a little.

And M is the mass of the object. F is the relation between the effective temperature and the virial temperature T is time. So at time T =0 the object has some radius R0. And R will be the radius at some later time T. So the object essentially contracts and the entire object according to this collapses at a time T =R0/C bar F to the power 4 M square. So having done this let me now leave the remaining problem to you.

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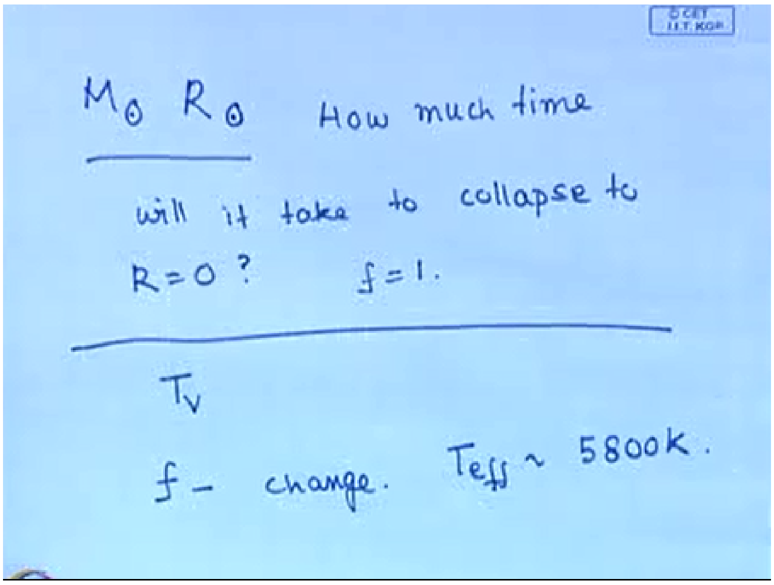

So you have to estimate the rest of the problem is left for you, you have to estimate the time it takes and object of mass of the sun and the radius of the sun to collapse. How much time in this model how much time will it take to collapse to R =0? Assuming F=1. When you do this you will find that this is ridiculously small time. So the question is then why how come you get such a ridiculously small time for such a massive object to radiate away all its energy.

And you can actually work out the reason if you estimate what the starting T virial is. If you estimate the virial temperature to start with you will find that this is much higher than the actual surface temperature of the sun. So to get an idea of what would happen to the sun about the Kelvin-Helmholtz time scale for the sun. So to get an idea what would happen for the sun if there was no source of nuclear energy you would have to change the value of F so change F.

So change F so that the effective temperature comes out comparable to 5800 Kelvin which we know to be the surface temperature of the sun and again estimate the time it would take to collapse. So these are parts of the problem that I leave for you to work out. I have done a part of the problem. I leave the rest of the problem for you to work out. You have to essentially put a numerical values and go through this exercise. I hope the problem is clear.

The next problem that I am going to discuss is again I have mentioned this problem it is not something new. The problem is as follows.

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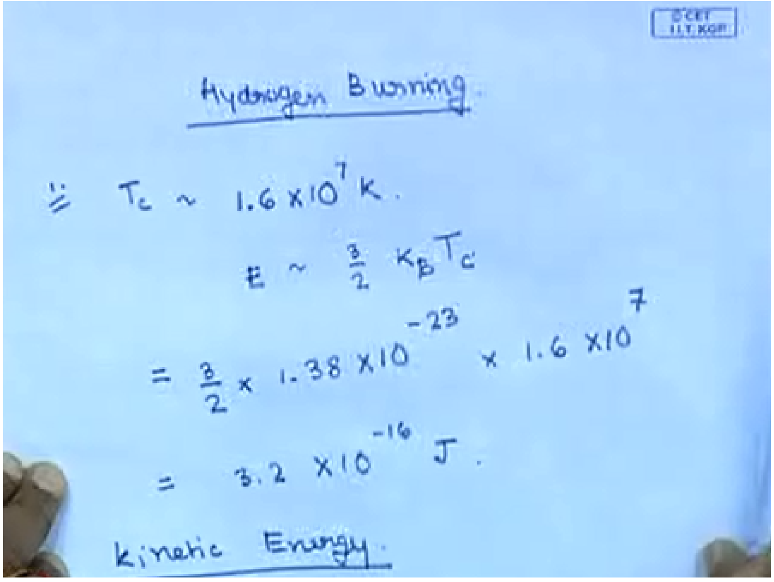

So the previous problem we had to do with the virial theorem. The next problem has to do with hydrogen burning. So the problem again has 2 parts. The first part we know that the temperature at the core of the sun is around 1.6 *10 to the power 7 Kelvin. So let us estimate the typical the average temperature, average energy of a particle of a proton at this temperature.

And we know from kinetic theory that the average energy of a proton of particle at this temperature is 3/2. So the typical energy of such a particle kinetic energy is of the order of 3/2 the Boltzmann factor * the temperature. So this will give us so the particles inside have a Maxwell Boltzmann distribution of velocities and so the kinetic energy also have to spread the typical energy kinetic energy is of the order of 3/2 KbT the average kinetic energy and we put in the values.

So 3/2*1.38 * 10 to the power -23 that is the Boltzmann factor * 1.6 * 10 to the power 7 that is the value of the temperature and this value if you put in these values and do the calculation it comes out to be 3.2 * 10 to the power -16 joules that is the kinetic energy of the protons.

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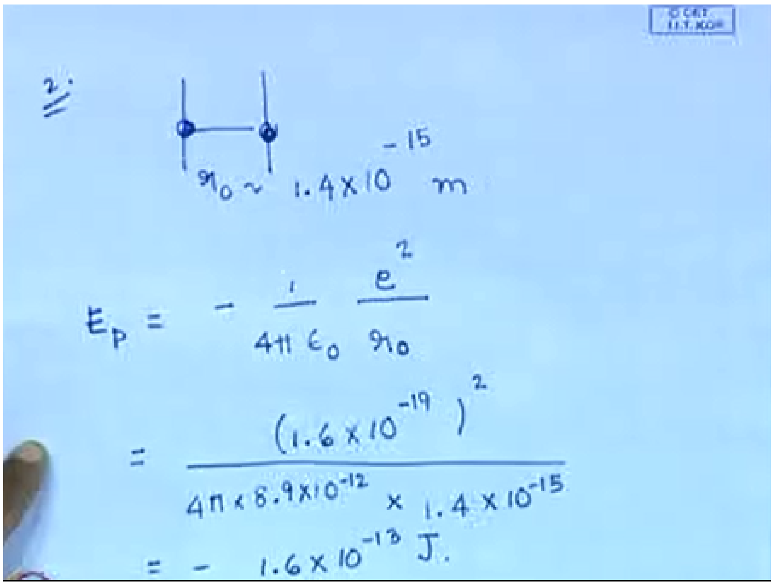

Now the next question is that for the nuclear fusion for hydrogen burning to occur. The first step is the PP collision. So we need 2 protons to collide to form a deuteron. So we have 2 protons and we know that at large scales there is the coulomb repulsion between any 2 charge particles which have the same charge and it is only if they can approach within a distance R0 of the order of 1.4 *10 to the power -15 meters of this order will the strong interaction come into play.

And they will then be able to merge and form a deuteron. So the question is what is the potential energy of 2 protons at this distance. So the kinetic energy has to equal to this amount if you want the protons to approach to overcome the coulomb potential and approach each other to such a small distance where they can become bound by the strong interaction. So let us estimate the potential energy of this.

So the potential energy of such a configuration is -1/4 pi epsilon 0 * the charge of the proton which is the same as the charge of the electron square E square/R0. So here again let us put in the values and we have the 1.6 * 10 to the power -19 that is the charge of the proton square in all in SI units so that will be in coulomb/4 pi * epsilon 0 which has a value 8.9 10 to the power -12 * the distance which is 1.4 * 10 to the power -15 all in SI units.

So you put in these numbers and the potential energy comes out to be of the order of -1.6 * 10 to the power -13 joules. So this simple estimate you can compare the typical kinetic energy is 10 to the power -16 whereas the potential energy is 10 to the power -13. So it is roughly the kinetic energy is roughly a 1000 times less and this is the problem which I had mentioned earlier.

So inside the sun the protons do not posses adequate kinetic energy for this collision to occur and for the reaction to proceed. If you think of them as classical particles, but when you treat them as waves there is a probability that they can tunnel through his barrier and for the reaction to take place and it is only because of this that you have the nuclear burning hydrogen burning going on in the center of the sun.

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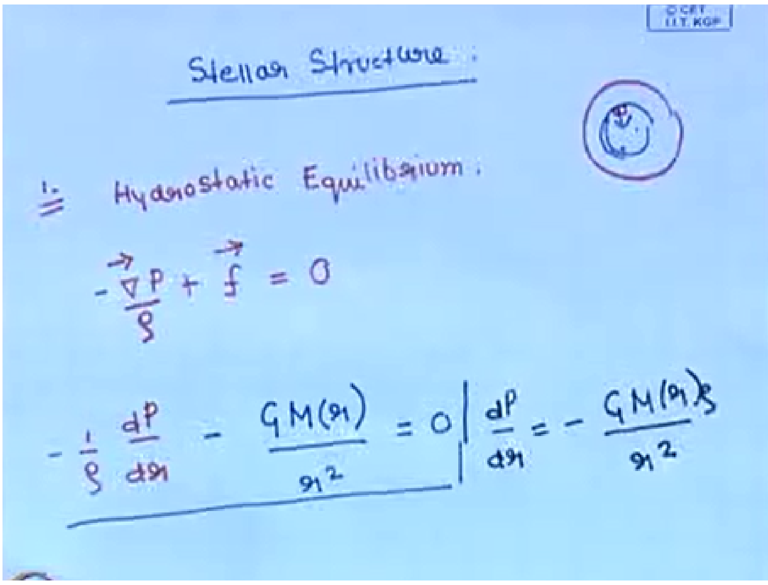

So having discussed these 2 problems let me now move on to the topic of remainder of today's class the remaining part of today class. So we are going to now move on to discussing the structure of stars' stellar structure. So this is what is called stellar structure. So we already have some idea about this. Now in today's class I am going to introduce the equations the basic equation that govern this so the structure of a star.

We think of star as a ball of gas a spherical ball of gas that is the starting point. So every star is essentially a spherical structure like this self gravitating spherical structure. So it is bound by its own gravitational force and it is by and large most stars are hot gas their plasma ionized hot gas. There are 4 equations governing the structure of a star. The first equation is the equation of hydrostatic equilibrium.

So it is the equation of hydrostatic equilibrium and so we assume that the star is in a static configuration. It is neither expanding nor contracting. We also ignore the rotation of the star and for such fluid we know that for any part of the fluid the pressure gradient across the fluid will balance the acceleration or the force per unit mass that is the main consideration. So if I consider any small element of this element the pressure gradient across the fluid must balance the force per unit mass of the fluid.

So here this fluid element every fluid element is being pulled inward by gravity. So this is spherically symmetric and every fluid element is being pulled inwards by gravity. The gravitational acceleration acts to pull these fluid elements inwards. So there has to be a pressure gradient. So the pressure inside has to be more and the pressure outside has to be somewhat has to be somewhat smaller and the difference in this pressures is what balances the gravitational attraction.

There is a factor of rho which I have missed out. So it is 1/rho. So everything here is spherically symmetric. So things are only function of the distance from the center the whole thing is assumed to be spherically symmetric. So we can now write this equation as DP/DR -1/rho + the acceleration per unit mass. So the question is how much is the acceleration per unit mass.

Now we know that if I have spherically symmetry only the mass inside this the total mass inside this is going to contribute to the acceleration of this fluid element. So this is going to be = –the gravitational acceleration is GMR/R square that is = 0. So we finally have the first equation which is essentially this and I will write it in a slightly different way. So the final equation that is the equation of hydrostatic equilibrium is essentially DP/DR = –GMR*rho/R square. Let me tell you that MR is the mass within.

So we are looking at a distance R away from the center and MR is the mass.

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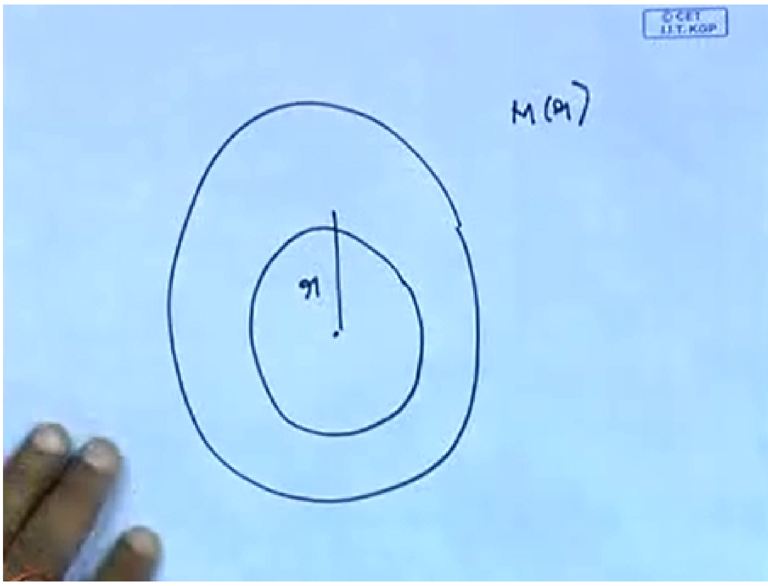

Let me draw a picture which we can use repeatedly now. So this is my star and we are looking at a distance R from the center and M of R is the mass within this sphere. So that is all that contributes to the gravitational acceleration over here and this is the equation for hydrostatic equilibrium till that is the first equation.

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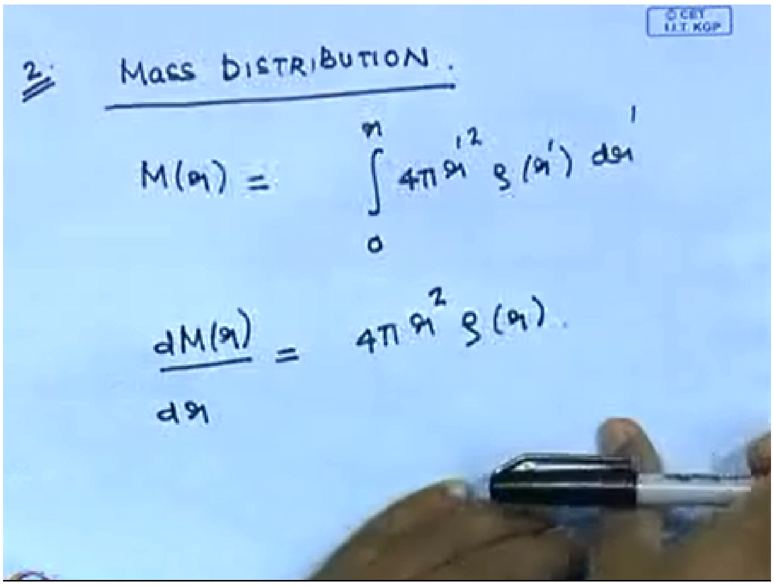

The second equation that governs stellar structure is the mass distribution equation. So the mass we know the mass within a sphere of radius R we know that this is the integral of the density 4 pi R prime square * rho R prime DR prime. So what we are doing is we are considering shells like this. So the thickness of such a shell is DR prime and the mass inside that is 4 pi R prime square rho R prime.

So this equation can also be written in differential form which is DMR/DR =4 pi R square rho R. So this is the equation that governs the mass distribution.

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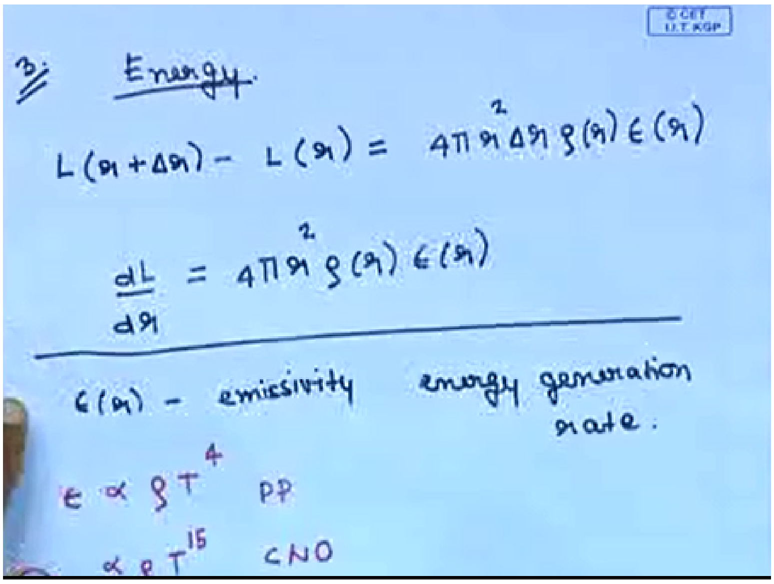

So the third equation in a stellar structure is the flow of energy. So it is essentially the flow of energy. So let us go back to this picture. Let us say that the luminosity at a distance R is L of R and the luminosity at a distance R+delta R is L of R+delta R. So the difference so let us just calculate the difference in flux between energy flux between here and here. So the difference in energy flux between this and this is the luminosity.

You can calculate the difference in luminosity. The luminosity is a total radiation that is coming out from this surface. So the difference in the total radiation coming out from these 2 surfaces that is the energy which is generated in the shell and the energy that is generated in the shell we write as the volume of this shell. So the volume of this shell is 4 pi R square * delta R * the density that will give you the mass inside this shell.

So into the density rho R * the energy generation rate or the emissivity of this material so epsilon R. So that is the amount of energy generated per unit mass of this material. So this gives us the equation that DL/DR how does the luminosity change as we go out from the sun and DL/DR = 4 pi R square rho R * epsilon R where epsilon R is the emissivity or the energy generation rate.

And this is something that we discussed just in the last class in the previous lecture where we discuss nuclear burning. So I told you that the energy generation rate is proportional to rho. So epsilon this energy generation rate is proportional to the density *T to the power 4 for the

PP chain. So the energy generation mechanism in any star is through nuclear burning that is the only source of energy generation.

And we saw that there are 2 processes by which this nuclear burning can occur. One is the PP chain where 2 hydrogen 2 protons collide to give you helium 4 and the energy generation rate is proportional to the density *T to the power 4 okay this is the rate per unit mass that is proportional to density to the power 4. There was another mechanism which was a CNO cycle and for the CNO cycle.

This was proportional to rho T to the power 15 I told you this also. So when you solve for the stellar structure one has to also take into account the fact that there is energy being generated in different parts of the stars. The energy generation rate is more effective in the center where the density is high and the temperature is high. It is less effective in the outer part where the temperature is low and the density is also low.

So one has to simultaneously incorporate this and that goes into determining how the luminosity changes as we go away from the center that goes into here. And we have written this as a function of R because both the density and the temperature are just function of R. In reality, it is essentially if you look at the physics behind it is a function of the density and the temperature.

But since the density and the temperature are themselves functions of R when you solve the stellar structure you have to essentially can think of it as a function of R. So that is the third equation.

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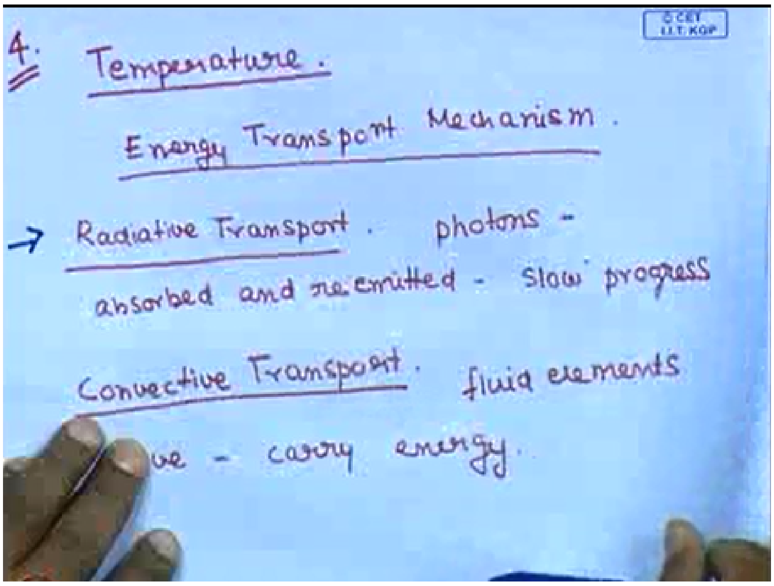

The fourth equation is the temperature so we need to know the temperature profile. So how does the temperature change inside the sun and this how the temperature changes is essentially determined by the energy transport mechanism. So this is determined by the energy transport mechanism and there are 2 energy transport mechanisms that are active inside stars.

The first one is radiation so you can have radiative transport. So in radiative transport what happens is that the photons they get absorbed and reemitted and they slowly make their way across the sun and they diffuse. So this is a diffusion process. It is a random walk we have discuss this. So the photons slowly make their way it is a slow progress through the sun. This is the radiative transport.

There is another mechanism which can also at play inside the stars the other one is convective transport. In convective transport fluid elements move and these fluid elements carry the energy and so the energy transport on the temperature distribution in the sun, the temperature gradient are determined by which of these 2 processes is active it depends on that and you can have 2 possibilities.

So the radiation you can have energy being transported through radiation or you could have energy being transported by the motion of fluid elements themselves. So there is a competition between these 2 mechanism and we shall discuss the criteria which one will be operational, but first let us just look at this. So in today lecture we shall just consider the radiative transport.

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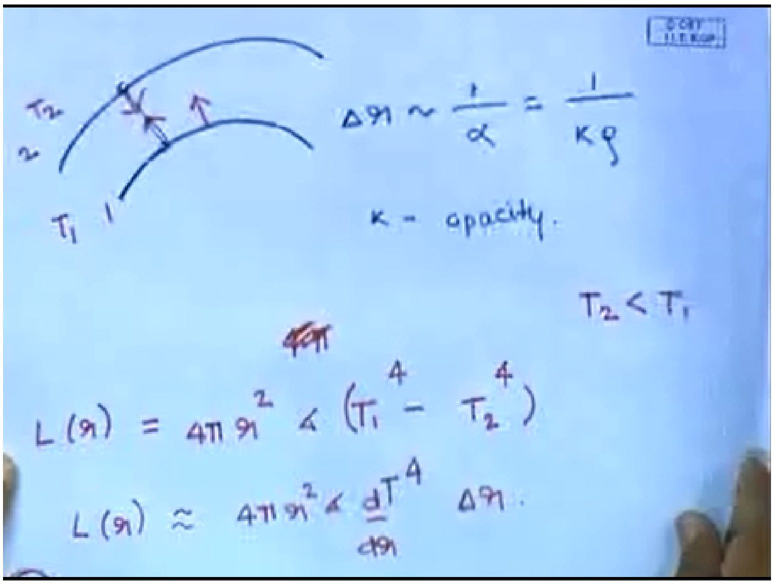

So we proceed as follows. Let us consider 2 shell of the fluid the 2 different radii distance delta R apart where delta R is the mean free path of the photon inside the sun and the mean free path we know the mean free path in any medium. The mean free path we know is 1/absorption coefficient alpha or we can write it as 1 by the absorptivity or opacity into the density.

So this is the absorption coefficient per unit volume that is how we had introduced it alpha and Kappa is per unit mass. Kappa is also called the opacity and it is related to this alpha which we have been using and which is the absorption coefficient through the density. So let us consider 2 regions, 2 shells a shell of thickness delta R which is of the order of the mean free path inside the star.

So you can think of the photon essentially propagating from here to here and then getting absorbed or scattered. Now let us calculate the energy flux which is incident which is being radiated by this surface. The energy flux being radiated out from this surface is we will call this 1, we will call this 2. The temperature here is T1, the temperature here is T2. So the energy flux that is being radiated out is 4 pi flux is going to be sigma T1 to the power 4 that is the flux.

Per unit area how much energy is going outward that is 4 sigma T1 to the power 4. The energy of flux which is incident in this direction from the upper surface is going to be T2 to the power 4. So the difference of these 2 where $T2<T1$ we know this. As you go out the

temperature goes down. So the difference in these 2 gives me the net outward flux which is the luminosity.

So if I multiply this with the area 4 pi R square I get the luminosity at a distance R. So the photon which leaves from here propagates only this distance typically and then get absorbed again. The photon that leaves from here it propagates this distance and then get absorbed again and the net flux over here, net luminosity over here is the difference in flux into the area which is 4 pi R square*sigma the Stefan–Boltzmann constant T1 to the power 4 –T2 to the power 4 and T1> T2.

So this is an estimate of the flux net luminosity at any point. So we can write this as approximately =4 pi R square * sigma*the derivative of T2 to the power 4 * delta R that will be an estimate of this quantity and the mean free path we know is much smaller than the radius of the sun is very small is of the order of millimeter so this gives us the luminosity and we can invert this relation.

So let me invert this relation to get the temperature gradient inside the star.

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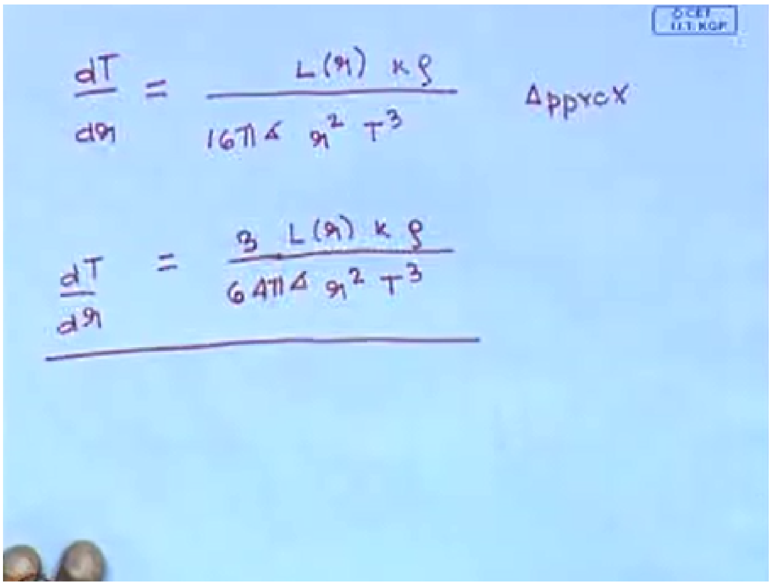

So the temperature gradient inside the star is equal to so what we are doing is we are differentiating this and then taking all the factors on to that side. So this will be equal to the luminosity at that point divided by I will get a factor of 4 when I differentiate this. So I will have 16 pi R square the Stefan–Boltzmann constant T cube. So let me write it like this 16 pi sigma R square T cube.

And then I have 1/delta R and delta R we know is the mean free path which is the 1 / Kappa rho. So we have 1/delta R so we have to multiply with kappa and rho. This gives the temperature gradient in the star. This is an approximate formula that we have derived. All calculation here has been rather approximate if you do a more detailed exact calculation what you get, this is approximate.

If you do a more detailed calculation what you get instead of 1/16 you get 3/64 pi sigma R square T cube L R kappa rho. Okay this is the temperature gradient. So this completes the 4equations which one has to solve in order to determine the structure inside the star. So let me just tell you what this equation tell us. This equation essentially tells us he can look at this equation.

So if you want to have a higher luminosity at a certain point in the star. So you want the luminosity here to be high then the temperature gradient over here should also be large or given the same luminosity if the opacity increases then again the temperature gradient has to be larger for the same luminosity at this point. If you have a higher opacity, then the temperature gradient has to be larger.

Let me now summarize what we have done to determine these structure of a star one has to simultaneously solve these 4 equations for the material inside the star. So we today discussed only one energy transport mechanism in the next class we shall discuss about convection and then also discuss how one goes about solving the stellar structure.