Astrophysics & Cosmology
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Lecture – 11
Radiative Transfer (Contd.)

Welcome, we have been discussing radiation for the last 2 classes and we started off by learning that it is an electromagnetic wave and if you are dealing with length scales that are much larger than the wavelength you need not bother about the fact that it is a wave, we can treat it as rays and the energy that is carried by a set of rays is quantified using the specific intensity I knew.

And we discussed various properties of this specific intensity in the first class. In the second class; in the last class, we discussed what happens when radiation propagates through a medium, so this is a phenomenological description we had one term that describes the emission and another term that describes the absorption and then we found out the general solution and a behaviour of that solution; the general behaviour of that solution.

In today's class, we are going to discuss thermal radiation, so what is thermal radiation? Thermal radiation is radiation that is emitted from material that is in thermal equilibrium, so the material is in thermal equilibrium and radiation is emitted from such a material then it is called thermal radiation. Now there is a flaw in the definition itself right in the start because if a material is in thermal equilibrium if it radiates, then it is losing some energy, right.

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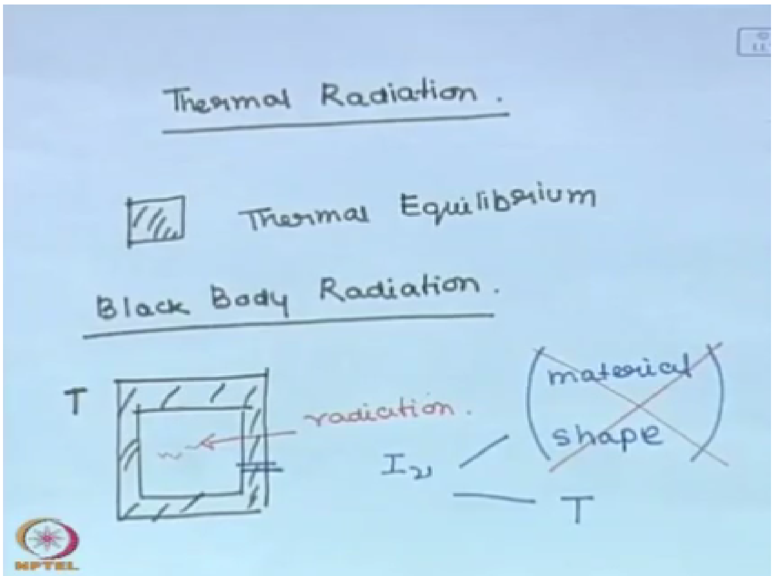

So, we assume that the radiation; the energy that is lost is either compensated or it is not significant compared to the energy in the media, okay. So, we shall be discussing thermal and the radiation here is emitted from some material, which is in thermal equilibrium, so the material is in; that is the material and it is in thermal equilibrium, okay and we start our discussion with a specific situation that is called blackbody radiation.

So, right at the start we are going to discuss; what is called black body radiation? So, let me first explain what we mean by a black body cavity? This refers; so just imagine a cavity like this and these are the walls of the cavity and these are the walls enclose a cavity and there is radiation inside the cavity so there is radiation here and the cavity is assumed to be at a temperature T, we maintain it at a temperature T.

And we wait sufficiently long, so that the radiation inside interacts successively with the; keeps on interacting with the walls of the cavity and through repeated interaction, the radiation inside is comes to equilibrium with the walls of the cavity outside, okay. So, the radiation inside this cavity is now in thermal equilibrium; it is in equilibrium with the walls of the cavity, okay. Such radiation that is in equilibrium with material at a temperature T is said to be blackbody is referred to as a blackbody radiation; this is a black body cavity.

The walls can absorb and emit radiation at all frequencies that is why it is called black; a black body, okay. So, the radiation inside is in thermal equilibrium with this material outside the material outside is in thermal equilibrium and a temperature T and the radiation after repeated interaction with the material is also now in thermal equilibrium with the surrounding with the walls of the cavity, then this radiation is said to be referred to as blackbody radiation.

So, this is radiation that has come to equilibrium with matter and a temperature T okay. Now, what we do is; we make a small hole somewhere here, let us say the hole is not; hole is just essential for studying the radiation, so let us imagine that we make a very small hole through which some of the radiation comes out, okay without disturbing the equilibrium inside. So, the hole, we are assuming is so small.

So, the amount of radiation that comes out, you are assuming is so small that it does not disturb the radiation; the equilibrium inside, okay. So, this radiation that comes out will have some specific intensity; Inu, right, so this is the specific intensity of the blackbody radiation. Now, the first thing that we are going to see is that the specific intensity of blackbody radiation does not depend on the material that makes up the blackbody.

Now, either does it depend on the shape, so this does not depend on the material that make up; makes up the walls of the cavity neither does it depend on the shape or any other property of the cavity, it does not depend, okay, so it does not depend on these, okay. It is independent of all of these things; it depends only on the temperature of the cavity; that is what we are going to see. The only property of the cavity that the specific intensity of this blackbody radiation depends on is the temperature.

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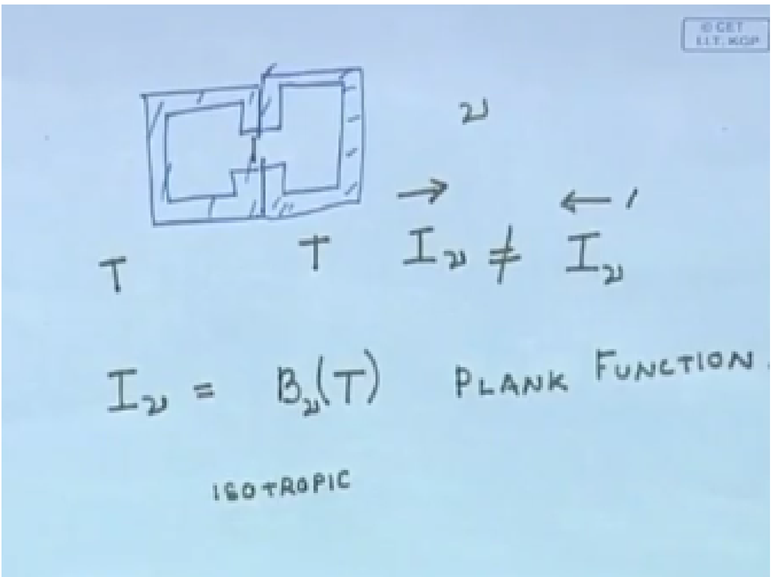

So, this is what we are going to see now and how do you arrive at this conclusion; to arrive at this conclusion what you do is; you consider 2 cavities. So, let us draw 2 cavities; 2 black body cavities and whose shapes could be different, the material that makes up these 2 different cavities could be different and we assume that there is a filter in between which only lets light of a certain frequency nu pass through.

So, there is a; there are 2 such cavities and there are holes in these 2 cavities, which are like this and the shapes of these 2 cavities could be different, the materials making up this cavity and this cavity could be different but they are linked together through a hole like this and we assume that there is a filter here that only lets light of a particular frequency nu pass through, okay.

Now, the amount of radiation that goes from the left to the right in any specific direction is going to be I nu and that is the radiation that goes from the left to the right and the radiation that comes from the right to the left in the same direction, let us say it is specific intensity is I nu prime and the same frequency. Now, if these 2 things are different then there would be net energy transfer from one cavity to another.

But both the cavities, we have assumed to be at the same temperature this is something I forgot to mention, both the cavities are at the same temperature, so this radiation is in thermal equilibrium and is at a temperature T, so is this radiation, this is at the temperature T this 2 is at a sub temperature T. So, if the 2 specific intensities are different, it would mean that energy would flow from one cavity to another.

Now, that would violate the laws of thermodynamics right. There are 2 systems at the same temperature, I cannot have heat flowing energy flowing from one to the other, right. They would be in equilibrium, so this is not true, so it essentially tells us that I nu is a universal function; is just a universal function of the temperature alone, it cannot be different in these 2 cavities, okay.

And this universal function of the temperature we denote by B and this is the Planck function, it is called the Planck function, so this is an universal function of the temperature and it is isotropic, we also see from this, okay. So, the blackbody radiation and specific intensity depends only on the temperature of the cavity nothing else and it is isotropic, it is the same in all directions.

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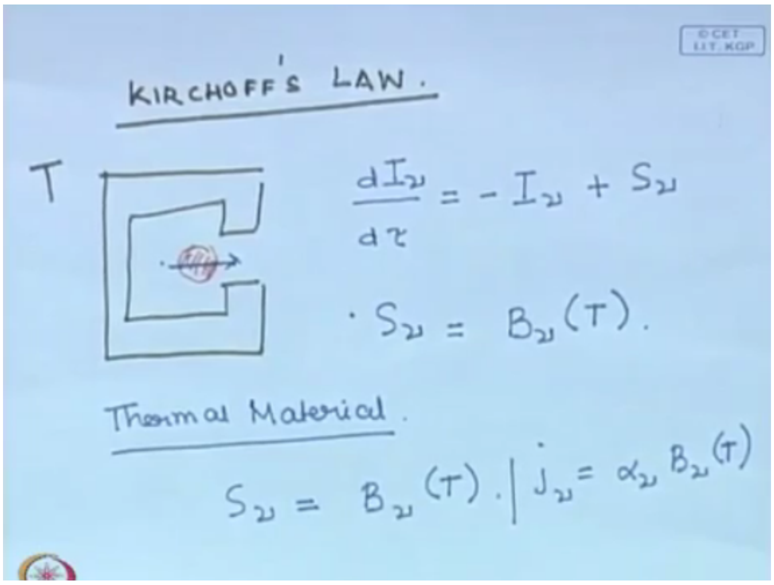

So, that is one property of the blackbody radiation, which we can arrive at just from your thermodynamic considerations, otherwise it would violate the laws of thermodynamics, okay. So, that is the first property. Let us, now derive another property; this is of the thermal radiation this is called the Kirchoffs or Kirchoff law, okay. So, let us again consider a black body cavity and a temperature T.

And let us assume that we have put some material also at some temperature T; at the same temperature T over here, so this material that we have introduced here is at the same temperature as the cavity and the radiation inside okay and let us write down the radiative transfer equation through this material, so the radiative transfer equation we see is dI nu, d tau = - I nu+S nu.

Now, note that even though we have introduced this material at a temperature T it still remains to be a black body cavity because this is at the same temperature as the rest of the wall okay, so this again is a black body cavity and the incident radiation on the left hand side, we know that it is the Planck function that it is inside the black body cavity, so the incident radiation we know is the Planck function, now what comes out must also be the Planck function.

Because this entire thing is still continues to be a black body cavity, so from this we can say that the source function for this; the source function for this material must be = the Planck function, right. If it were more than the Planck function, then the specific intensity would increase when the radiation passed through this and if it were less than the specific intensity would decrease but introducing such a material this still continues to be a black body cavity.

So, the specific intensity cannot change, it must continue to be the Planck function, so it tells us that the source function for a thermal; this is a thermal medium, thermal okay is there a temperature T. So, for any thermal emission thermal radiation, for some thermal material, material in thermal equilibrium; this is a material in thermal equilibrium at temperature T. The source function = the Planck function.

Or it gives us a relation between the emission coefficient and the absorption coefficient; we have seen that the source function is the ratio of the emission coefficient to the absorption coefficient. So, it tells us that j nu should be = alpha nu * B nu T, that is a property of thermal; for thermal material, right. So, what have we learned till now? We have learned essentially 2 things.

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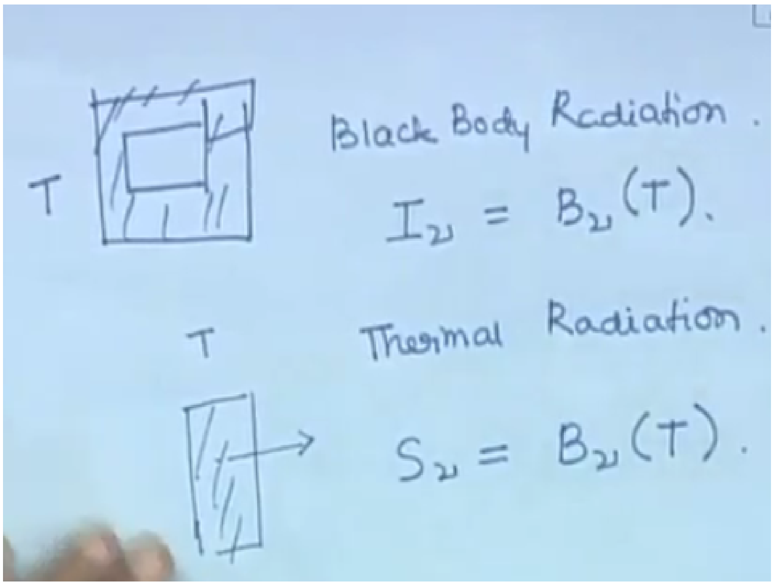

So, we have 2 concepts now, one concept is as follows. Let me just briefly summarize what we have learnt. One concept is that of a black body radiation; let me forget about the wall hole in the cavity and the black body radiation is a radiation that is in thermal equilibrium, so this is called black body radiation, the radiation here has come to thermal equilibrium with material at thermal equilibrium at some temperature T matter it.

And we saw that the specific intensity of this radiation is the Planck function, which depends only on temperature universal and that is the one thing. The second thing is thermal radiation, which is different from blackbody radiation; which could be different, so thermal radiation we have some material here, which is at a temperature T in thermal equilibrium at a temperature T could be a gas at temperature T, could be some material okay a piece of glass at temperature T whatever.

And radiation that originates from this will have a source function which is B nu, which is the Planck function but the radiation need not have come to thermal equilibrium with the material okay, the source function is the Planck function. The radiation itself need not have come to thermal equilibrium with the material, if it comes to thermal equilibrium with the material, then the radiation also; the specific intensity of the radiation also becomes the Planck function.

But this is a more general situation, so here the source function of the material is the Planck function such a radiation is called thermal radiation, okay and let me ask you, what is the condition? So, what is the condition that this should; the specific intensity should become equal to the source function, we saw it in the last class when discussing the radiative transfer equation we collect what we had considered this particular problem, right.

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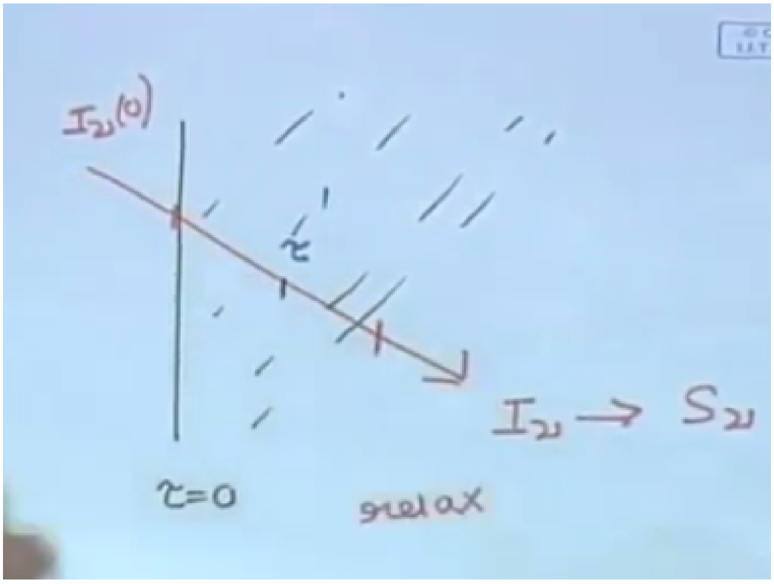

So, we had considered this particular problem in the last class where there is radiation propagating through a medium which has the same source function everywhere, the incident radiation could have some specific intensity I nu.

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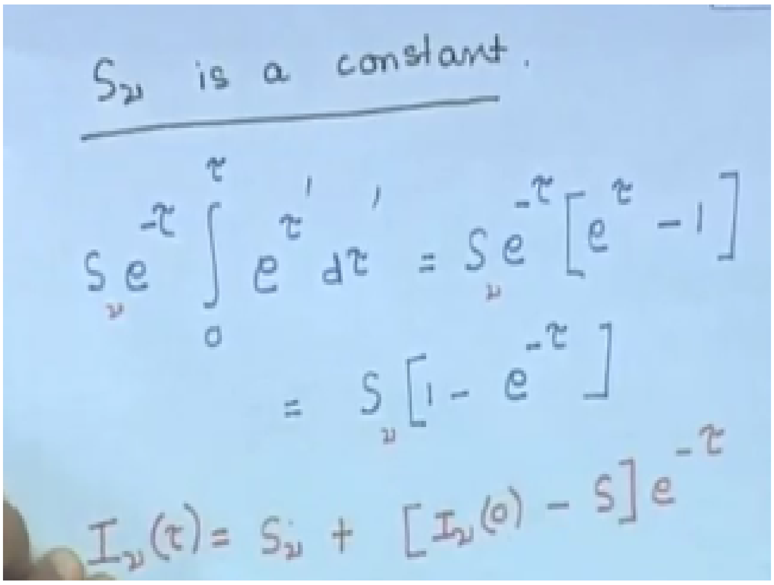

As it propagates inside, the specific intensity changes and inside the material, the specific intensity is the source function plus this term. Now, as the optical depth tends to vary becomes larger and larger what happens? This specific intensity tends to the source function okay and finally given adequate optical depth, this becomes equal to the source function, so if this is thermal medium, a thermal material any radiation that is incident inside on this provided it propagates for an adequate amount of optical depth in this material.

It will finally become the; it will come to equilibrium and it will become the Planck function, it will become blackbody spectrum basically blackbody radiation okay, that is an important lesson which has to be borne in mind, right. Now, let us derive a few properties of the blackbody radiation of the Planck function essentially or blackbody radiation starting from pure thermodynamic principles, okay.

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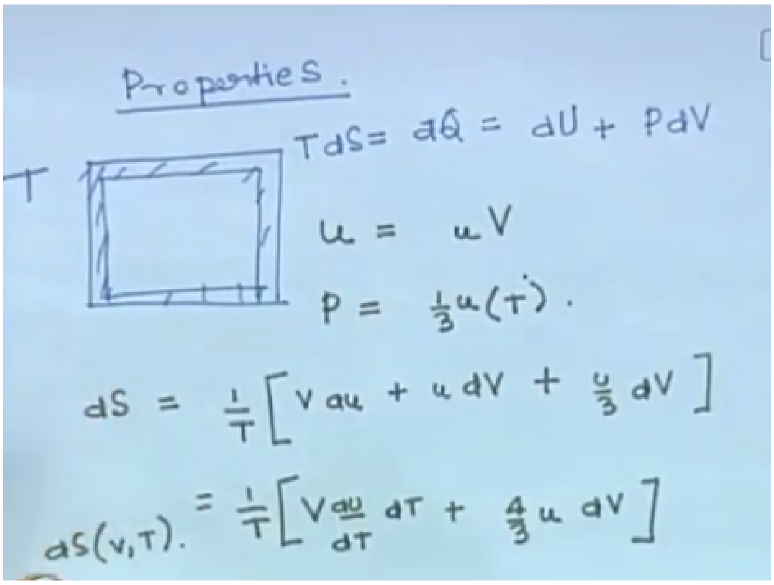

So, we are going to start from pure therm; we are dealing with black body radiation and we are going to start from thermodynamic principles. So, let me draw the picture again we are drawing the same picture quite a few times, so properties of the blackbody radiation and these are properties that can be derived purely from thermodynamic consideration. So, this is a black body cavity, let me draw it again at a temperature T.

So, there are the walls of the cavity and there is radiation inside; we are dealing with the radiation inside. Now, the radiation inside this cavity can be treated just like any other thermo dynamical system, right, it is a thermo dynamical system, it is a thermal equilibrium, so we can treat it just as any other thermo dynamical system and we know from the first law of thermodynamics, that the heat that is gained by the system is the internal energy.

So, the heat; see the internal energy goes up, the heat has to; either heat has to flow into the system or there has to be work done on the system, that is the first law of thermodynamics okay and we know that this is not a perfect differential but the entropy we can write this as TdS and the entropy is a state function okay, so we can write it the first law in this way. Now, for the blackbody radiation, we have seen; what are the properties of the blackbody radiation that we already know.

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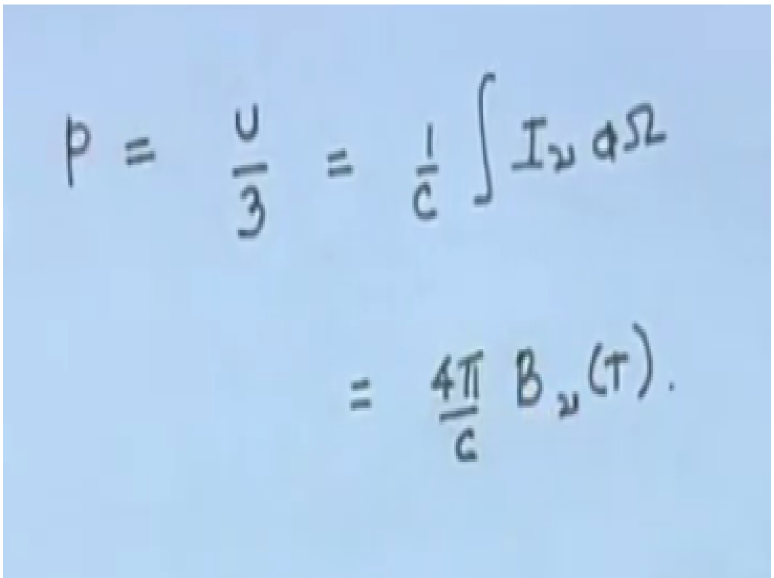

So, there is one property okay a property which I should have mentioned but I did not the fact that this radiation is isotropic, for any radiation we saw that the energy, let me do it here; for any radiation, the pressure we saw the pressure = the internal energy density of the radiation divided by 3 and we have seen this right in the beginning in the first class lecture on radiation.

The pressure of the radiation is the energy density divided by 3 okay. So, the total and internal energy of the system; we can write the internal energy density into the volume of the system and the pressure we can write as 1/3 the internal energy density, this we have derived in the first lecture and this we are writing the total internal energy in terms of the density into the volume.

So, using this, we see that the differential of the entropy is 1 by the temperature into; then we have 2 terms that come from this, the internal energy can go up due to 2 reasons; one is if the density of internal energy goes up, so I could have Vdu+ udV + the entropy could also increase due to this increase in the volume and the pressure acting on that, so that will be u/3 dV.

And another thing that I should mention, so there are 2 things that I should mention the second thing is that the total internal energy density; the total internal energy density, so the energy density of radiation is equal to this also we have derived; it is equal to the integral of the specific intensity over all solid angle all directions. This we have to divide by c, right and so here since the blackbody radiation it is isotropic.

So, we know that this is = 4pi/c * B nu T right. The specific intensity of blackbody radiation; for blackbody radiation this is the Planck function, so we know that we have this relation. So, what we see is that the pressure and the energy density of this Planckof this blackbody radiation both depends only on the temperature, so both of these, so the internal energy the energy density here u is just a function of temperature alone.

The pressure 2 is just a function of temperature alone okay; all of these things depend only on the temperature. So, from this we can write this as 1/T, right. Now, the entropy we know is a function of; we can write the entropy as a function of any 2 of the variables PV or T, so we write it as a function of V and T right, so from this what we have is that the derivative of the entropy with respect to temperature at a fixed volume; derivative of the entropy with respect to the temperature at a fixed volume, we can read off now that is V/uT * du dT.

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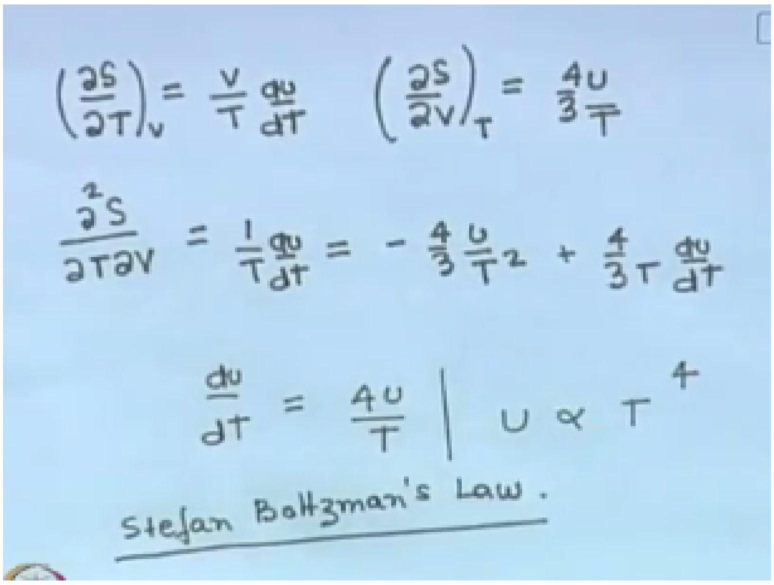

Similarly, the derivative of the entropy with respect to volume at a fixed temperature is 4 third u. Now, if I differentiate this with respect to V and if I differentiate this at a fixed temperature if I differentiate this with respect to T at a fixed volume then the resultant should be equal right because the resultant then should be del square S del T del V. So, we are going to differentiate this with respect to V.

And if I differentiate this with respect to V at a fixed temperature, what I get is that; this is = 1/T du dT; Sorry, this should be 4/3u /T and we are going to differentiate this with respect to temperature, so it should be equal, right. The order of the derivative does not matter, so if I differentiate this with respect to volume it should be equal to this differentiated with respect to temperature.

And what we have if I differentiate this with respect to temperature is -4/3u /T square+4/3 1/ T du dT okay, so we can combine these 2 terms take this onto the left hand side and what we have is that; no not these 2 terms sorry, these 2 terms, we can combine these 2 terms take this onto the left hand side and what we have then is that du dT = 4times u/T, right. We have just taken this onto the left hand side, so 1/3 just remains.

And here, I have 3/4, both pickup minus sign, so the 3 cancels out what we are left with is this, which tells us that U; the energy density is proportional to T to the power 4 at the energy density of this radiation of this blackbody radiation is proportional to the temperature to the power 4, from purely thermodynamic considerations okay and this is called the Stefan Boltzmann law basically.

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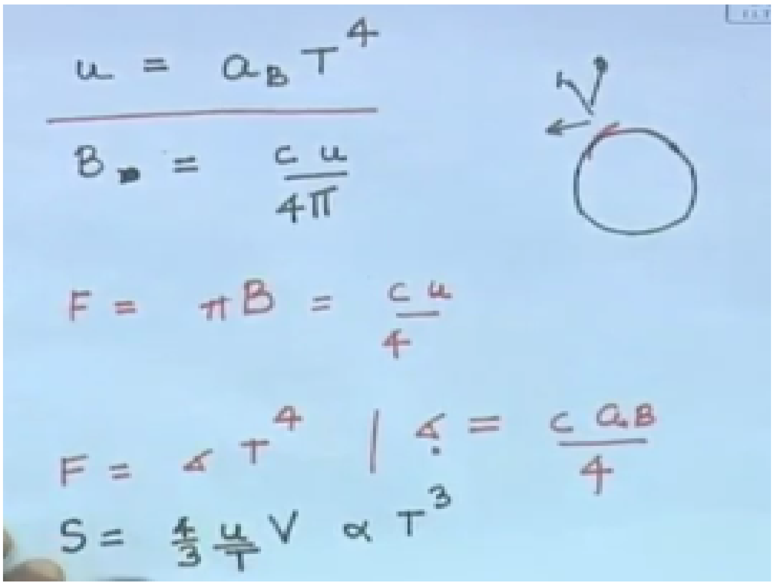

Well, this is not exactly what is referred to as Stefan Boltzmanns law but we can derive it I will just show you how to derive it just now. So, let us write this internal energy the energy density of this radiation as aB, T to the power 4, where aB is a constant; the constant of proportionality that is the Stefans constant, whose values we cannot determine from these arguments okay, just tells us that the energy density is proportional to T to the power 4, right.

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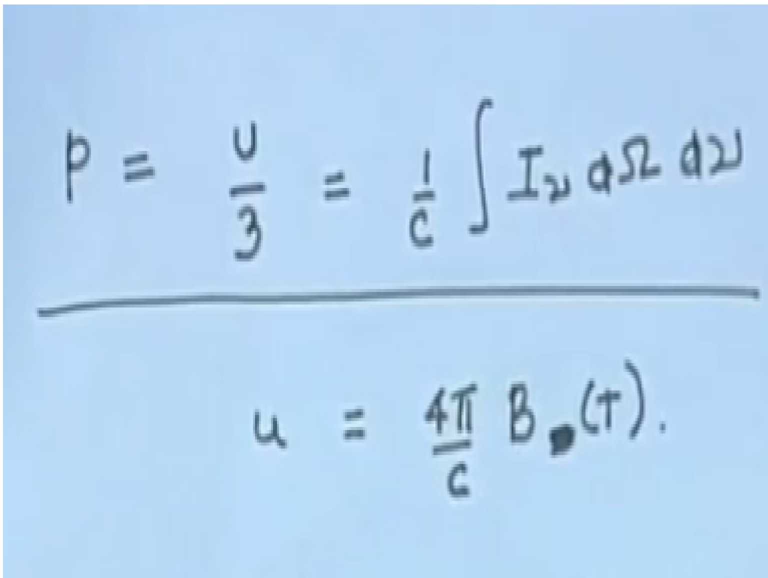

It also tells us so once you know the energy density you know this, so you can work out what this is going to be, so this will be; no this is U, this is not the pressure okay. So, the U = this; the energy density = this; 4pi/ C the pressure is 1/3u okay, so what it tells us is that the specific intensity or the brightness, the u; the frequency will not be there this will not be there it is integrated over all frequency, right.

So, there is a frequency integral here also, so what it tells us is that the integrated bright over brightness; the integrated brightness is specific intensity = cu/ 4pi okay, so this is the specific intensity of this totally integrated overall all frequencies of this radiation of this blackbody radiation. Here we have not the specific intensity; the intensity okay, integrated overall frequencies.

Now, we had also remember considered a problem where we had a spherical surface emitting in isotropically in all directions with the brightness B and we had calculated the flux at different distances from this and we had seen that at the surface, the flux was = pi*B and that is easy to understand you have to just integrate over half basically okay, so this was what we had seen that the flux = pi * B, so the flux = c times u/4.

So, this is another; this is actually what is called the Stefan-Boltzmann law, the flux of radiation from any surface the surface of a blackbody at a temperature T = sigma T to the power 4, where this sigma = c aB/4 and this is called the Stefan-Boltzmann constant and this is referred to often as the Stefans constant okay but the values of these constants cannot be determined from these thermodynamic arguments, okay.

What we can determine is the fact that the radiation behaves like this and you can then do experiments and determine the value of these constants from the thermodynamic from this thermodynamic approach okay. We can also calculate the entropy density, once you know the or the total entropy because once you know the internal energy as a function of temperature, you can also determine the entropy as a function of temperature okay.

And the entropy density comes out to be; so the total entropy in the system; let me write it, the entropy in the system I leave it for you as an exercise, the entropy = 4/3 energy density/ T into the volume okay. The entropy is 4/3 the energy density by the temperature into the volume, so this is proportional to T cube, right because it is the energy density by the temperature, so the entropy goes as T cube and that energy density goes as T4.

And the constant for the entropy is this or if you want the entropy density then you can just ignore the volume okay, so what have we done till now let me briefly recapitulate before we move on to more detailed discussion. So, until now we have first defined what is thermal radiation? What is blackbody radiation? And then we determined certain properties of the blackbody radiation.

Its specific intensity is a function of temperature alone, it is isotropic. The source function for thermal radiation is the Planck function, which is also the specific intensity of blackbody radiation and then we showed that the total energy density of blackbody radiation is proportional to T to the power 4 okay. Another one or 2 more very, okay useful things before we finish.

So, let us now consider, so we have that is what we have shown before we finish this particular topic, let us now consider the situation where I make; I do some adiabatic transformation on this black body cavity okay. Let us consider a situation, where suppose I make the cavity expand adiabatically, what do you mean by adiabatically? There is no exchange of heat and dQ is 0, so the entropy is conserved okay.

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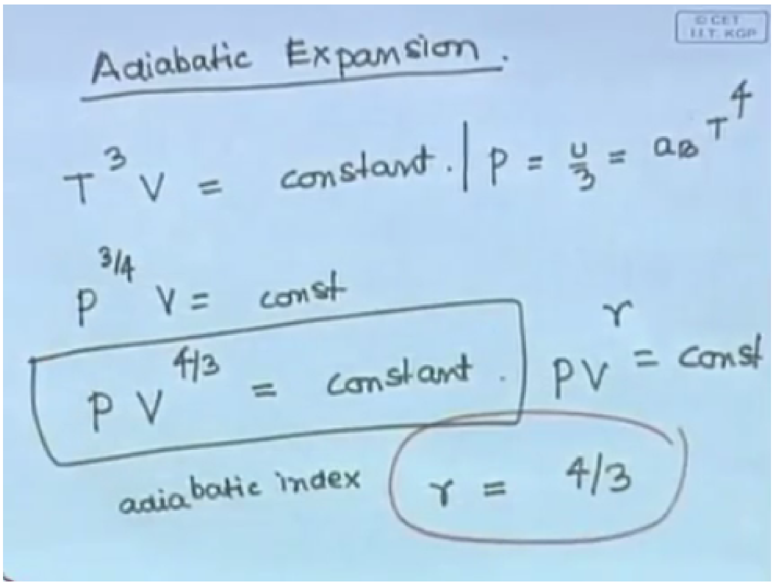

If the entropy is conserved, so under adiabatic expansion, the entropy is conserved and entropy I just told you and you can easily verify is; 4/3u by the temperature; the energy density by the temperature into the volume, so what it tells us is that T cube*V is a constant right and we also know that okay so it is convenient to write this relation in terms of the pressure and the volume okay so this is one relation.

This is also convenient quite often to write it in terms of the pressure and the volume. Now, we know that the pressure is u/3 and we know that this is proportional to T to the power 4, so we see that this relation can be written as P to the power 3/4 * volume is a constant or P volume to the power 4/3 is a constant okay, this is very convenient. Now, we have something called the adiabatic index which you have possibly encountered P V to the power gamma is a constant.

So, here the adiabatic index gamma is 4/3 okay, this is a thermodynamic medium where the adiabatic index is 4/3 okay and this is quite a useful thing that we shall be using quite a bit in this course okay. So, this more or less completes the macroscopic description of thermodynamic description of radiation this is as far as you can possibly get. Now, we shall move on to a microscopic description; the brief idea of the microscopic description of the blackbody radiation.

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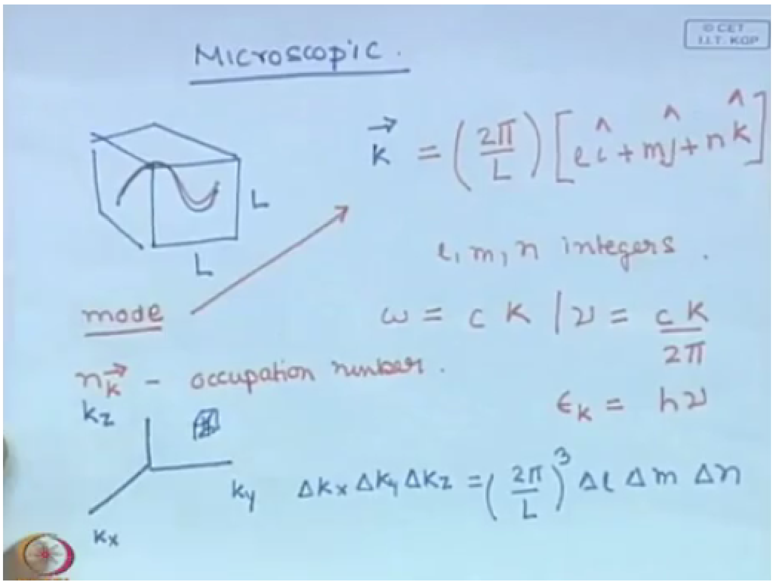

And how you can determine these quantities, the Stefan constant aB, which tells you how much is the energy density or how you can determine sigma, the Stefan-Boltzmann constant okay. So, for this you require a microscopic description so let us look at; briefly look at what some aspects of this, so we are going to look at this so microscopic; so we let us start off by considering a volume; a cube like this of side L; all 3 sides are L okay.

So, a volume is L cube and we assume that there are electromagnetic waves inside this cube. And we will assume periodic boundary conditions okay, so if you have periodic boundary conditions then the waves that you can have; the wave number of the waves that you can have is going to be restricted, right and it will have to be such that one entire wave fits into this box, so this is one possibility or you could have 2 wavelengths fitting into the box etc.

So what you can say is that the wave vector for all possible electromagnetic waves inside this can be written as 2pi/ L into; let us say li+mj+ plus nk, where l, m and n are integers, okay. So, the possible wave vectors for the electromagnetic waves that you can have inside this box are not unrestricted and they are of the form that the wave vector should be 2pi/ L, l*i+m*j+n*k, where l, m and n are integers okay and the frequency of the angular frequency we know omega = for electromagnetic waves in vacuum, it is the speed of flight into k or we can straight away write down the frequency which is of interest to us.

So, this is going to be ck/2pi, so each such k is we are going to refer to it as a mode of the electromagnetic radiation okay, so this is one mode one set of integers is one mode of the electromagnetic radiation okay. So, these are the possible modes of the electromagnetic wave that are allowed inside such a cubic box okay and corresponding frequency is here and we know that these are photons.

So, each mode can have number of photons and the number of photons in each mode we will denote by nk that is the occupation number of that mode okay which is a microscopic relatively microscopic thing here, so this tells us how many photons are there in any particular mode. So, the entire electromagnetic radiation in this box can be decomposed into different modes and nk is the occupation number; the number of photons in each mode.

And it will be different for each value of k okay, further the energy associated with any mode let us call it epsilon k = h with one photon is h nu, where nu can be calculated for that mode like this right, so the way we imagine you think of the electromagnetic radiation in this box; the electromagnetic radiation in this box can be decomposed into modes, each mode will have some number of photons that is the occupation number given by the occupation number.

Each of these photons will have some energy okay, so the total energy of electromagnetic radiation in this box is to be thought of like this. I hope this picture is clear now let us look at the K; the direction k vectors the k space okay, so this is the wave vector space kx, ky, kz and let us ask the question how many modes are there in some volume element delta kx, delta ky, delta kz, in this volume element how many modes do we have?

And it is quite clear that this will be = 2pi/L cube delta L, other way round actually we want the number, so we have this relation that delta kx, delta ky, delta kz will be delta l*delta m*delta n right, if I just increase kx only this is going to increase, if I just increase ky only this will increase, if I increase just kz only this will increase. So, if I want to count the number of modes, I have to take the increase in this into the increase in this into the increase in this.

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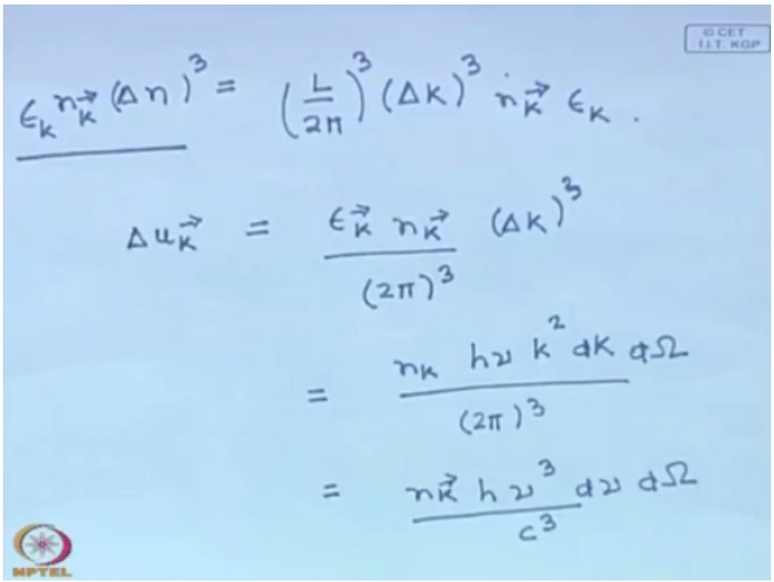

The product will give me the total number of modes right, so what I can say is that delta; the number of modes is going to be; let me write down the number of modes, so that is going to be, so the number of modes in this volume element in this space of wave vectors that this is the volume element in the space of wave vectors delta K cube right, it is delta kx, delta ky, delta kz, if I divide this by multiply this by L cube and divide by 2pi cube, this gives me the number of modes that are there in this volume element in the wave vector space okay.

That is essentially delta l*delta n*delta m right, so this is the number of modes. Now, I want to calculate the number of photons, so I would like to calculate the number of photons in this range of delta k. How many photons are there in this range of delta k? okay, so what I have to do is; I have to multiply this with the occupation number of the modes that are there in that volume.

And if I want to calculate the energy; the total energy of the photons the modes that are there in this volume of dQk, then I have to multiply it with another factor of epsilon k, the energy of each photon right, so epsilon k, this gives me the energy of; this is the energy contained in the photons that lie in that region of the k space okay and we are interested in the energy density of the photons that lie in that region of k space.

So, if you are interested in the energy density then what you have to do is you can just divide by this L cube okay and the energy density of photons that have that lie in that region of the k space, you can write now as, let me write it here, so delta e; delta let us say, uk = the energy density that = nk, that is the occupation number the energy we know, so let me write it here.

So, it is the energy into the occupation number of those modes into the volume occupied cube by 2pi cube okay right. Now, it is convenient to work in spherical polar coordinates, so think of these wave vectors; these wave vectors in spherical polar coordinate, I can write in terms of the solid angle subtended by these wave vectors d omega*k square dk, so I can write this delta k cube as k square dk.

The energy of each photon, I can write as H nu, I have the occupation number divided by 2pi cube * d omega, the solid angle and so the wave vectors I have written in terms of the magnitude and direction and the direction I have decomposed into solid angle like this okay. Now, we know; we want to write everything in terms of frequency, so let me write it in terms of frequency.

In terms of frequency, so I can replace k/2pi nu/ c, so the k can be replaced by 2pi nu/c, so if I do that I will have 2pi cube coming from this, so that is gone and so this will become the occupation number into h nu cube d nu d omega/c cube okay, so what have we calculated? We have calculated the energy density of the radiation that is propagating in this frequency interval d nu that is propagating in this per unit volume that the energy density that is propagating in this range of directions d omega.

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$$u_\nu(\Omega) = 2\frac{h\nu^3}{c^3} n_{\vec{k}}$$

$$I_\nu = 2\frac{h\nu^3}{c^2} n_{\vec{k}}$$

And in the frequency interval d nu, so from here, we can straight away identify that we had defined this specific energy density omega remember this is equal to and the specific intensity Inu =; so you have to just multiply it with c, so the specific intensity is going to be h nu cube/ c square nk, okay right. So, we have seen this is essentially the relation between the specific intensity and the photon occupation number okay.

This relates the number of photons that are propagating in a certain direction with a certain wave vector to the specific intensity of that ray in that direction with the corresponding frequency okay, so this is a specific intensity of the light in the same direction as this wave vector, the same frequency corresponding to this wave vector. They are related like this okay and you can put in a factor of 2.

To account for 2 polarizations if you wish assuming that the light is not polarized okay. So, let me stop over here for today and we shall continue on this; on this microscopic approach and see how we can derive properties of the blackbody radiation from this.