Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Week - 02

Lecture – 09

So, we will go to the next equation, we have lots of equations, but Poisson, Hamels and wave, I finished three. Now diffusion equation is, I am sure everybody knows this, dt pi equal to Laplacian, I am putting the diffusion coefficient to be one, we choose like that. But I want to diffusion equation with a source, so we put a source here. So, this is a heater, in fact this is a good example, we have heater in our room and I look at the temperature in the room. So, this is a heater and phi is temperature. Now heater, imagine to be a point heater, so somebody just heated at t equal to 0, basically flare, one flare came and disappeared, it, it x prime at time t prime.

So, this flare will basically conduct heat and we are looking for the solution. This is a straight forward problem. So, I again go to Fourier space, it is nice to work with Fourier space. So, I will have both omega and k.

So, you agree with this, the solution of the G k omega, where d by dt will give us minus i omega. So, I replace this by e to the power i k dot r minus i omega tau actually, tau equal to t minus t prime and r equal to x minus x prime, these are change of variable. So, I do not know how to carry the x prime t prime, these two nonsensical. So, if I do the d by dt, I will get minus omega, you know, this one. Yes, so these make Green's function.

For wave equation it was minus omega square, but here is minus i omega, is a minus i omega. So, please keep in mind, it is not plus i omega. So, let us do the time integral, d omega integral first. So, where is the pole? So, d omega integral here, the pole is i omega equal to k square, omega equal to minus i k square. So, we can put the pole, its negative axis, this one, minus omega, minus i k square, sorry, and this omega real, e minus 0 omega.

For this integral, which side should I close the semicircle? Opposite, upper or lower? Tau positive, so let us assume tau to be positive. So, tau positive, Jordan Lemma, lower, lower you should lower. So, I will get contribution from the pole. In fact, just straight forward, I have to just make slight change in this omega. So, I need to write omega minus something, omega minus, remember the integral is z minus z naught integral, f z.

So, always you should make z as my, so this minus of omega has to pulled out. So, I leave this part, it is exercise, I mean you just fill the steps. So, G k tau, for tau greater than 0 is minus, exponential minus nu k square tau, tau remember is t minus t prime, is exponentially decreasing in time, tau greater than 0, for tau less than 0 it is 0, tau less than 0 I need to close from the top, and then it has no contribution from the pole, so I get 0. So, this is my g k tau. Now, I can go to real space, I need to do the k integral, straight forward, I mean I hope this algebra, which you can do it easily.

So, I do this in 1D first, let us do 1D, 1D is very easy to write, so this is the integral. Now this is, we do not need a complex algebra, for this, this 1D stuff, so we use this integral in the left hand side, this one. These are formula I think is worth memorizing or you can keep in your notebook in page 1, we will have lots of formula in fact, I can give a cheat sheet, these formulas are required everywhere again. So, you know this, this integral d k, d equal to 0 is a Gaussian, but when d not equal to 0, then you can always convert it like that, in fact the idea is to convert this to exponential and pull out k. So, this one everybody knows, this is a Gaussian, the square root pi by this guy, but for here I need to just, in fact the idea is to rewrite this as a Gaussian and something outside.

So, remember that I am integrating with k and this is not of the form k squared, so we write k minus k not square and exponential minus k not square will come out. So, the k not is basically this part and if this is standard thing you can look at in Wikipedia, you will find this derivation, I will not do it, but this is the idea, the exponential part, this is root pi by a integral d k and this is a constant which come out of the integral and that gives you this. So, this integral you can be done by using this formula, you can see that what is a, a is minus nu t nu tau by 2, 2 times that, 2 times that, compare this, this a and b is tau, yes, just plug that in, this is my integral minus r squared by 4 pi tau and 4 pi nu tau, I actually I need to make this thing that I have put in diffusion coefficient here, this is my notes, I put the diffusion coefficient not equal, well basically I had ignored it, but put the diffusion coefficient which is minus nu, nu Laplacian squared delta function. So, the diffusion coefficient is nu which is sitting here, in fact we need that for the next slide. So, this is my answer, so if I have a flare in time in this room, my temperature will spread like that, it will be it, of course what does it mean, it will be basically spread out Gaussian, will like a Gaussian and the pre factor for the amplitude.

So, if you wait for long time the temperature will heat, will meet you, I mean will approach you this heat wave, it is not a wave actually but heat diffusion. So, this is, in fact you should understand the interpretation, my source is at, right now source is at origin at time t, I start like that and then I look at the temperature, how it spreads in the room at r and at time t, this is the amplitude of temperature. This interpretation you should keep in mind, I am going to come to the next slide and show you how it can be used for quantum mechanics, source at r equal to 0 at t equal to 0 and response at r t, response is temperature. In three dimensions we can just do the 3D integral, like we have been doing in the past and so this algebra I will leave this part and answer is here. This 3D is pretty similar except this is minus 3 by 2, earlier it was square root, it is minus 3 by 2, okay, is just following the same logic, okay.

There is one trick actually, let me just tell you the trick, this dk is going to give us a k squared and 1 k will cancel by cos theta, so there is a k squared by k, this one k is coming here, okay, this one is important. Now I get k exponential i k r and this part, in 1D there was this k was not absent, this k was not there, it was absent. Now I have this k, so what should I do with this k, okay. So this k is nuisance actually, I mean I can, of course you can do with parts and you can do jiggle with it, but it is a nice trick. This in fact I saw in Matthews and Walker, this is a nice book, I am not sure whether you know this, math methods and this trick is attributed to Feynman, okay, I do not know who is, this is a very beautiful trick.

So this k I want, so what do I do? No, no, this is this trick, well there are many ways to do this, like by parts we can manage it, okay, this is also possible, but this is nice trick, okay, there is another one way to do it. So keep only the exponential part, okay, now do the derivative with r, if I do a derivative with r what will I get, i k will come, imaginary you can put it outside, okay, imaginary is a linear operator, you can compute something and do the imaginary part or you can do the imaginary part and compute, so the imaginary will come, commute. So d by dr of this will give you i k, right, so k cannot be pulled out, this is a dk integral, so k will be inside, but i I can cancel here, okay, so this is a d by dr divided by i, this one is exactly same as that, other than 2 pi square, okay. So now this integral we know how to do it, this formula, okay, after I get it then I can, I have to just do the d by dr and this d by dr basically gives you that minus three half, okay. So this is a nice trick you should use it often, okay, this is a beautiful trick, okay.

So the difference between 1d and 3d is that this is minus three half, in 1d it was minus half, okay, so that is the only difference. Now Schrodinger equation, we can do the Green's function for this as well, okay, what is the difference between Schrodinger and diffusion equation? This i there, okay, that is the only difference, okay. So how can I solve this problem? I already solved that one, so I can rewrite this as d by dt, I can pull out i, so I can multiply both sides by i, so minus i, so if you put minus i this becomes minus mu, minus i that, okay. This and my coefficient this what I had, so my nu has to replace by minus i, minus i nu, diffusion equation is d by dt minus nu k square, nu Laplacian square, yes, this is my diffusion equation. So what is the difference between

Schrodinger equation and a diffusion equation? I just replace this nu by i nu and I can just use that formula, well I am just waving my hand a bit, this is called analytical continuation, okay, but we ignore all the math subtleties, it works, in fact I will show you the answer.

We can just as I said this is exactly same way, well I should put a minus i in the right hand side as well, okay, but delta function part is minus i will come and so that is it. So instead of nu k square, i nu k square, and this is my Green's function for this equation, okay. Easy you know, I mean we just straight forward use the diffusion equation here. Now Feynman path integral, okay, you can, you have done it for free particle in your course, that is reasonably complicated derivation, what is Feynman path integral? So exponential i action sum over all path, so I go from 0, 0 to xt, okay, it is a free particle, so classically you would have gone in a straight line, right. So I start from t equal to 0 and I need to reach x at time t, it will be just a straight line, of course with certain velocity, which will be x by t, that is a classical path, but quantum mechanics says well all paths are possible.

So this particle goes like that, goes like this, so all paths are there, right, that is what is a Feynman prescription. So I have to compute over all paths and then compute the action, okay, this is a bit of algebra and integrate over all of them, okay. So action is integral Lagrangian dt by h bar sum over all paths, this is the Feynman path integral, which is nothing but the Green's function, it is exactly same Green's function, the GRT, okay. If you do not believe me, let us compute it, what do I need to do to convert it to Schrodinger equation? I have it in my next slide. So what is Schrodinger equation? I h bar d by dt equal to Laplacian h bar square by 2 m, right, psi, okay, h bar, h bar cancels, so I get I partial dt equal to minus h bar by 2 m plus m, okay.

So what is my nu? Compare these two, nu is h bar by 2 m, correct, I need to replace nu by h bar by 2 m, so nu equal to h bar by 2 m, okay. So Green's function is exactly same thing, we just need to put exponential I r square by 4 h bar by 2 m, so these two will cancel, h bar m t 4 I pi nu will be h bar by 2 m, these two will come minus d by 2, okay. You just look at 1D formula and this is what we get for 1D, okay, and this is the Feynman-Powell's interpretation. So what is the interpretation from this language, Green's function language? Start the particle at x prime at time t prime, so this is the position x prime, you know, in this room you can just choose a position x prime and at time t prime and what is the amplitude, probability amplitude, this amplitude for the particle to reach x at time t, okay, and this is the answer, okay. You do not need to compute by integrating over all paths, you can do it, use the equation and compute it, Green's function, is the same Green's function, you can, if you do not believe me, just look at your notes or Wikipedia or book, you will find the same, same stuff, okay.

So Green's function and path integral are giving the same answer, in fact it should, okay. So I hope I convinced you that the Green's function is very, very important ingredient for all these problems, including field theory. Okay, so yeah, this is about it, so this is probably amplitude, so this is not like temperature, this is not measurable, this is a probability amplitude, so this is what Feynman writes, xt x prime t prime, okay, this is not a wave function either, wave function does not have delta function in the right hand side, please keep in mind, these are wave function, I should not use probability psi, I use something, in fact this is Green's function basically, okay. Path integral is not the wave function, is the amplitude for going from one position to the next position at different times, okay. If I want the wave function, then what do I do? Yeah, so that, these are separate stuff, okay, I mean this is, once I know the Green's function, then for given potential, I can integrate, okay, I can integrate for a potential, so from free particle we can go on, but I think we will skip that part, okay, I do not want to make things more complicated, but I think you should be happy that we can get this Green's function and relate it to Feynman path integral.

Okay, so I think I will stop, yeah, okay. So let us summarize, I mean for Green's function as I said is very, very important in lots of physics fields and specially field theory, okay, you see that when we proceed and this is nice thing, I hope you appreciate this, Green's function is propagated and end. So I think we are almost only 10 minutes, so I think I will not start a new topic, I will just end here. So next thing we need to do is functional, okay, Lagrangian for fields we need to do that, so we know Lagrangian for particles, but we need to do it for field and Hamiltonian for fields, I will introduce that, that is the next task. So next week I am not around now, so next week is holiday for you, but please go through this algebra and do the homework, I am not asking you to submit it, but you should do it.

No, we are not changing, limit it, we are shifting the pole. No, no physically this Fourier space is not physical, well for me I would say there is some physics in it, physical reality of that, but that we will not do it right now. Shifting the pole is, so this was the wave going outward, no, so you look at example, first example, let me first save this, because I think this, okay, so this saved, this poles, remember this pole, now we chose, we chose like this for wave which is going out, now I want wave to come in, then what should I do? In fact, yeah, we should just need to change the pole, this guy, this guy will go down and this guy will go up, okay, this will do the trick. So, this is a different boundary condition, no, in fact it is possible, there is a something which is attracting the waves, wave need not always go out, wave can come in as well, no, so that will correspond to wave, well, if you want incoming waves and you just change the poles, flip it, the right will go down and left will go up. If I want a standing wave, then what should I do? So, it is combination of out and in, so both should be either up or down, so there will be sine or cosine, so you have to shift both like that and this is boundary condition, I mean at infinity what is the nature of wave, is it going out or coming in, that is the boundary condition, so that is the trick.

So, if you are doing in real space, then boundary condition has to be specified at t equal to 0, here we are not doing it, in fact we are doing quite a bit of hand waving, we are not too worried about boundary condition on the wall, like all of Jackson problem which is reasonably complicated, it is reciprocity theorem, Green's condition, boundary condition at the walls, they are reasonably complicated, I mean I think you have to think quite a bit, but you should really think about them and understand the basic logic. Let us stop recording, I mean the sum of it is. Thank you.