Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Lecture – 65

So, here it turns out that if you make a change of variable grad h s U, if you do this change of variable grad h s U, then this is a minus sign minus grad h s U, okay. So, then it turns out that we can get a new equation for U and the equation for U is this, okay. So, you just take a gradient of this equation, okay. So, in fact you can obviously see this one and this one here and if I take a gradient of this, so gradient h is already U, so this is U dot, it turns out it is U dot grad U, so this is Burgers equation, okay. It is like Navier-Stokes equation with pressure being 0, okay. Now we showed in previous lectures that Navier-Stokes equation respects Galilean invariance, that means the equation remains unchanged if I go from one initial frame to another initial frame and the transformation is you add a velocity right U naught when I go from one frame to other frame and that invariance leads to condition that lambda must not change under scaling, okay, this I showed in the class.

So, the same argument works here and that proves that lambda remains unchanged under RG, so this is unchanged, okay. So, our idea would be to renormalize nu and D, okay. D is the coefficient, so eta x t eta x prime t prime is 2D delta x minus x prime delta t minus t prime, okay. So, D is the pre factor, strength of the noise, eta squared is essentially 2D, okay.

So, that D is renormalized. So, let us see how to do it. Now please note that this is non-equilibrium system, right, because there is a time dependence. So, this system has frequency, there is a time, okay. So, this is not same as Wilson RG which had no time, okay.

So, that is a difference. So, we are going to do RG for a non-equilibrium system or which is system is, has time dependence, okay. Well, if you look at in some detail for white noise eta, KPZ can be argued to be degree in equilibrium, okay, but I think I will ignore that part, that is the subtle part, but well basically it is time dependent equation, I am going to do RG, okay. So, frequency is a parameter here, okay. So, this is my equation.

So, we can write down, we can do this similar thing as what we did for Navier-Stokes equation, okay. So, I want to convert this to Fourier space, okay. So, let us convert it to Fourier space. So, I am going to write it here dh by dt h of k, right. So, h of x will become h of k, in fact h of k t, we will use h of k t.

Well, now I think I am going to use k omega, in fact the idea is to use k omega. I am following standard procedure, okay. I am going to make some deviation from what I did for Navier-Stokes. Navier-Stokes I did with t and k, but KPZ paper that Kardar-Parisi-Zhang paper, we use frequency and k. So, d by d t will give you minus i omega, correct.

I mean, we write h of x t as integral d k d omega in fact by 2 pi to the power d plus 1, d for our k and 1 for omega, then h k omega e to the power i k dot x minus omega t, okay. That is my transformation. So, I just substitute it here and take the derivatives. So, I will get minus i omega here h of k omega and this you take it to the left side. So, new Laplacian will give you what? So, here minus nu k square, right.

When I take it to the left hand side, I get plus nu k square u k omega, okay. Noise will be just straightforward, nu x t becomes eta, sorry eta x t becomes eta k omega, right. But, but with this non-linear term will be convolution, it is a product, so it becomes convolution. So, that is going to be lambda by 2 grad h. So, this you can call it grad h multiplied by grad h dot in fact, okay.

So, now I am going to replace h by e to the power i p dot x minus omega p t. So, here the argument will be I should replace this by p in fact and this omega by omega p. Frequencies, there are three frequencies, one for k and well there are three independent frequencies, but I am going to write this for one for k is like paired, k is paired with omega. But p I am going to pair with omega p and q I will pair with omega q, okay. So, grad will give you i p, right.

So, gradient gives i p know, h of p omega p and the other guy will give you when I take a gradient when that h is function of q and omega q i q h q omega q, okay. So, these are vectors i p and i q. So, that gives you i times i will give minus 1 minus p dot q, this is integral integral h p omega p h q omega q and I need to do this integral, d p by 2 pi to the power d d omega p by 2 pi. And what is the condition on this APQ? Conservation of linear momentum that gives you the condition k equal to p plus q exactly like Navier-Stokes and omega equal to omega p plus omega q, okay. So, these are condition.

Now, these are non-linear equation it has become even worse, it has become a convolution, okay. So, what do I do with that? Okay. So, the idea is to do it perturbatively, okay. The way we did it for Navier-Stokes very similar scheme. So, this 2 h I am going to write as these two lines h p, h p here and h q.

So, in fact, is p 4 vector, I will put 4 vector, p 4 vector means p vector omega p, okay and q also 4 vector h q and what is the coefficient in front is p q p dot q minus sign, right. So, this is my Feynman diagram, okay. If I assume h n this these two h guys h omega h p hat and h q hat to be random, well what kind of random we will assume kind of Gaussian, then if I average them out h p, h q well then I will get 0, okay. But before that I need to do some since I am doing RG then I need to separate the scales, right. We did that before know.

So, RG what is the RG procedure? We divide this wave number space into weighted shells in fact log bin shells. In log scale they are linear but in real space it is power law. So, this is k n plus 1, this k n and this k n plus 1, okay. They are supposed to be equal, okay and this k n minus 1 and k n we write as b to the power n, right. This is what we did it for Wilson RG.

Now, following Wilson RG procedure, now I am going to rewrite this equation for less variable and greater variable. So, this is greater variable blue one and less variable and I am going to coarse grain for wave numbers in this band. This idea is clear? No, we did it coarse graining. Coarse graining means high frequency modes or high wave number modes which are small scale modes. I am going to average out those modes.

So, the equation new equation which I am going to write as, okay. So, since my, so let us let us write in the next page, okay. So, let us write in the next page, okay. I think we will write in the next page, okay. So, let us, this is also filled up.

Yeah, so that is what I am writing it here, okay. So, here I made this k space and we have like that. I am going to extend it here. So, k n to k n plus 1, okay and this part I am going to average. So, we rewrite this equation for h as h less is h less.

We will have h less h less, then h less h greater, then h greater h greater, right. Of course, we will have h less of p h greater of p h less of q, right. So, there are four possibilities. So, I have this k equal to p plus q, okay p plus q, but p can be small and q can be large or p can be large, q can be small or both are large.

So, depends where is p and q. By the way, please keep in mind that k can be here, p can be very large and q can be like that. In fact, this corresponds to greater greater, p can be tiny, but p and q can be large. So, there all these possibilities exist. The idea is how much they contribute and that comes from calculation, okay.

So, this is the integral. Now, we have this integral, this one d p omega p. By the way, I do not write d p d q. If I write d p d q, then I need to do put a delta function. So, we vary only p because p is not a independent variable, q is k minus p. So, my I normally prefer this, I do not write d q, otherwise I need to put a delta function, okay.

So, we have d p d omega p divided by 2 pi to the power d plus 1, okay. So, this integral, so the the circle equation inside the red circle or inside the red oval actually not circle, that is same as original equation, right. But these are new, these three are new terms and they will lead to some correction to nu. The way it happened for Navier Stokes or it happened for Wilson theory. So, the parameter R which is connected to the temperature which was changed because of this greater, greater term, okay.

Now, if I assume that greater is a random variable, but less is not a random variable, then these two averages will be 0, right. So, this will become less is less is not averaged, but greater is average. But so, for example, look at this term, we write this as less is not the average to h q, this comes out and h greater p and h greater p is 0 because h is a random variable with 0 mean. But what about greater, greater? So, greater, greater is a tricky part and let us write this one in the next slide. So, I am going to write, so right hand side RHS which will give us non-zero value is integral d p d omega p by 2 pi to the power d plus 1, this is annoying to carry it out, but h greater than p, h greater than q, four vectors, okay.

And this is a p dot q with a minus sign. So, we write this as this 2 h, h of p, h of q and infront is minus p dot q and of course we need to integrate. Now, to zeroth order this is 0, right, because if I do the average p is not equal to k, so that will give you 0. So, zeroth order is 0, but I want a non-zero contribution.

So, I go to the next order. So, to get a next order what should I do? I am going to expand by Green's function, h p I am going to expand as a Green's function, okay. So, if you recall h p will be what? h p, this is a Green's function inverse G naught inverse, bare Green's function, that is function of original viscosity, that is original nu. So, this was remember is about minus i omega p plus p nu p squared h p. So, we write this as Green's function inverse. The right hand side will be again the non-linear term and noise.

So, look at the non-linear term it will be lambda integral. So, now I should use a different variable r and s, right. So, let us use a different variable r and s right inside minus lambda, well minus plus lambda, then r dot s is the minus sign, r and s are two wave numbers and we have p equal to r plus s, right. That is what convolution means and inside will be h of r, r 4 vector, h of s 4 vector and now I have d r, r 4 vector divided by 2 pi to the power t plus 1. r 4 vector means I have corresponding frequency as well, ok, d r vector d omega r, ok.

So, this is what is the Green's function. So, we write is the Green's function like that and the Green's function will have 2 h. So, 1 h will go like that and 1 h will go there, ok. Now, this technology is very similar to what we did before. So, this h of q 4 vector. So, this must be h of minus q, right, to give you non-zero value.

But look, so now we can, in fact we can figure out what should be this guy. So, if r is going in, so r is minus q, ok, minus q vector, then what should be s? You can easily guess what should be s, not guess, you can, you can derive plus s. So, from here I say that s equal to p plus q. And what is p plus q? Is k.

So, this h of k less, ok. Now, this guy is minus r dot s. Well, r dot s is nothing but q dot k, because r is q and s is p, s is k. So, q dot k. So, all that follows from this conservation stuff. So, which will correct the viscosity? The loop diagram.

So, correction will come from this loop diagram which is Green's function and correlation function, ok. In fact, there will be minus sign, because I need to take it to the left hand side. So, delta nu, delta nu of k, ok, which is, there are integral d of p4 vector. I am not writing 2 pi by d plus 1, ok. Then we have p dot q, minus minus becomes, well, forget about the sign, I mean, that detail stuff we need to do it yourself, q dot k.

Well, I mean, forget means, do not focus on it. You need to of course keep it in detailed calculation, but let us focus on the idea. And it will be G of p, 4 vector, right, G of 4 vector. Whether G of 4 vector is what? G naught, they use, they are not using self consistent procedure, they are using the original G naught. So, G naught of p will be what? 1 divided by, so this is inverse, right, so 1 minus omega p plus nu p squared, ok.

Now, we have correlation function C of q. Well, they are omega p as well, ok. Now, what is C of q? So, we can expand this one. How do I compute C of q, which is hq and h of minus q? We written like this. So, here, instead of p replaced by q, ok.

So, this G naught inverse will go to the right. So, just focus on, this is h of q, h of minus q. So, I am going to write h of q, ok, is g naught of q, right. So, this, this guy will go to the right and the non-linear, and the noise term. Do not put the non-linear term now, because that will go to higher order.

Put the noise eta q, 4 vector, ok. Now, G naught of minus q eta of minus q, ok. So, eta, eta will give you what? Eta q and eta q minus q will give you d, right, that strength of the noise. So, if I do all this stuff, I will get from this eta and this eta, I will get the strength of noise d, integral dp and there are, this p is also there, p dot q, q dot k and the three Green's function, G naught of p, G naught of q, G naught of minus q, ok. So, this is a term I am going to get, ok. Now, we make the assumption that this G naught has low frequency, ok.

So, that will give us low frequency means this omega will go away. That is the assumption made in this calculation. So, it essentially gives a nu, k squared, nu p squared, right from here. So, the three G's, so the one will get 1 by nu, 1 by nu, 1 by nu. So, outside will get d by nu cube and integral dp and this integral is not done for all wave numbers.

So, where should the integral be? It is a shell, right. In Wilson we did for lambda by b to lambda. So, this integral is done from k n to k n plus 1 which is k n to k n plus 1. So, which is width of b, is this ratio is b, ok. So, there is a integral which is done and ok, some numbers will come, ok.

Is the idea clear? And that is going to correct the viscosity. So, that is the next slide. So, we have this three Green's function I wrote, right and this is the two etas and all that. So, so I already, so by the two lambdas, one in the left, one in the right. So, this lambda square d and that three Green's function gives you nu cube and the frequency is small, ok. So, this is by our nu less, nu correction, this is the correction, this guy.

Lambda square d by nu cube and because I am doing volume integral of this of this sphere then this K d is a surface area of the sphere, right. Surface area multiplied by thickness and this part is the thickness part, ok. Now, that detail I will leave out, that is the nu, but this is coarse graining right now. But if you recall Wilson procedure we also need to do rescaling, right.

Remember there is a rescaling. So, after coarse graining the system has become, this is the thing I so, we have this big system which had many many squares. In fact, they are 16 squares for half a coarse graining it becomes 4 squares. So, my squares become big. So, I need to rescale it to make it like small earlier square and that scaling is like b to the power something and we know for nu it is b to the power z minus 2. The factor which we derived earlier in fact, exactly same factor b to the power z minus 2.

If you recall the new Laplacian H had the b to the power z minus 2, right, then z equal to 2 and that is a disaster, right. So, I wrote that. So, there is a pre factor for b which is b to the power z minus 2 and we make this thing that b is equal to e to the power L. We did it for earlier know I mean for Wilson theory as well as for mass renormalization. So, if I substitute it then this looks pretty nice that b is expanded as exponential b to the power z minus 2.

In fact, I have done it here e to the power z minus 2 times delta l and that is expanded for small delta l is this. Now, you substitute this nu less, nu less is nu times z stuff. So, this guy will also give correction. So, overall I get in fact I need to substitute here. So, this guy is 1 minus in fact this is called g squared, okay, this is called g squared, g squared K d and integral, this integral, okay, this integral is basically 2 minus d by 4 d.

This part and this part is whether this dl will come here. This integral also there is a thickness and thickness is dl and this is delta l here, okay. And I think this part is plus, this part is plus by some calculation there is a stuff and this part is 1 plus delta l z minus 2. So, to leading order in the leading order for delta l, I will get z minus 2 from here and this part will come here, this k d g squared 2 minus d by 4 d. So, and there is a nu in front, okay, this nu is here, okay.

So, d nu by dl is nu z minus 2 plus k d g squared in this direction, okay. So, nu is in fact if this was not there, then it would be z minus 2, but this is the correction by non-linear interaction, okay. This is for nu, got it? Now, so this is the Wilson flow equation, right. Remember we go to fixed point and do Wilson flow equation that is what we got. Thank you.