Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

Week - 11

Lecture – 60

So, I did cover how to do field theory for fluid turbulence, that was the topic for last lecture, so I think two and a half lectures, two lectures or two more than two lectures. So there is I followed one scheme called self-consistent scheme and that was essentially popularized by Zhou and McCombs, Zhou's group and McCombs group. But there are several there are many schemes, field theory I did discuss there are ways to do using equations or using genetic function. Like we did Wilson theory that was I did two methods, I did using partition function the formula and then one using equation. So both are perturbative but there are two schemes I said I showed. So similarly for turbulence also there are several other schemes.

So I will show you two more schemes but I will not do the calculation. So let us look at one scheme. So this is called functional form of a dynamic, basically functional field theory. It is very similar to partition function.

But it is quite tricky, so I will get to that. And this is basically proposed by these authors. So this is one paper by Martin, Siggia, Rose and there is another paper by De Dominicis. And there is a pretty nice research notes by here in this site. So let us look at, so this paper are cited here as shown here. So Martin, Siggia, Rose I am sorry this the third author has been not present and this Dominicis and Martin. So you can look at these papers but I mean these are difficult papers to read but I am just putting it for reference. So the equation, this is a generic equation. So we have, it is a time dependent equation. So it is a dynamical thing also the dt partial dt of t with of phi with t.

So and this is a, it could be linear plus non-linear term and there is a noise. So a Navier-Stokes equation how does it look like is dt u vector equal to minus nu grad square u plus u dot grad u plus we put a force, noise is like a force, okay. But it is supposed to be random. For calculation we make certain approximations. So there is a force which is like a noise which is function of x and t, okay.

So these two terms are put in here. So v phi is combination of linear and non-linear term, okay. Now this is a differential equation but I want to write a in terms of generating function. Now please note that this is a dissipative system. You understand now? The energy is dissipated because of viscosity.

So integral of u square is not conserved when viscosity is non-zero. Everybody knows this, right? So people normally have misunderstanding that we cannot write Hamiltonian or Lagrangian or partition function for non-dissipative system. It is possible, okay. There are ways to do it which I am going to show you. We can write down generating function for this equation and that is a trick which Martin, Siggia, Rose proposed and there are, there are simpler discussion on this topic, okay.

So it is interesting idea, you know, we do not need a non-dissipative system for Hamiltonian. We can write a partition function or we can write a generating function for this. We do not call it partition function, we call it generating function. So one assumption which is made is that this noise is delta correlated in time and space, both space and time which is a very drastic assumption because most of the noise is not delta correlated, right? In this room if I take this fan, fan has a certain correlation in its forcing, right? I mean two blades of the fan, they are connected in time as well in space. So the forcing is definitely in real life is connected, correlated in x and time, but calculations become more complicated.

So it is assumed that they are delta correlated, delta correlated is here. If x equal to x prime and t equal to t prime, then only this is non-zero, otherwise it is zero, okay. So this assumption and given this assumption, well, we make one more assumption, okay. There is one more assumption. So this is a random, phi is random field, but it can have some correlation, that is what we are after.

Zeta is uncorrelated, but phi is a dependent variable, right? Phi depends on zeta and it could have certain correlation, but we assume that this zeta function, this noise has certain distribution. Now this is Gaussian distribution, right? You can see that zeta square and this, by the way, lambda is the zeta square, lambda is zeta square, no? And this function of x and t. So this is non-zero, lambda is non-zero when x is equal to x prime and t equal to t prime. So look at this form. It is exponential minus zeta square by gamma.

So does it look familiar? So in statistical physics, what is a, what is a probability Boltzmann distribution? It is eta, e to the power minus beta E, right, energy and beta is 1 by temperature, 1 by kBT, right? Well, I should divide by partition function, I mean normalization factor. So this probability will not become, probability must be equal to 1 if I integrate over all zeta square. So this, one has the same form, right? Because energy is like zeta square for the noise, for the bath. So heat bath, forcing is creating a heat bath and its effect is on the system which is phi. That is the connection.

In fact, lot of theories are motivated by statistical physics, this non-equilibrium theories, okay. So I hope this is clear. Eta I am replacing by zeta square which is like energy and beta is 1 by temperature and what is the temperature? Temperature is energy, right, measure of energy which is zeta square, zeta square is energy, average energy, okay. So we put it here and this function of x and t, okay, but so I integrate over dx and dt. So given configuration, so I have a certain zeta as function of x.

So what is the probability for this particular configuration, right? So in statistical physics what do you do? Given a configuration, I compute energy and the probability for that particular configuration is e to the power minus beta e by z, right? So this is precisely same formula. Well, motivated by that formula, zeta is not exactly same, I mean there is no heat bath here, it is a dynamical system, but we are making an analogy, okay. So this P of zeta, probability of given zeta and then we write action, okay. So action is function of phi and this secondary variable. I am going to show you how the secondary variable comes. I will prove it quickly, not skipping some steps, but I will show you this phi tilde, okay. So this tilde has gone up in the in PPT.

So phi and the secondary variable phi tilde and that action it is of this form. So this phi tilde, I phi tilde and this is the linear, well this is the dynamical part. So d phi partial phi partial t equal to v phi. So I just put it this dynamical part without noise. And the noise part comes here, how does the noise part come? 2 gamma phi squared, okay.

I will show you how to get that, okay. And I integrate over dx and dt. So these dx is a vector, okay, in volume. Is that clear? So this action, I am going to prove it in the next slide. Once I have the action, then I can write down the generating function.

So how do I write this generating function? Exponential of minus s, this s1, s1 plus you recall the source term we write, J phi, the source term, that is useful for computing the correlation function, okay. And J tilde phi tilde, there are two variables phi and phi tilde. So we write two source terms, J and J tilde, n phi and phi tilde. So this is for source term which is finally set to 0, okay. Generating function will always be write this source term.

So computing the correlation function in terms of J is easy, okay. And you integrate with D phi, this is the functional integral, D phi and D phi tilde, okay. So if I want to take the correlation function, I take this one, delta delta J, right. So what will that give me? It will pull phi, right. Yes or no? If I take the derivative of z, generating function with J, what will it pull? This is exponential, no? So exponential of any x multiplied by alpha, so alpha will come out when I take the derivative.

So I take derivative with J, then phi will come out, phi will come out here, integral same, this phi will come. If I take another derivative with, of this with delta J tilde, then what will come out? Phi tilde, okay. If I integrate this, then I will basically this is phi phi tilde, correlation. So this generating function is with source term is useful for computing the correlation, okay. Well, basically I need to divide this by, if I want to normalize it, divide by the probability with this part we divide, okay.

So I am just giving a sketch because this discussion is definitely quite complicated. But look, for a dissipative system I am able to get a partition function or a generating function. So how do I prove this action? So let us look at the proof, a quick proof, okay. So my zeta is the like noise and it has probability p of zeta, this is exponential minus zeta squared by 2 gamma, right. I put in the previous slide that was well 4, 1 by 4, okay 1 by 4.

Why 1 by 4? This is half e to the power minus half, right, and normally we write and this gamma there is a factor 2 sitting there, okay, this factor 2 that is why we put 4, okay. So it, there is certain motivation for that one, okay. So and we put a delta, so basically this equation is sitting here, that equation, phi partial t minus V phi minus zeta. So given zeta what are the available, what is the solution of phi, okay. So this will give you non-zero value only when the solution phi satisfies the equation, right.

And then I integrate D phi, okay and integrate zeta. So right now my integration, functional integration with phi and zeta, okay. Now delta function is not very easy thing to handle, no. So we replace this by exponential because I use this stuff, e to the power ik x dx is integral delta k by 2 pi, okay. I am ignoring all that, 2 pi delta, okay.

So 2 pi and all I am ignoring, okay. So I replace delta by integral, okay. So I put the integral. So this is my field and I put phi tilde. Now this is a new function phi tilde. So I integrate this with phi and I am doing the integral, it dx dt and so basically I got, get rid of delta function by this, this, this formula, okay.

Instead of this one I am using phi tilde, okay. Fine. Now we got in this term this p zeta is exponential minus zeta squared by 2 gamma, right. I showed you that part and so, and this also integrated with dx and so this part is there.

I am going to push it in. So we have term e to the power minus zeta squared 2 gamma and this is zeta, so this I plus I, there is a 4 there, sorry, there is 4 gamma, this is I, phi tilde, sorry I, phi tilde zeta. Can you see that? Right. So this, this one and this, this one. So what should I do? This is integral with zeta, integral d zeta.

So this is not quadratic yet. Please recall e to the power minus zeta plus some phi tilde squared integral d zeta, what is that? By, by 2, 2 gamma, it is square root pi, no? Square root pi times 2 gamma or pi or something, okay. So this Gaussian integral has a solution and that has a number. So you want to write in that form, so that this zeta is taken care of. I do not want to integrate over bar. Is that correct? So motivation, so this d zeta integral I want to finish.

So that part is done here. So you put zeta squared and minus 4 i phi tilde zeta and we rewrite, how do I rewrite this? So this zeta, well I mean there will be, you can convert it to quadratic form. What should I add and subtract? So this zeta squared, this is phi tilde zeta. What should I add and subtract? Proportional to phi tilde squared, no? And subtract phi tilde squared. There is a pre-factor in front, yes? So these together, I mean with all this gamma and so on, will be of this form. Well, there will be gamma, some gamma will be there, okay.

I am not going to do the algebra. So and that is square root pi to the power, okay, so this square, so it is a number. A number can be taken off. Number does not affect our functional integrals. So that, so I add and subtract, so this part gives you a number, but this part gives you a function, okay.

In fact, the gamma will come there. Gamma phi tilde squared, this 2 will also come. So that is how this term comes, minus 2 gamma phi tilde squared and the other term, this remains as is, okay. So this is my action. Well, this is generating function, sorry. So action is this one without j, without source term, right.

So this is what precisely I had written earlier. Action is exponential of this object and because of the noise is 2 gamma, phi squared has come, okay. It is complicated, but we can write the generating function for dissipative systems, okay. So once you have the generating function, then you can compute correlations, the way we did it for phi-4 theory, okay. So let me just do little bit of free Green's function and free correlation function. So just to demonstrate, I can do it using equation, but it can also be done using this generating function, okay.

Generating function is bit complicated. So let us try to do it for the equation, using equation. So the equation was d by dt of phi equal to linear term. So forget the nonlinear term. So nu grad square u, okay, phi, so grad square phi.

I am using a linear term, okay, plus zeta, all right. So this is a linear term. What happens to this in the Fourier space? So d by dt becomes minus i omega phi, k omega. So it is function of k and omega. So often we use both k and omega space.

So t becomes omega under transform and x becomes k. And this becomes minus nu k square phi of k omega, okay. So I take this to the left hand side, equal to zeta of k omega. So we can write down the equation for the Green's function now. So the linear operator is minus i omega plus nu k square G of k omega.

Right hand side is 1, right, the delta function. So delta thing becomes 1. So Green's function is called G naught is free. The free Green's function is G k omega equal to 1 by minus omega plus nu k square. This we have seen it before, no? This is a Green's function, we have seen it in our earlier lectures.

So for diffusion equation, this is a Green's function. Now how do we compute the correlation function? Okay, it is straightforward. In fact, from this equation I can compute the correlation function. How do I do it? So take this to the left hand side, minus omega plus nu k square phi k omega equal to zeta k omega. Okay, yes, I mean this is just straightforward algebra. The phi k omega equal to zeta k omega by minus omega plus nu k square.

Now what is the correlation function? So phi k omega phi k prime omega prime equal to, so zeta is a random variable. So zeta k omega, zeta k prime omega prime, so this term gives you the first term and second term gives you the second term. But of course I need to put what is there in below. So minus i omega plus nu k square minus i omega prime plus nu k prime square, this one. Now it is a delta function in real space, but it is also delta function in the Fourier space.

So the second term at the top numerator is what? It is a 2 gamma delta k plus k prime delta omega plus omega prime. We had it before. Okay, divide by now, so minus i omega plus nu k square. Now we replace omega prime by what? Minus omega, minus omega, right? Because of this condition.

Omega plus omega prime is 0. So that becomes i omega plus nu k square. So k prime is k. So what is this? 2 gamma divided by omega square plus nu square k 4, delta function. So this is my c of k, this part.

And of course the delta function will be the product. So my correlation function in Fourier space which is the free correlation function for the free part, it does not include the nonlinear term. It is this. Now does it sound some well? I mean we had done it for Langevin equation, right? And I derived, well I did not derive it, but I demonstrated fluctuation dissipation theorem. So we have Green's function and we have correlation function. We can easily write down, so correlation function C of k is 1 by omega imaginary part of

Green's function.

You may recall that I wrote something like this, right? And n the temperature, k B T was there, okay, right? At this temperature. So the temperature is coming 2 gamma. I would ask you to please show this, okay, given these two formulas. But it is straightforward, I just took imaginary part of g k omega as straightforward, so you will get this. So it is very similar to the fluctuation dissipation theorem for Langevin equation, okay.

So you see equilibrium, well equilibrium or near equilibrium theories are being used in statistical theory for doing this non-equilibrium field theory. Now of course we can do the same thing using generating function, but it is, well, okay, so well I mean, okay, I hope you can see this because it is kind of on top, okay. So we can write down this part, the argument of S is a matrix form, okay. So I think what I should do is, unfortunately I did not, okay. So basically this matrix is the action, well, so this has this part inside and from this I can derive the Green's function and correlation function, okay, which is exactly same as what I have done it here, okay.

So this part is quite complicated, so I will leave that part, okay. So how to do it from generating function is definitely involved, but I just wanted to just show you some formulas, so that once you, if you work on it then you have some familiarity, okay. So in fact this is, this part is same as that, right, Green's function, d by dt by minus L, L is the linear operator which is this part. So this part is same as minus, this minus i omega plus nu k square. So this is the Green's function and the other one, this delta 1 1 is the correlation, okay.

So this machinery gives you that, okay. And okay, so this was, so I should have done in the next page, but I put in the same page, that is the problem, okay. So this calculation is done by De-Dominicis. So after quite a bit of tedious algebra, this is called RG equation, in fact this is called, this is called Callan-Symanzik, okay. So using this equation is derived that Ek will go like this and this gamma is function of k or x and t, right. So gamma is chosen as function of k omega and it is k to the power minus y, okay, chosen like that.

So this forcing is k dependent and that y appears here, okay. Now this is still not phi third, no, it does not look like phi third, right, this is function of y. But if you put y equal to 4, then what do we get? If we put y equal to 4, this becomes k to the power minus 8 by 3 plus 1, which is k minus phi third, right. So for y equal to 4, we recover Kolmogorov's theorem, but with the strange way to do it now and because we will not get Kolmogorov's theory for arbitrary forcing, which is expected, Kolmogorov's theory comes when I force the large scales, okay. And k minus 4 is pretty steep forcing, if you plot k minus 4 in log-log scale, it is going to be steep. So by the time I reach k equal to 10 is going to be 10 to the power minus 4, 10 minus 4, which is small.

So wave number 10 already forcing has become weak, so that is one intuitive way to say that steep forcing gives you Kolmogorov, okay. But we have to put y equal to 4, otherwise you will not get Kolmogorov, okay. So by the way Kolmogorov theory says whatever forcing I use, but that is at small way numbers, I will get phi third spectrum. And so this proof is not really Kolmogorov's, proving Kolmogorov's theorem.

But it is okay, field theory has certain assumptions and you have to live with it. So this is this theory. Second very popular paper is by Yakhot and Orszag, 1986 this paper. And this paper is based on differential equation, okay. So we have, so I am going to do another equation, I will not do Yakhot-Orszag theory, but I will do another famous equation called KPZ equation and the formalism of KPZ and Yakhot-Orszag is very similar, okay.

So I think I will skip this part, Yakhot-Orszag, but it is a very famous paper, okay. So that uses noise, but it does not use the functional renormalization or this functional approach or generating function approach, it uses differential equation. In fact it starts from minus i omega plus nu k squared u k omega u i, tensor notation. And the non-linear term which is convolution u p u q, we write four vector, so p has omega part, some integral and the noise q omega, okay. So then you write down correlation function, Green's function.

So it is a pretty long paper, but it is done using differential equation, but with noise. What, so can you see the difference between what I did in the class in detail and this approach? I said I am not worried about the noise, noise at the large scales or small wave numbers. If I look at the inertia range, I am somewhere here, in the inertia range this term is 0, okay, that is what I assume in the inertia range. And then I model what is correlation function and assume it to be Kolmogorov. And viscous term also the nu I modulate using dimensional analysis, this goes as k minus four-third, okay. And we substitute it and then we got this, I mean our theory finally says that this nu k is good and I can get the pre factor nu star by calculation, okay.

So our calculation is self-consistent which is in fact the algebra wise is simpler, okay. And we can get all these constants which are quite difficult in this other approach. So I like this Zhou and this self-consistent approach may come, okay. So let us look at the summary. So computation is done, can be done by generating function and there are lots of papers but I am skipping due to lack of time. So the basic idea is, well this is more complex of course definitely I would say myself I am not very comfortable working out this approach.

But generating function is very useful and a lot of field theory is done by generating function. So you have to follow this approach if you want to follow, use generating function. Thank you.