Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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So, far we performed both RG and flux calculation for shell model which I did it in full detail. So, that was simple so, it would be done, but the Navier Stokes equation is bit more complicated and, but I will show you how to proceed. I will not exactly show you all the details, but you will get an idea ok. So, this is the Navier Stokes equation just to recap the u is the velocity field non-linear term and as before lambda is equal to 1 and that is unchanged under RG. So, we do not worry about it we only renormalize the viscosity. And this external force we assume the external force is active at large scale ok.

We need this because we are assuming steady state otherwise my energy spectrum will decay with time which is not which will make it more complicated ok. So, you want the spectrum to be fixed. So, I can do ensemble average or ok. So, we need that forcing, but that is at large scale.

So, for the inertia range calculations I will not have f u. So, it would not complicate my life in this scheme. So, I must say that the scheme I am following is by Zhou and his co-workers and McComb. So, these are the two groups which propagated this way of RG. There are other RG schemes I will do one more maybe not today, but I will not do RG full, but I will show you how it is done by genetic function ok, which is you can easily see that is more complicated, but I am following this people's scheme ok.

And grad p will add something to u dot grad u, but for my calculation I will say that ok I will ignore that term ok. I am not ignoring it, but I am hiding it. If I ignore it then I will get buggers equation which is a very different physics, but I am hiding it to simplify my algebra and we assume incompressibility. So, this theory will not work for incompressible Navier-Stokes. So, in Fourier space we will work in Fourier space.

So, this is Wilson RG we motivated by that. So, that equation in Fourier space is this. So, viscous term has come to the left. So, nu k square and this is a kinematic viscosity the original viscosity and this is a pressure term and the non-linear term has become a convolution right k dot uq up and this is integral. For shell model we did not have the integral we had just three terms, but now I had to work with the integral which is which makes life bit complicated and is the force F u which is acting only at large scales ok.

So, we again go back to our RG procedure. So, imagine that I am in the inertia range I am in the inertia range and I am here at this shell. Now, my shells are I do it by hand for shell model they are all single variable for every kn, but now I have many many variables, but S before kn plus 1 by kn is b ok, but there are modes sitting in between or there in fact, the 3D or 2D will focus on 3D. So, there are many many modes in between a magnitude. So, it is a 3D.

So, there are in fact, a full lot of modes in the shell and the inner radius is kn outer radius is kn plus 1. So, there are k square times the distance between the shells ok. So, we denote that by k greater symbol. So, this is to be coarse grained after this operation I am going to average out and basically see the effect of this coarse graining on viscosity. I di is exactly the same as before except we have more calculations ok.

So, these are the modes which I am going to coarse grain that means, I am going to average and they will not appear in the renormalized equation ok. So, as before I am going to this non-linear term. So, so if I look at the filter I look at any u n in between the less region this is to be retained right this modes are to be retained. So, when I look at any velocity mode in this regime. So, this will be less this will be less ok pressure also will be less, but the non-linear well pressure I will just keep quite right now, but here there are product of u.

So, what are the possibilities? Two u's are in the left of cane less, but there are possibilities that both u's are greater or one u is less other u is greater or this is greater this is less there are four possibilities right. So, it is a convolution. So, this integral will involve all combination of u's less greater greater less less ok. So, my equation if I divide in this part I will have less less I said I am hiding pressure now. So, I am going to keep that then there are three more terms.

Now, if you look at this part of the equation other than pressure gradient it looks same as before ok. These are annoyance right these are something well I got some new term what will do with that. So, I need to model these terms A B C I need to model them ok. Now, if I do simple model let us do simple model ok. So, if I average them coarse grain means I average the u grade mode.

So, simple thing is well I am going to write this as in the similar shear model I write this u less this is not to be averaged. So, I am saying that they are not quite connected very strongly. So, I am going to write this u greater of p and what is this? This should be 0 because I assume to be randomly fluctuating field u is random there is no mean. So, this is

0. So, these two terms go away and we left with this term out of three only one term greater greater and I need to compute this greater greater term and see whether there is it is a non-zero value ok.

So, what do I do? So, I follow very similar scheme I do to first order. So, I am going to write the Feynman diagram. So, what is the pre factor? Minus i k dot well I will just write it as k ok pre factor and to use to use this is uq and this is up greater. If I do average here since p is not equal to q I will get 0. So, I go to the next order how do I go to next order put minus ik.

So, then we just the two possibilities expand using Green's function to each of them, but one at a time. So, let us expand this Green's function and this u greater of q. Now, they will they will give you two u's correct. Now, what is the condition here at this point? p plus q must be equal to k that is the condition for convolution right this is g of p, but what is the condition here? So, I had two Fourier modes R and S R plus S must be equal to p right. Now, of course I need to integrate over all all p's, but I am looking at one tri one p and one q.

Now, if I want to get a non-zero value then I need to get them to come together and give non-zero values. So, this will be u of r greater and what should be condition on r and q? Imagine that well not imagine you know that u is a real field. So, u greater u of r must be complex conjugate of u star of q right I need that to give non-zero values, but for real field u star of q is u of minus q. I am not doing it here, but for real fields this is a property for any real field f of q f of q equal to f star of minus q this property of Fourier transform for real f. So, I put a complex conjugate here I do that.

So, that means r must be equal to minus q correct is that clear to everyone? Now, look at this condition. So, r is minus q I am just rewriting it here plus s equal to p what should be s implies s equal to p plus q and what is p plus q? k. So, this must be k. So, this is u of k great know why is it great? Because this calculation is giving a term proportional to u of k and that I am going to bring it to the left and that is going to correct the viscosity ok. So, this what happens under this field theory where I do this one loop diagram or with this perturbation I get a term which is proportional to u of k ok.

So, this is what we get the second term is also similar, but the Green's function is above and correlation function this is a correlation function. So, this is Green's function correlation function and this is u of k. So, in fact, we had very similar thing two loop diagrams for shell model and idea is exactly the same. So, these loop diagram and these loop diagram will correct the viscosity.

Of course, this is in the right hand side when I bring it to the left hand side I need to put a minus side clear. So, let us write this stuff. So, I have in the nu bar k squared u less of k from the right hand side I bring this to the left. So, I get two Feynman diagrams this one plus this diagram and both of them are proportional to u less of k. But if you look at this equation now.

So, I am going to go back. So, this term is gone and this term I bring it to the left. So, what does the equation look like after bring it to the left. So, let us write that d by dt of u of k less plus these guys equal to right hand side is what minus i k dot uq up integral less less. So, now, this will look exactly same as original equation if I can connect these two to that. This looks like exactly same equation, but my viscosity has to be modified.

So, I write that nu n at the nth step n squared is nu bar k squared plus these two diagrams ok. Now, these two diagrams are to be computed. So, what does the two diagram what do the two diagram look like. So, this is let us look at one of them dp integral. I have Green's function which is p of t minus t prime there will be integral dt prime, correlation function of q t minus t prime ok that is what we will get.

There is another term with p q interchange. So, this will be dp and by the way there are two ks k square will come in front ok. I can also tell you that this case will come in front. So, this Green's function is exponential minus nu p p squared t minus t prime that is a model I assume this form. In correlation function I write as c of q exponential minus nu q q squared t minus t prime ok.

Now, of course, this u of t prime k because this is u of t prime at a k number k less, but then we use same Markovian approximation to convert t prime to t. What do I say why do I say that this pre factors are large. So, the integral is basically coming from t prime near t that is argument for Markovian approximation. So, this is replaced by uk of t ok that is what I want this u of k of t right this is a time t. So, there is a Markovian approximation and I need to integrate this.

So, I get k square integral dp c of q divided by nu p p squared the nu q q squared and this is my nu n times k square plus nu bar k square ok. So, k square k square cancels and I need to solve this integral, but nu bar is small ok. So, we get rid of this. You know because it is big because it is grows with the small k when I decrease k it is grows more and more. Now, how do I solve this problem because c is I do not know what is c what is nu, but then I go back to again self consistent idea of McComb and Zhou.

So, we write this c as Kolmogorov constant epsilon two-third q minus five-third. Well, this is a modal energy. So, I divide by 4 pi q squared ok. This detail I am hiding right now and the viscous term nu p p squared exactly like what I did before nu star q root of k Kolmogorov epsilon one-third p two-third.

Plug this in I get integral of. So, basically now I get Kolmogorov constant will in fact cancel. One Kolmogorov constant square root comes here, one square root comes here, Kolmogorov constant epsilon cancels everything basically gets cancelled and we get some function of a constant integral. But the integral is not trivial this integral involves some change of variables and some massaging, but by doing this algebra I get nu star again computed by if I do the integral properly ok. If you are interested I can share some notes or share a paper which has this these details. This is a nice book by Leslie or you can look at my archive paper I have paper on this one in archive 2005 ok.

So, that has this formula for integral I hope to write a paper on newer scheme. So, if I do all these steps then I get nu star and it is absolutely 0.4 ok. So, lot of detail algebra I get 0.4 which is similar to what we get for shell model.

So, this is amazing know that this field theory of different models are giving a pretty much correct nu star ok. So, this also depends on B the ratio of kn plus 1 by kn, but we normally take this ratio B as b kn kn by kn plus 1 around 0.7 ok. I did this calculation for this and other authors also do around 0.7.

So, this is a RG for Navier Stokes equation. So, I hope you got the idea. So, the idea is that we write down the Navier Stokes equation we coarse grain then we get the original equation where there is a term in the right hand side which is which is which does not look like the original equation, but then we solve that or rather we simplify it using field theory and the term after simplification looks exactly like this form which is similar to the viscous term. So, that means, this term will correct the viscosity and then of course, we need to do the integral and make some approximations and then we get nu star to be 0.4 ok.

This part is so ok ok. Now, let us look at. So, this is the equation I had written before and you plug that in and that will give you nu star in 0.4 to 0.5 degree and it is possible to compute is quite difficult to compute nu star. I told you how to do it for our shell model. A similar thing has not been attempted as far as I know for the Navier Stokes equation.

So, one good idea would be to try well we did this stuff, but our simulation only phi 12 cube and that was not really conclusive what should be nu star, but to get nu star from computer simulation. So, we need to compute basically idea is that look at this uk u star u of k t u star of k t prime this correlation divided by mod uk t mod square average. Exactly same as what I did for shell model and this quantity should be decaying with time. So, that should give us nu k k square t minus t prime, but there are complications. So, it turns out other than this is a oscillating term ok.

So, we know that this oscillating term ok, omega t minus t prime and this oscillating term is connected with the sweeping effect which I will not discuss right now, but this oscillation also come. So, it is a new feature of Navier Stokes equation and that complicates this stuff. It is similar thing is what some of you are trying. So, magnetic field will oscillate right. So, that is what happens here as well and, but only way I believe we can try nu star is computing this correlation function and from nu k using RG of this model we should be able to get this nu star ok.

And this needs to be done ok. Now, let us go to the next flux calculation.