Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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So, the next is field theory of RG I did. Now let us compute the energy flux that is also important quantity which I told you what is energy flux. Energy flux is energy coming out of the sphere of radius k naught ok and it turns out we can do it using field theory of course to first order. So, there are approximations I did not emphasize that, but let me I will tell you what is first order ok. So, you will just give me one minute I will explain for this flux calculation ok.

So, for shell model of course for I wrote the formula for the Navier Stokes equation which was in terms of velocity field u of in fact u of k dot sorry u no complex conjugate now for this k dot u q and then I need to sum over imaginary part of this and sum over giver modes the one which gives should be in the inside the sphere right this is inside giver is inside the sphere giver a receiver must be outside the sphere right. So, energy going from the modes inside the sphere to modes outside the sphere ok. So, we will get to that that is my discussion after this flux calculation ok. So, I will not do the detailed calculation for Navier Stokes, but I will show you how to go about doing it I mean the sketch of the proof.

But for the shell model we can show this is a proven by various authors before. So, we have shell model please recall that these are shells and I am at shell kn. So, we again look for the energy going out from the shells left of kn kn and left of kn to modes to shells beyond kn. So, energy received by kn plus 1 kn plus 2, but it is a local model in the sense interactions are only local. So, kn interacts only with two more shells in the right.

So, we do not need kn plus 3 right because kn can only interact maximum only up to kn plus 2. So, we stop there ok. So, there is no integral there is just a sum there are two terms ok. The two terms are interaction of n minus 1 n and n plus 1 this is the term ok the two of them are complex conjugate in sabra model. So, there are three modes interacting n minus 1 and n plus 1 it is coefficient in front and there is imaginary part this imaginary part ok.

And the next term is un un plus 1 un plus 2. So, tried interaction the three shells. So, this one is so I mean you can put this stuff here n n plus 1 n plus 2 n minus 1. So, this one is n minus 1, n, n plus 1 these three for the first term and second is this three. Now, this one can be computed in numerical simulations there is some analytical work, but you can easily do it in computer simulation.

I simulate the shell model and by initial condition this initial value problem given initial condition I can get un as function of time all the un's and then I can compute them ok compute the flux and it turns out flux is constant in numerical simulations. But let us try to see whether we get constant flux in the inertia range for using field theory which is not RG. So, what I am going to do is not RG, but it is the field theory. So, first order perturbation. So, it will become clear when I write the diagram and again I am going to do self consistent theory that means for the spectrum I am going to put Kolmogorov spectrum and for the viscosity I am going to use that what I derived using field theory ok the RG.

So, I am going to put that what I know the answer and then I saw well flux is constant this was self consistent. So, using the answer you show that answer is consistent it satisfies the equation ok. So, please recall that I force at large scale and there is a dissipation in small scale if I do not force there is no dissipation then that is a equilibrium theory ok. In equilibrium theory flux is 0 ok. So, we need non equilibrium that means there should be energy flow is constant or not we do not know, but we will show that it is constant ok.

So, this formula I had written before and it is a very simple interpretation. So, if I like take flux at two shells which are separated by slight distance dk. So, shell 1 instead of shell 2, two shells and imagine the distance between them is dk. So, there is difference in flux. So, this is the term multiplied by dk I can multiply by dk right.

So, that will be the difference in flux right there is a differential derivative. So, how can flux change? Flux can change in two ways I inject energy into the shell put in energy which is fk by external force non nonlinearity external force is required nonlinearity. So, nonlinearity will not change it ok. So, that we know that nonlinearity is giving you this fluxes that is that is already taken into account. So, external force can change the flux in fact it will increase pi k plus dk right.

If I inject energy then the flux for the bigger radius will be larger right because I inject energy, but this is a dissipation as well viscous dissipation and viscous dissipation will act like a sink. So, if there is dissipation only no injection then the flux at a bigger radius will be lower right. So, it comes with a negative sign. So, this is a dissipation and this is a injection by external force, but for the shell model we assume that forcing it at is at a very low wave number small wave numbers small shells well small wave number cells ok. So, in the inertia range fk is 0 here fk is 0.

So, this is gone in the inertia range we can also assume that dissipation is weak why because your nu k squared Ek is a dissipation rate 2. So, this k squared factor for small k inertia range of k is not too large. So, then this is small. So, this is also gone. So, flux is to be constant that is what we want.

So, let us compute the flux now ok. Now, so remember I had this interaction was un minus 1 un un plus 1 3 modes right 3 shells. If the pre factor the vortex is 2 a 3 kn minus 1 there is a there is a vortex this is a pre factor. Now, there are 3 of these arrows. So, they represent the velocity field.

Now, let us assume this velocity field is Gaussian. So, what is the property of Gaussian variables or what about odd products u u u 0 non 0 or 3 of them if you multiply this was a mid-sem problem you have multiplied 3 of them what will I get 0 right because odd number of well I assume that velocity the u's have 0 mean. So, all odd ones will give you 0, but even once you can write as a product is sum of 2 point correlation that is this well all possible ok. So, that well I wrote same way u then this will be factor 3 you see same u otherwise you have abcd and abcd, acbd, adbc three. Now, to first order we have to write or to 0th order this is 0 if I do the averaging.

So, flux is computed as an average flux will fluctuate right this pi will fluctuate for different different times, but we do average in fact ensemble average that means I have prepared lot of shell models and I compute this flux at different at a given time, but they will give different numbers differential models, but then I have average. So, it looks 0, but then this is a problem I mean I want non-zero flux then what should I do say well that is where you make this Gaussian variables quasi Gaussian ok. And you say well I am going to go to the next order and I am going to expand this velocity field in terms of Green's function ok. So, I am going to I have three velocity fields they are all expanded one at a time. So, if I expand this u n plus 1 this one so the Green's function and so remember in the last class.

So, the right hand side is 2 u's right. So, it is going to be Green's function times 2 u's correct. So, in fact if you write this is g t minus t prime u t prime u t prime ok integral t prime t prime goes from 0 to t this is how it will be I mean this property of the solution. So, un plus 2 un plus 1 sorry this un plus 1 will be this where the pre factor in front pre factor and of course the three un's but it turns out only one of them will give non-zero values by this property that un minus 1 should connect with un minus 1 star otherwise you have to get 0 ok. This is exactly same as what I did for RG.

So, u n plus 1 remember u n plus 1 had how many things one term to the both of them to the right one is one in the left one in the right and one term is both in the left ok. So, what are the values here u n plus 1 will be both in the left will be u n and u n minus 1 right both left will be u n n minus 1 one in the left one in the right will be u n is in the left and u n plus 2 in the right and third is both in the right. So, u n plus 1 u n plus 2 u n plus 3 ok. Now I am going to multiply this by these guys right. So, which of the three terms will give non-zero values will this guy give non-zero values.

So, you will have u n plus 2 u n plus 3 ok then we will have u n star and u n minus 1 star. So, now these are Gaussian variables ok. Gaussian I did it here I mean I said this non-zero I expanded it, but then I will say well now I got quadratic term I will use Gaussian property for the for this use. Now look if I do this pair what is it 0 right if I do this pair what is this u n plus 2 with u n is 0 if I next is what u n plus 2 that and these two together choose any combination you will get 0 because u n must be multiplied by u n star to get non-zero value. Shell n and shell n plus 1 if I multiply I will get 0 because they are two different random variables, but if the same random variable complex conjugate I get non-zero value.

So, which is the term among the three will give non-zero value this multiplied by these two because I have u n minus 1 here this u n minus 1 multiplied by this and this u n multiplied by that. So, only one of the three terms will give you non-zero values and these are here. So, this is u n u n star will give you this temporal correlation two type two point correlation c n t minus t prime and u n minus 1 will give you u n minus 1 t minus t prime ok. I hope this is clear. So, Feynman diagram basically just makes the calculation very simple.

I do not need to write all this I can just do it by drawing the diagram. As a pre factor there is a pre factor as I said there is a there is a vortex term this is a vortex term ok. So, this coming from the first diagram which is well this is the first diagram where I expanded u n plus 1, but as I said I need to expand other terms as well other fields as well. So, u n expansion will give you this Green's function and two correlation function ok. There will be c n minus 1 c n plus 1 and third one is I expand u n plus 1 using Green's function u n minus 1 sorry u n minus 1 I get this ok.

So, I have three Feynman diagrams and I know this values they are one Green's function and two correlation function ok. So, I will delete these guys this is rough work now and I am just going to. So, I am going to just write down this formulas equations for first term ok first term and these are the expressions. So, this is what well one of the terms I am looking at ok. So, there are three more Feynman diagrams for the second term flux has two terms right in the previous thing.

So, first term I am doing this, but in that I am going to expand only n plus 1 is using Green's function. So, n plus 1 is Green's function this one and the three terms, but out of these only these guy is giving non-zero value right because that will connect with n this one will connect with n and n minus 1 will connect with that and the pre factor will be I a 3 k n minus 1 which is here ok. So, I just substitute this u n plus 1 this u n plus 1 is substituted here ok. So, that is two correlation function this and that and Green's functions. So, what is this integral? So, we get product of these two these are ok, but I take the imaginary part ok.

So, this is I sitting there which is good in a take imagine well these are all real variables ok for shell model they are all real variables, but this I sitting there. So, I am taking imaginary part. So, I get a 3 squared a 3 here a 3 here k n minus 1 k n minus 1. So, I get a 3 squared k n minus squared integral. Now the Green's function is this ok that is Green's function and these are going to give me two correlation functions which one this multiplied by that and this multiplied by this only one there is no three terms only one term because other two terms are 0.

So, if I do that there I will get well I need to also integrate if I integrate the exponentials I will get this term below right integral of exponential will give the pre factor will come below that this is one pre factor from the two correlation functions I will get these guys here and the two correlation functions gives you the give you this and these are for unequal time, but unequal time correlation is written as product of equal time correlation which is c n and c n minus 1 this is c n and this c n minus 1 above. So, this term is the value of this guy this Feynman diagram first Feynman diagram ok and these are number no I mean a 3 is a number k n a minus 1 is a number everything is a number except we do not know what is c n and nu n minus 1, but I substitute Kolmogorov formula ok and for nu n what I derived with RG. So, that is what I will do ok now this is one Feynman diagram, but I need to do it for five more. So, other two diagrams this is from the same diagram I just reproduced this two are I4 and I5.

So, I1 and I2 was for RG. So, I am using I3, I4, I5 for these fluxes and there is another diagram right another set of diagrams for the second term of flux I am going to use at I6, I7, I8. So, there are six integrals six Feynman diagrams ok and denominator is this. They come from the time dependence part of Green's function and correlation function ok. Now, the other diagrams we just follow the same procedure what I just did. So, this is that n, n plus 1, n plus 2 second part of the flux formula and just follow exactly same ideas. So, this is not square this is pi n 2 the second term which is this one n plus 1, n plus 2, n.

So, this is a pre factor vortex and when I do the expansion I get this vortex and. So, I6 I7 I8 ok. Now, I need to sum them up right. So, the six integrals. So, if I just add the all the six integrals I will get the flux ok.

So, let us look at the total this is that integral ok. So, remember the integral at correlation function is correlation function and below is a nu n. So, let me just do it here actually that is a do I have space there well let us do it here. So, the right hand side has cn cn plus 1 let us look at one of the terms divided by nu n k n squared ok and there is a kn squared right.

So, I just look at this term. Now, what is a k dependence of this function this part this is k dependence know. So, kn squared these are cn is in shell model is k n minus 2 kn minus two third not minus five third ok. I described that in the last class I need to divide by kn to get the minus five third this is kn plus 1, but this is a factor b will come ok. So, but this again k kn minus two third right. So, these are numerator k dependence denominator k dependence what is that denominator k dependence.

So, nu n is kn minus four third and the k squared I multiply then I get kn two third. So, what is this k dependence constant right. So, this two will cancel with 2 by 3 times 3. So, this is constant k to the power 0.

So, this will happen for each of the terms. So, every term is constant in k ok. When I sum it up I will get constant in k ok, but each of them will have give some b factors ok. So, that is why you get this b formulas kn plus 1 kn plus 2. So, all that will give you this b. So, numerator is function of b denominator function of b and we get a number, but what is how are we getting this ok.

Now, let us go back again. So, two Kolmogorov two correlation functions. So, two spectrum will give you k Kolmogorov squared right and this viscosity nu n will give you what nu star square root of k Kolmogorov. There is a pre factor if you recall I said this square k Kolmogorov was used to cancel things for RG, but it will sit there. So, overall what is the factor for Kolmogorov k Kolmogorov three half by nu star.

So, this is a pre factor for the flux. So, that is what you get k Kolmogorov three half by nu star ok. So, flux is constant this is a proof that flux is constant by field theory, but I need to do first order I made this diagram which are only two first order we can do two higher order as well, but that will be two complicated diagrams, but that is how is called first order perturbation. I expanded one of the legs within one Green's function, but for example, we had we had one Green's function and I connected like that right, but then I could also expand this as a Green's function. So, I can do one Green's function like that and two legs ok. So, this leg can connect with that and so it is pretty complicated diagrams will come.

So, two Green's function and more correlation functions. So, that I am not doing it here ok. So, so flux is constant, but now we can do it here. By the way epsilon also will cancel know. So, this should be this should have been cancelled. So, right hand side will get epsilon there is a typo there should be epsilon why go back.

So, two correlation function will give what epsilon each of them is epsilon two third. So, they will give epsilon four third epsilon two third into epsilon two third and what is new independence of epsilon. So, new n is velocity times length. So, velocity has epsilon with epsilon one third ok.

So, that will give you epsilon one third below. So, this together will give you epsilon ok. So, this epsilon in the right which I had missed out. So, this epsilon will cancel with that ok. So, what do I get from this algebra? So, this object is a constant function of B of course. So, with that I can get k Kolmogorov three half by nu star is equal to 1 by this whatever this constant is 1 by this constant.

Nu star is known for a given B nu star is around forgive I mean order 0.5. So, from this I can compute Kolmogorov constant correct. So, that is how you compute Kolmogorov constant using field theory and that is what. So, this is numerical simulation that flux is indeed constant and which matches with the field theory prediction ok.

I can compute Kolmogorov constant by same method for different B's and Kolmogorov constant for is basically according to theory it is around 1.7 this number is around 1.7 this is 1 and this is 2. So, it is around 1.7 of course, for small k a small b this is called b pre factor well b is the ratio of k n plus 1 by k n is b ok.

So, it depends on b, but they are between 2 point something to 1.7 or 1.6 ok. But if I do numerical simulation then my numbers are consistently lower the numbers are around 1 or around 0.95 something around 1. So, this is something which not it is not clear of course, there are assumptions made in field theory.

The numerical solution is I expect it to be more more reliable ok. In field theory we made some assumptions and these assumptions are probably not correct not all of them and so, this 1.7 when cell model is Kolmogorov constant is order 1 ok. So, the factor of 1.6 between the two.

So, this is how we compute Kolmogorov constant. Now, these are all non equilibrium solutions, but they are equilibrium solution. In fact, which is quite straightforward. So, the idea is that if I start with random initial condition for the shell model then we checked numerically that it remains random. So, this correlation for any un ok un plus 1 they are all well anyway this should be 0, but even the triple correlation this guy is also 0 identically 0. Look for the non zero shell well I mean if I compute this for non-equivalent model when I force at large scale and decay large form scale this is not 0.

We saw in our simulation this is not 0 this one, but if I start with random initial condition I get flux to be 0. This is that nu equal to 0 Euler equation well Euler equivalent not Euler equivalent flux is 0 for equilibrium solution. So, it turns out for Euler equivalent this guy is give you 0. So, flux is 0 and it is possible because un are all delta correlated they are all white noise basically. So, even triple they remain Gaussian they are remain Gaussian they are not quasi Gaussian.

So, this is this comes in turbulence theory quite often that the variable u's are not Gaussian in equilibrium theory u's are Gaussian. So, they become non Gaussian and there are deviation for the Gaussian and that gives you flux if they remain Gaussian then you will get 0 flux ok. So, these are very important contrast for the flux ok. So, detailed balance is respected this equilibrium solution this is exactly same as thermodynamics. So, given shell model is not that it will always be non equilibrium solution for some parameters it gives equilibrium solution ok.

So, now let us summarize. So, first order self consistent computation of energy flux ok computation has come twice ok we did that and we get non zero values. When I force at large scale and dissipation in small scale a constant energy flux we can compute Kolmogorov constant by this procedure which I outlined and the number is around 1.7 by theory for equilibrium solution the energy flux is 0 ok. So, this is basically coming from the flux calculation which is not RG.

Now, let us summarize the field theory application to two shell model. So, non equilibrium theory right when I have non zero flux a correlation function differs from Green's function right we saw that in our computation both of them are similar form for time dependence, but one of them was correlation function was Green's function was simply this. So, there is a big guy sitting there again ok and of course, these are time dependent when I had I was doing equilibrium theory it was there is no time dependence ok. So, there is that is main there is a major difference if you recall our correlation our Green's function correlation functions for phi-4 theory was this the at least a free Green's function, but if I g naught of k, but if I do include all the terms then this m is changed, but Green's function is basically argue that m is modified, but it is of the same form ok. So, we can see the Green's function is very different I hope you remember this one after the course. The time dependent and if I the decay with this decay rate viscosity we found this which is changing with time energy flux is constant we compute the Kolmogorov constant in the field theory, theoretical predictions are close to numerical results except Kolmogorov constant, but it turns out if I do Navier Stokes equation I get pretty much same convergence.

Shell model has this Kolmogorov constant is lower than DNS that means, lower than what we get for Navier Stokes ok. Two more thing coupling constant remains a coupling constant I should use better English is unchanged under RG and that we do the symmetry and equilibrium solution is zero flux and this nu n is not altered it remains constant. Thank you.