Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Alright. So, we will so far we dealt with equilibrium systems ok. Statistical physics equilibrium we are doing, where we are looking at equilibrium properties of phase transition. I did not do many calculations, but even quantum field theory lot of as long as you assume Green's function and correlation function are equal time is not playing a role. So, then we solve equilibrium ok. So, now, we will go into non equilibrium.

So, here things will be dynamic, things will change in time. So, we start with Langevin equation. So, today I will illustrate one example Langevin equation which is time dependent. So, let us first look at review the properties of equilibrium system which we discussed in this course so far.

So, equilibrium system time does not appear explicitly. So, I if you recall when I was doing phi-4 theory I just turned off d by dt of phi 0 right I had turned this off. So, you are looking at only the equilibrium properties of phi-4 theory. This is useful studying for many things like equilibrium stat mech, a phase transition and some quantum field theory ok, which I did not do lot of examples, but most of lot of things can be done with equilibrium framework. For example, even superconductivity will be equilibrium framework, but if you are looking at conductivity quantum conduction then that will be non equilibrium ok.

So, I think it will become clear when I give some examples. So, for equilibrium system G(k) is C(k) I if I showed it well demonstrated using an example. So, Green's function and correlation functions are same and is not time it is not we do not have frequency dependence on this omega was not there ok. The G(k) or time dependence we do not have it and detail balance is respected. So, energy flux or energy transfer from one to other is 0 ok.

Non-equilibrium systems time will come explicitly ok. So, if I were doing with phi-4 theory then I had to worry about this d by dt of phi. So, this will be like this phi minus phi cube plus Laplace and phi. So, this phi d by dt will be playing a role. So, I had to write phi a function of x well k and omega.

So, wave number k which we used to do before, but the frequency will also come into play ok. So, system will evolve in time a Green's function and correlation functions are functions of time ok functions of time G(k) is not same as C(k). So, in fact, we will have G(k) omega ok. I will give an example in today's class itself when frequency comes and things are different. Detail balance is typically not respected sometimes it may be, but in general it is not energy flux is not equal to 0.

There will be energy transfer across scales or across it could be real space itself. For example, in conduction when heat is flowing from. So, this hot region and cold region heat flows like that know. So, this heat flux and that makes it non-equilibrium right because heat is going from left to right. Higher it will be in equilibrium then what goes from right left to right should also be equal to right to left and they should cancel, but for non-equilibrium there will be a one direction of energy heat flux or point of flux and so on ok.

So, we will deal with I am going to introduce Langevin equation. So, Brownian motion everybody is aware of. So, I will so, Langevin equation describes Brownian motion or motion of a this particle in a background of atoms and molecules. So, these black dots are molecules and this red dot is red block is some dust particle or pollen ok. And please keep in mind that this system is there is no mean mean velocity hydrodynamic velocity.

So, it is not like atmosphere where there is a hydrodynamic velocity. This is at low temperature where things are I mean the water is still and if you put a pollen particle or some light particle it will jiggle around. You can see this movie in Wikipedia it is nice movie it moves slowly. So, we have to see there is a difference the speed of the molecules are very large speed of sound, but speed of this yellow particle is pretty small ok. It is like in fact, it is hydrodynamic velocity, but it is viscous regime is a small velocity ok.

So, how do I model it? So, let us look at let us model in the viscous regime. So, we start with the Navier Stokes equation this is Navier Stokes equation. So, the velocity of this u. So, you think of this u as the velocity of this yellow particle I am making only motivation. So, it is in the background I think I will not say I will take back I will just say that this we have to model particle in a in the background of a viscous fluid ok.

So, there is a derivation with the boundary condition know I mean you might have seen might have seen this derivation a cylinder and there is a flow going around the cylinder then how much is the force acting on it ok. So, this derivation we will not do it here it is too complicated, but I will state the result. So, imagine so, one thing is to keep the body fixed and the flow coming from the left or flow is at rest fluid is at rest and the ball is falling or ball is moving with some velocity. So, what is the drag force acting on this on this particle ok. So, the total drag is the one component is proportional to the velocity v gamma v.

So, this is a linear drag or viscous drag other one is v squared turbulent drag this comes from this part. So, if you ride a bicycle with very high speed then it will be v squared, but if you are riding slowly or very slowly then it is gamma v ok. So, we will stick to the regime where we this approximation is good and we will drop the v squared term. So, the force is mass time gamma u, u is velocity of the particle in the background of the fluid ok. Now, the derivation which I am skipping it is 6 pi mu r u mu is a dynamic viscosity and r is a radius of the sphere, a radius of the particle is treated sphere and that is thing.

So, at least I am sure you are aware of this formula we may not know the derivation, but this formula you may be aware ok. So, we are going to use this formula. Of course, the velocity u is not constant velocity, this velocity u is changing with. So, this particle in the yellow particle in the background will be just getting kicked around like that ok. So, it is random motion, but so, we will have to model the random part as well, but losing the motion the it experiences this drag force.

So, this simple equation first proposed by Langevin models this particle motion quite well ok. So, this part this is acceleration which is proportional to the that viscous acceleration or viscous drag and this zeta is a random kick random force whose average is 0 ok. So, this is a noise which is kicking it, but average of zeta is 0. Sometimes it is from the right to left, left to right, bottom to top, top to bottom. So, it could be in any directions no particular direction a force will be employed.

Time average is 0 yes, time average of zeta is 0 of the force acting on it, but in statistical physics for ease of calculation we assume ensemble average. Now, ensemble average is that you have to create lot of similar systems ok. We have instead of one system where we have this fluid and a particle moving around we have 1 million of them and of course, all of them will behave differently because it is a different system, but we will do average of that and then look at velocity. So, this is ensemble average ok. So, ensemble average makes it makes calculation simpler.

So, this is supposed to be ensemble average. So, if I do ensemble average then the zeta average is 0. So, we will get this and it has a very trivial solution right. It is a linear equation for u t average and the solution is exponential in time. So, this is the average initial velocity e to the power minus gamma t.

So, if this particle slows down, but this slowing down is for short time. This gamma is a so, this viscous force is pretty strong ok. So, so gamma is gives you the inverse time scale right. So, you can look at from here the time scale is 1 by gamma. So, gamma t so, we will assume that gamma is large or time scale is small.

So, kicking is getting kicked very very often ok, but on the whole it has certain drift that is what I am going to derive in a while, but when it gets kicked it will move because of it has been injected some initial energy or the energy during the kick, but because of viscous drag it is going to slow down. So, this is slowing down because of the viscous drag ok, but then you will get another kick. So, then it will again get bump of speed and again slow down and so on. So, it does not have constant speed, it just keeps getting jump drop jump drop jump drop like that and this for short time, but now we want to see the long time behaviour ok. The long time behaviour I will I what we let us look at the position of the particle.

So, to get the position let us start with this r square, I am not really interested in this isotropic system know it can be going any direction is equally probable. So, we look at r square and d by dt of r square by 2. So, what will that be? So, take the derivative r is a vector. So, r dot r dot dr by dt right.

So, there will be u. So, that is r dot u. You take another derivative, if I take another derivative what will I get? d square by dt of r square by 2. So, r dot u plus r u dot, r dot is u right. So, the first term in the right hand side becomes u square u square right u dot u. The second term is r dot u dot u acceleration and acceleration is minus gamma u plus zeta ok.

So, now what is this guy? This r dot u minus gamma, but I know what is r dot u? It is d by dt of r square by 2 ok. So, you plug that in. So, this is replaced by d by dt of r square by 2 plus r dot zeta. This is it by right hand side.

Now, I do the ensemble average. I do averaging. So, if I do averaging r is changing of course, changing, but changing slowly, but zeta is changing fast. So, if I do average of this what will I get? 0 no because r is changing slowly adiabatic approximation. So, this will give you 0. So, we have two right hand side has only two terms not three and this is a first term here and second term is here and u square I take it to the right hand side.

Well, it is already in the right hand side, but I am taking this to the left hand side ok. So, this will give us how r square changes the time right. It is a second order equation, but it is easily solvable is linear in r square. So, what is the solution? Homogeneous part and a particular part and homogeneous part will be constant plus exponential minus gamma t. So, the two parts two homogeneous solution right second order.

So, that is what I write it here. So, this is exponential part and so, you try e to power m t that is and find m and one m will be 0 other one will be minus gamma and the particular part is. So, we let us try the solution which is r square is linear in t. So, this will give us a constant and this will give 0. So, linear in t satisfies the equation and particular solution is unique. I am just doing it by guessing of course, you can do by other tricks as well, but this is a particular solution.

Just plug it in here, you will find the left hand side will give you u square ok. Now, how do I determine c1 and c2? I get it from the initial condition. Let us assume that it starts from origin. So, at t equal to 0, r square is 0 and also I need another initial condition. So, let us assume that d by dt of r square is also 0 ok.

So, I will not do the algebra, if this please put these two initial conditions and that will give you the solution like that ok. So, this is the exact solution of that equation for the initial condition given here ok. At t equal to 0, you can see that this becomes 0 and this becomes 0. So, r square is 0. So, initial condition is respected, but of course, I have to do that derivative as well more algebra, but you can verify.

Now, for short time I can what short time what happens? So, which will short time is basically pretty uninteresting right. I mean this is too much jiggle, I am looking for long term behavior ok. So, short time I can expand this part 1 minus gamma t and then what happens? This will be 0. So, you go to the next order term gamma square t square by 2. So, that will basically give you t square and what does what does it give you? What does it mean r square is going as t square right.

If I do this, then this will cancel with this this and I will get t square. So, it is like a ballastic right a short time before the kicks, it is going with constant velocity for short times, but if you go for t bigger than gamma 1 by gamma, then the kicks are trying to negate is ballastic motion ok. Is that clear? I mean for short time it is going with some velocity before it gets another kick. So, when this is going nicely happily, but this is for poor guy, he just gets boom when atom comes and hits it, then it gets what happened then it moves in some other direction, but on the whole for very long time. So, t much bigger than gamma, I am looking for other limit t much bigger than 1 by gamma.

So, this is going to 0. So, and t is very large, so 1 can be ignored. So, this is the term which is nonzero. So, r square is so this 2 I am taking the right hand side 2 square by gamma 1 gamma cancels with gamma square. So, this is the thing. So, r square is proportional to t looks good know, does it are you happy with this? This is a random walker right r goes square root t in time t.

Had it been ballistic it would have gone straight t square, but because of this collisions it is losing direction once in a while. Well, very often actually not once in a while, but on the whole it is drifting away from the origin. So, r square is proportional to t and this is a proportionally constant. Now, u squared is a in a heat bath. So, this fluid is at a constant temperature and there are lot of molecules.

So, this our particle will be in heat by thermal equilibrium. Of course, temperature of the this this blob is same as temperature of the molecules right that is equilibrium that is what it means. Temperature will become equal if this particle was too hot then it will transmit to the particle molecules and vice versa. So, its temperature is t. Well, it is a single particle what do you mean temperature is a single particle, but idea is that half m v squared of this particle is half k B T, but it is three dimensional.

So, v x squared plus v y squared plus v z squared in the left hand side so I have to put 3 here. So, m v squared is 3 k B T. So, u squared is 3 k B T by m. So, that is what I put it here.

So, 3 times 2 is 6 k B T by gamma m. k B is a Boltzmann constant. Now, k B you may know that k B is Rydberg constant by Avogadro number and gamma m is the viscous force I can use 6 pi eta r, r is radius of this particle. So, put that 6 is cancelling with 6. The numbers we should not worry too much about these pre factors, but irrespective 6 is cancelling with 6 here and we get this formula. So, r squared is proportional to T and this coefficient is called diffusion coefficient.

So, diffusion coefficient is proportional to temperature and is inversely proportional to the viscosity. So, this is the formula and who came up with this formula first? Who derived this formula first? Einstein. So, he is 1905 paper one of the famous papers that is why it is called Einstein relation. Now, Einstein used this formula to get Avogadro number. So, how do I get Avogadro number? So, this of course, he used an experiment.

He did not do the experiment himself. So, the experiment D was reported, R was known for thermodynamics, temperature was known, eta was known for water, r estimated the molecule radius. So, n can be obtained from from this. So, from the experimental data n was determined. He was off slightly, but he was quite on dot. So, this how the Avogadro number was computed for the first time and this is a good formula.

I mean this is a very powerful formula. So, you can get diffusion coefficient from microscopic quantities. So, this is Einstein relation. So, in electric field, so the equation will be changed a bit. So, instead of this random force we put a constant electric field and this is also very easy solution. For short time the homogeneous solution will be important, but for long time only the particular solution will be important and particular solution will come by u will be constant by equating these two terms.

So, by equate that two term you will get ud is equal to e by m e by gamma and this is the drift speed. So, electric field, so compute conductivity you use this model. I think Drude's formula something like that and this mu is called mobility, the coefficient in front of e. So, the drift velocity is proportional to e and the proportionally constant is called mobility and mobility is connected with d right.

So, d is k B T by gamma m I derived in the previous slide. So, I just adjust these parameters and so on. So, diffusion coefficient is k B T by e electric charge e multiplied by mobility. So, we can do quite a bit of stat mech by this simple model Langevin equation. Looks interesting and looks very promising as well and this equation is non equilibrium.

So, in fact, that leads to diffusion. So, if you have a bunch of particles near the origin then these particles. So, instead of one particle you put lot of particles, but then what will happen to this particle because of each guy will drift, but they will go in different directions. Some of will of course, try to come back, but on the whole you will find that the this particle would be spread out. You can solve it in fact, the diffusion equation is a good model for this process and here it is changing with time.

This density is changing with time n is the density of particles. So, this is non equilibrium. Now, the question is, what about the atmosphere in this room? Can we model particle in this room with some of it is invisible, but can we model particles in this room using Langevin equation? Great, yes. So, there is a nonlinearity that makes a diffusion equation not applicable. So, we assume in the beginning recall this particle was assumed to be viscous moving in a still fluid. The background was only the molecules which were randomly moving, but molecules in this room is not randomly moving.

So, there is a random motion, but there is certain hydrodynamic motion because of AC and our breathing. So, it turns out this is a hydrodynamic motion and the thermal motion is 300 meter per second, but the hydrodynamic motion is possibly around 1 meter per second much slower is averaged out velocity. So, it is a collective motion of the molecules because of various factors in this room. So, there are in fact this room is turbulent, there in this room is turbulent and the assumption breaks down.

So, that is not going to work. So, Einstein relation does not work in this environment. So, we will get to that bit later and that formula is not applicable here. So, in atmosphere D from Einstein relation is not valid and the D for this room is this well actually I have no epsilon, epsilon is the energy supply rate. So, we will discuss turbulence we will come to that. So, in turbulence we have in this room there is energy supply by the air condition and our body you know heat.

So, that energy supply to the to the atmospheric gas here that epsilon is coming here and so this diffusion coefficient is function of L, function of the size of the vortex. It is not constant like Einstein relation, this one diffusion coefficient. It turns out the diffusion coefficient will change with L. So, this is the formula which we will derive later in the course and so it is not it is more complicated and the diffusion r square, r square was proportional to t in Langevin equation, but for atmosphere it is not t.

It is we know that and we can derive it is t cube. So, this requires more work. So, the derivation of we need to bring in hydrodynamics and try to derive this relation.