Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

Week - 08

Lecture – 46

So, we will do bit of this symmetry breaking is a very important concept in particle physics as well as in condense matter physics. In fact, all sorts of physics we will symmetry breaking. So, I will explain what that is and Goldstone mode. So, this is what I will have three modules, but we start with this this module first. So, discrete symmetry. So, there are many examples, but the one which I have done so far I will take that example which is Ising spin.

So, Ising spin Hamiltonian is without magnetic field. If there is no external magnetic field then it is minus j Si Sj, j is a coupling constant which is coupling between two spins right. So, we have spins on a lattice and this spin interacts with neighbors. So, this spin interacts with four neighbors this is.

So, we compute total energy by summing over all spins, but interaction is only with the nearest neighbors. So, that is our Hamiltonian. Now, this is Hamiltonian is a scalar. Now, you can see that this object S i S j take value plus and minus 1 right or right it is up spin or down spin. Now, you can see that it is symmetric under if we change the spin to all I mean change the sign of a spin of every all the spins then the Hamiltonian remains unchanged right.

Of course, we have to change sign of all spins. So, symmetric under S i to minus S i right. So, it is called up down symmetry. So, that means, if I have Ising system where if spins some are up and some are down then you compute energy. Now, we just reverse the roles down becomes up and up becomes down the energy will still be the same right because this product is symmetric under my S to minus S ok.

So, Hamiltonian is symmetric, but what about the solution? Is the solution symmetric under S to minus S? No because ferromagnet in this either they are under after phase transition either they are up or down right all of them are up or all of them are down. That means, the symmetry is broken right. So, the symmetry is broken in a ferromagnetic solution of course, paramagnetic solution is symmetric I mean half are up half are down, but in a Ising spin the ferromagnetic solution has this symmetry. Symmetry is broken ok is that clear. So, this is called symmetry breaking your Hamiltonian respects symmetry, but solution does not of course, this is a discrete symmetry it is not a continuous symmetry because it is either 1 or minus 1 right there is no in between.

So, it is a discrete symmetry breaking ok. So, this is a simple example of course, there are many many examples, but like so there are this is a up down symmetry or you can have parity. Parity is always plus or minus 1 no. So, that that kind of aspects, but now let us look another example which is again discrete symmetry it is does not it is not continuous yet, but I this example is very important example. So, you take a scalar field phi and I write down the Lagrangian.

So, this Lagrangian is a relativistic Lagrangian ok. So, let us stick to relativistic. So, this means d by dt of phi squared minus grad phi squared right this is Lorentz scalar ok. So, we will do a Higgs mechanism where we need this done for relativistic. So, I am just going to keep that.

So, and U is a potential with this function of phi and we take phi 4 is phi squared plus phi 4 right. So, this we have seen it for statistical physics example we did full theory renormalization group. Now, is this Lagrangian symmetric under phi going to minus phi yes symmetric no phi going to minus phi they are all quadratic or quartic. So, that unchanged. So, Lagrangian is symmetric, but what about the solution? Now, look I am also telling you that we are looking for a asymptotic solution we are not looking at time dependent solution when the system is reached equilibrium and we are looking at that state.

So, the potential so, we will look for constant solution ok. So, I can write down equation of motion and this equation of motion will have set d by dt to 0 ok. So, this is the potential. So, where will it like to go the system will go to minima. So, for constant solution this derivatives will go to 0 right and we set not putting and with it is not the u phi to 0, but derivative of u phi with phi is 0 minima ok.

So, we want to set du by d phi to be 0 ok. So, take the derivative of this. So, what is the derivative of du by d phi? So, mu square phi plus lambda factorial 4 is 24 lambda derivative of phi to the power 4 will be 4 phi cube. So, this becomes 6. So, this is the solution the equation not yet the solution.

So, now what is the solution for this equation phi equal to 0 and phi squared equal to minus mu squared by lambda or phi equal to square root of this plus minus ok. So, this we have done it this is the phase transition Landau theory. So, when mu squared is positive then both these terms are positive, but lambda is always positive ok we assume lambda to be positive. So, u phi is for both mu squared and lambda positive this is the potential, but when mu squared become negative then it becomes inverted parabola and, but of course, for very large phi it should go to infinity plus infinity. So, we get this and for this case the stable solution is this ok for the left case stable solution is phi equal to 0.

In fact, the only one solution when is it mu squared greater than 0, but for mu squared less than 0 there are 3 solutions this one this one and this one, but among them which is stable or which are stable the one at the lower minima right they are stable. In fact, you can prove it these unstable ok. So, this proof I will not do, but you can do this stability analysis these solution is stable for mu squared negative, but this is phi equal to 0 is unstable and that is the theory of Landau transition ok. So, these are now what does this example tell us my Lagrangian is symmetric under phi going to minus phi, but my solution is not ok, but these again a discrete symmetry you are changing the sign ok. Now, let us look at continuous symmetry now.

So, here I need to make this expand this field phi ok. So, this is a picture of what I want to do. So, I take 2 scalar fields phi 1 and phi 2. So, so again Lagrangian has the time derivative component and the space derivative component, but we set it to 0 right we are looking for constant solution and u phi. So, x axis is phi 1 and y axis is phi 2 ok.

So, earlier I was plotting phi is a axis, but now I have 2 fields. So, I plot phi 1 phi 2 now my potential my Lagrangian is this ok. So, these are the 2 they have 2 fields. So, I write down exactly same form as a scalar, but I add them. Now, my potential u is again similar form, but except instead of phi squared I have phi 1 squared plus phi 2 squared and here it is phi 1 squared plus phi 2 squared ok and I put a minus sign ok.

So, I am following the book by this grifted amateur book. So, this minus sign is put. So, here for mu squared positive we will get a nonzero solution ok, but that is up to you can choose either way. So, what is this how do I get the solution? So, by the way this system is better to use phi I mean basically phi r which is square root of phi 1 squared plus phi 2 squared. This is assume with the symmetry you know.

So, this angle well I should call it theta not y. So, this theta if I choose some angle theta for between this one this my potential is unchanged. So, it is better to use phi r as a variable then my potential is minus half mu squared phi r squared plus 1 by 24 lambda phi r to the power 4 ok. So, the analysis is pretty similar except now phi r is we will get value for nonzero value for phi r and what is the value for phi r or nonzero value? So, that means, the radius. So, you can see this is a hill now you go down and the distance of hill from the minimum of that surface.

So, the surface is like a is called Mexican hat or any hat not any hat, but you in India also you can find this hat now. So, this is a hat where you have this circle for which where it is the function is minimum right that the circle is here. And what is the radius of the circle? The radius of circle is will be finding the minimum of this and phi r will be pretty similar. So, that is what did I get? So, I take the derivative mu squared phi r plus 1 6 lambda phi r cube. So, 6 this is a minus sign 6 mu squared by lambda ok.

So, that is the radius of the circle ok. So, but important point is now it has. So, when mu squared is negative when this is this quantity becomes positive and this becomes positive mu squared is negative will have both the quantities positive and this you do not know Mexican hat anymore it will be paraboloid not paraboloid is some paraboloid will be only this part ok. But it is a pi squared plus phi 4, but it is some function like that which is minimum at 0. But as soon as I make mu squared positive for this example then I get solution which is unstable solution at the hilltop or any point on the circle right.

Any point in the circle is the solution the system can lie there and happily sit there forever agree. There is a solution I mean. So, my solution is independent of this angle theta. In fact, that is a stable solution any point on the circle ok. So, this is called continuous symmetry ok.

So, my solution right now has a well first again before going to that. So, what is the continuous symmetry of the Lagrangian? So, phi r ok. So, this part I would write. So, here this angle theta my Lagrangian is independent of this angle theta right. So, if I write in terms of this angle.

So, phi 1 will be phi r cos theta and phi 2 will be phi r sin theta ok. So, when I sum the squares is become any amount of theta. So, that is a continuous symmetry this Lagrangian is independent of theta ok. And by the way my solution will pick one theta when I look for the asymptotic solution it would not be everywhere it is not a quantum system it is a classical system. It will find some solution some some theta ok that is where it will it will find itself ok.

So, my solution for this case will be phi r and you can choose any angle and theta I may choose theta equal to 0 the stable solution ok. Now, here my symmetry is broken why? Because my solution was my Lagrangian is symmetric under theta transformation ok independent of theta, but my solution is 1 theta ok. So, this is symmetrically ok it is alright. So, this is another example which is given in mechanics books.

So, we keep a vertical rod ok. Now, you apply force at the top if you apply small force it will remain there no problem, but if you apply too much force then it will bend along some direction right. So, that direction will this point the symmetry the vertical rod has symmetry with is equal to symmetry it can be anywhere. Well it is it is not function of phi, but after it is bent it has symmetry is broken ok. So, I hope you understand that this continuous symmetry for the Lagrangian, but the solution breaks the symmetry ok. So, and one more point that if I take a solution at this point somewhere here then if I shift the particle along the circle.

So, this is circle of minimum u and is not looking a circle. So, let us make a slightly better circle. So, if I start from here and I shift it here either change in potential energy no right this is the same potential. So, here in fact, there is when I go along the azimuthal direction there is no change in energy. So, you can equivalently write you can write down the Lagrangian in fact, by the way in terms of the new variables and here for this part ok.

So, I must say that ok. So, for this part mass is 0. So, which is a mass term here in in our Lagrangian? So, I think it is important to make this emphasis this is a mass term ok. So, mu square phi square the coefficient of phi square is mass term. Now, I can rewrite this equation into the phi r and theta right.

So, I in fact, I rewrote like that. So, you can see that phi r square has a mu square coefficient in front, but theta is a new variable. So, theta square has nothing in front in fact, is 0. So, what is the coefficient in front theta square which is 0? So, that means, the mass corresponding to this azimuthal rotation in the Lagrangian is 0 ok. So, this was 0 mass along the azimuthal direction and this is called Goldstone mode ok breaking a continuous symmetry leads to massless excitation ok. So, my solution is now here and this excitation is massless and this is called Goldstone mode and particle associated with this symmetry is called Goldstone boson.

It has a 0 mass that is what you should remember it is no mass like photon, photon has a 0 mass ok. Now, let us look at another field which in fact, we need it for our later slides complex scalar field. Right now what I should have 2 fields phi 1 phi 2, but I am going to combine them and I say well let us make a field psi and psi dagger is for one component object will be just complex conjugate right. So, instead of phi 1 this is phi 1 square plus phi 2 square in the past, but now I just write a psi star psi, but in practice we write a psi dagger psi ok. So, it could be multi component right I mean it could be a vector.

So, psi dagger psi and we can also generalize it easily to fields rather not complex quantum fields ok. So, this is a pretty similar except now this is psi another is a psi dagger. Now, complex field has psi real plus psi I psi imaginary ok. So, these are the 2 components you may think or you can say psi and psi star you do not need to write this psi real plus psi imaginary psi and psi star are 2 independent fields. In fact, we done that kind of thing in the past when we derive the equation of motion we can use derivative with psi star and get equation for psi ok.

So, either you can treat psi r and psi imaginary as independent variables which is more complicated I mean you have to rewrite this is psi real plus psi imaginary or treat psi and psi star as a independent variables ok. Now, what is the solution for this object? So, first this Lagrangian is symmetric under this transformation psi exponential i alpha yes. Because look at this term. So, we will get first psi e to the power i alpha and what about psi dagger I get psi star e to the power minus psi dagger e to the power minus I alpha. So, e to the power minus alpha e to the I alpha will cancel alpha is a function alpha is function of x, but it is a it is a single object function not a matrix ok.

So, they will cancel similarly here also this is symmetric under this transformation then this also will be symmetric what about these guys you can look at it here. So, del mu psi e to the power i alpha. So, it will become del mu psi e to the power i alpha plus now mu will act on that psi e to the power i alpha del mu alpha ok. So, what do I get it the i alpha will be there and I get del mu psi plus del mu alpha ok there this part is problem ok. Now, so alpha has to be constant for this case alpha has to be constant because this will respond the symmetry.

So, I think let us start this part from scratch from beginning. So, let us take this Lagrangian which is generalization of last example not generalization it is a re rewording I mean change a variable here it is psi dagger psi and psi dagger psi squared. So, this Lagrangian is symmetric under this transformation, but alpha is a number ok. So, you can see that psi dagger will be e to the power minus alpha and this is e to the power alpha they will cancel if you rotate by an angle of the every field all the fields. So, it is a global transformation ok and that is symmetric, but what about the solution whether this alpha is like that azimuthal angle know the with example of theta I was giving you is similar to that that angle, but solution will be well this is the our potential is very in fact is identical to what the Mexican had what we got there.

So, here I will have to write psi real and psi imaginary ok and my minimum will be on a circle and whose radius will be square root mu square by lambda ok what is square square root 6 mu square by lambda ok. So, that is identical I do not need to do that in detail, but we get similar results for this case. Thank you.