Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Lecture – 40

Okay, so far we were discussing renormalization group of spin system, right, phi-4 theory, renormalization of phi-4 theory and I have computed, I basically showed you how the two parameters R0 and U0, they were changing with under coarse graining and rescaling, right. I did that before the, in the last class. Okay, now this is a parameter R0 and U0 are changing under this RG operations. Now we are going to look at fixed point that means what are the values when, well, I will show you what is, what is meant by fixed point. Okay, so this is what is called RG fixed point and this was first discovered or initiated by Wilson and Fisher. And this is taken from this material by Kopietz.

So, just to recap, we were working with phi-4 theory where the free energy is this. Okay, so the parameters, phi is the spin field where R0 is the parameter, C0 is the parameter and U0 is the parameter. And I showed you how R0 and U0 change under two operations, one was coarse graining, so we averaged over spins, right. And then when we averaged over spins and these blocks became big blocks, then we have to rescale it so that it looks like original spin system.

Okay, but of course in this process R0 became R prime and C, C0 was unchanged. Please keep in mind the C0 was unchanged but U0 also got changed. And what were the formulas? The formulas for R was, R prime is after the operation, coarse graining and rescaling. Earlier was R0, okay. So the correction is sitting here.

Under coarse graining and this B squared came because of rescaling. Okay, and Zb you can ignore it for this course, Zb is 1. And U prime, the coupling constant for phi-4 theory, it started from U0 but then became U prime and this is because of rescaling but this is a correction due to coarse graining. Okay. So please remember the B was a parameter, no.

So the block size was enhanced by factor B. So my earlier spins were like that but then I took B blocks, size of B block. So let us say our initial size was delta, then under coarse graining it became B times delta. So this was B times delta.

Okay. So B was the scaling parameter, coarse graining parameter. Okay. So it turns out that in field theory, instead of using B, we use this formula, e to the power L. Okay. B is greater than 1, right.

The example we did was, D was 4. The block size was, blocking was 16, 4 by 4 into D. But here, if you use B is equal to e to the power L, L will be greater than 0. L equal to 0 corresponds to B equal to 1 which is identity operation, no coarse graining.

Okay. So L is greater than 0. Okay. So in fact, we see that this equation become bit better with L. So instead of B, we use L variable and Zb is 1.

Okay. So I am going to write a differential equation. Okay. So basically, I am going to say R prime minus R0. So it went from R0 to R prime.

Okay. So that is my, I am going to write this is my change in variation divided by B. Or B is e to the power L. Okay. So that is a subtraction way I am doing.

Okay. So Rl, so please remember that instead of B, I am using l. So this is algebra which I am skipping right now. So this, this implies R prime minus R0 divided by Dl. Okay. And I am using l to be small.

Okay. So then this becomes dRl by Dl. Is that clear? Okay. So this is a differential equation now. And the right-hand side will be 2 minus eta L and dRl like that. Eta is, okay, so right now, okay, so this is what we get.

Zl is 1 and B is e to the power l. B square will be e to the power 2l. Eta l is 1. Eta l is 0 right now. Eta l, this is, this is 0 and Zl is 1.

Ignore eta l for time being. Okay. I mean, these are fairly more complicated stuff which we will not do in this course. And same thing for UL, we get this equation. Skipping algebra, this is what we get.

Okay. Eta l is 0 and Zl is 1. Now I make another change of variable. Still this equation look reasonably complicated. We make another change of variable.

Rl is converted to Rl bar. Okay. Which includes C naught and lambda naught 0. Lambda 0, what was lambda 0? It was the highest wave number, highest wave number. And then we brought it down to lambda which was lambda 0 by B.

You said, okay, you remember this one? So this was lambda 0 and this is lambda. And lambda 0 by lambda is B. Okay. So we make another change of variable. Now I am not going to do the algebra.

Then the equations appear like that. Okay. This is my RG equation. These are called renormalization group equations. So how RL bar is changing under RG operation which was combination of coarse graining and rescaling.

Okay. So what is the correction under this operation? If I was not doing, well basically these are the important contributions. These are the corrections. Okay. So this is the differential equation. Now I would like to study what is the, I will study the properties of the differential equation.

And those properties will tell us about the properties of phase transition. Okay. Now that is a crucial part. That is why we will, for which Wilson got the Nobel Prize. So from this we can say something about the phase transition.

In fact, very interesting aspects of the, about the phase transition. Okay. So first thing is, so if you have studied dynamical systems, we will look for the fixed point of this equation. What is the fixed point? Fixed point is when, fixed points are points where if you are sitting at those points, you will not, you will remain there forever.

Okay. That is a fixed point. Now my definition, so if I am staying at that point, I will remain there forever. So what is the condition I should impose on this equation to obtain the fixed points? I should set the derivatives to be 0. Right. So if I set dl, dRl bar to 0, then Rl is not changing. So you set both these equal to 0 and that will give you the fixed point solution.

So look for Rl bar and Ul bar, for which these two equations are 0. Okay. It turns out there are two fixed points.

Okay. And that is what we will study. In fact, one fixed point you can easily guess. What is one fixed point? 0, 0 is a fixed point. No? If Rl is 0 and Ul is 0, then it will remain there forever.

Okay. So that is a fixed point. And please remember one more point. Okay. Now since we are doing lot of complicated stuff, R was, R0 was T minus, small t, which is T minus TC. What is T minus, what is T is temperature and TC is a critical temperature.

Okay. There is a proportionality constant. You may put some constant here, alpha. But it is proportional to T minus TC. So T is a temperature. So what do I do when I go from paramagnet to ferromagnet? I decrease the temperature slowly, slowly, slowly, slowly.

I reach the critical temperature. That is a fixed, that is a transition point. And I decrease it even further, then I reach ferromagnet. So T equal to 0 or R equal to 0 means I am at the fixed point.

I am at the transition point. Okay. You have to just follow the thread somewhat. So this is one fixed point and U being 0 is the non-linear term is basically, well I am sitting at a point where non-linear term is not playing much role. If I look, by the way this is a, well we are going to go around the fixed point.

Okay. And we will study the properties. So these are, there is one fixed point and the other fixed point is slightly non-trivial. Okay. So let us study those two fixed points. One is called Gaussian fixed point, which is what I said 0, 0.

Okay. And as I said, this is, in fact you can just have a look and you find that 0, 0 is a solution. Okay. This is two, two solutions and we choose a parameter called epsilon, which is defined as 4 minus d.

Okay. So I am going to work with d greater than 4. d is a dimension. Okay. So in the integral you have K to the power d. So d is a dimension of space. So we live in three-dimensional space or if you are working with 2d spins and it is 2d space, d equal to 2 or it could be four-dimensional space.

In field theory you are allowed to go to any, any dimensions. Okay. And four-dimension has very interesting properties as you see right now. So epsilon is 4 minus d.

If d is greater than 4 then we get epsilon is negative. Okay. So now here I linearize, linearize around fixed point. So this is standard thing in nonlinear dynamics.

Okay. Linearize means I make a perturbation of Ul. So I make this thing Ul minus Ul bar star. Okay. So this is a fixed point, Ul star and this is my small parameter.

So I call it delta Ul. But right now Ul star is 0. So my delta Ul is same as Ul. Okay. Is that clear? So this is what we do. So here, so if I do this, so look, look at the first equation. So first equation Dl Rl bar, this is the perturbation around the fixed point.

By the way my fixed point is here, 0, 0. I make this as x axis is R bar, y axis is U bar. Okay. Now I am here, I, I go slightly beyond the fixed point.

Okay. Now I do linear perturbation. So I am going to drop the nonlinear terms. So here is going to be 2 Rl bar, half Ul bar and 1, 1 divided by the way this is a nonlinear term, no? Nonlinear, correct? I mean this. So how do I linearize? I take this to 1 plus Rl to the power minus 1 and we do Taylor expansion. If I do Taylor expansion, what will this become? This term is 1 minus Rl bar.

So Ul Rl bar is nonlinear, right? Your product is nonlinear. So this is approximately 2 Rl bar plus half Ul bar. Okay. So this is what do I keep? This is valid if you are close to the fixed point.

Okay. What about the second equation? Dl bar, Ul bar. So first term is epsilon, 4 minus 2 is epsilon, epsilon Ul bar. Second term, what happens to second term? Is quadratic and below if I do Taylor expansion, I am going to get, I cannot get something below quadratic, correct? So the second term is nonlinear already. It cannot become linear.

There is no linear contribution, so this is 0. Okay. So this is my linearized equation. So we can write down the linearized equation in a matrix form. That is my next object, right? How do I write this matrix form? So I write Dl Rl bar Ul bar equal to Rl bar Ul bar. Now what are the matrix coefficients? So first will be 2, right? 2 Rl bar and half.

Below 1, 0 and epsilon. So this is my matrix. Now what is next? So given a matrix, you find the eigenvalues. Okay, that is, this is very standard operation, linear algebra, eigenvalues and eigenvectors. Okay. Now some simple properties which I will just state it right now, but then we will, I will not compute the eigenvalues by hand, but I will just tell you the result.

So this is my fixed point. Now this 2 by 2 matrix, in general, it will have 2 eigenvalues and 2 eigenvectors. Sometimes eigenvectors become 1, okay, not 2. But always it has 2 eigenvalues. It could be degenerate or non-degenerate.

In fact, you can guess the eigenvalues here. What is the eigenvalue for this case? Some quick, weighted mathematicians. So the trace is 2 plus epsilon and determinant is 2 epsilon. So the eigenvalues are 2 and epsilon. It is in fact very straightforward. So eigenvalues are 2 and epsilon and the eigenvectors we need to check, we need to compute.

Now the simple properties, if the eigenvalue is right now, epsilon is negative, right, this epsilon is negative and so one eigenvalue is greater than 0 and one eigenvalue is less than 0. Okay. So let us say, so it turns out if it is positive, then what is the solution for the positive eigenvalues? In fact, this is what we were discussing sometime back in the lab. So x dot is alpha x, x prime. Along the eigendirection, alpha is the eigenvalue, lambda, let us put lambda eigenvalue.

So here if lambda is positive, then it will grow, right, x will grow. In fact, x will go as e to the power lambda t x0. If lambda is negative, then x will shrink. So here this, I start from somewhere, it will go towards the fixed point. If lambda is positive, then it will go away from the fixed point.

So one is unstable direction. If lambda positive, then it is unstable direction for the eigenvector and if lambda is negative, then it is a stable direction. Okay. So this is a standard non-linear dynamics language. You might have done this in your, in your classical physics, I believe, I do not know.

So this is how we will do it for this. It is exactly same language. So I am just going to tell you the result. Okay. So the two eigenvalues, 2 and epsilon, for 2, the eigenvector is 1, 0. So this is my Rl direction and this is my Ul direction.

1, 0 will be horizontal, no? x component is 1 and y component is 0. So this is the one eigenvector, this one, corresponding to 1, 0. This is eigenvector in both the direction. I mean 1, 0 is a one of the eigenvectors.

One of the vectors you can put, instead of 1, you can put alpha. Alpha can be any number. That will satisfy the, that will basically satisfy the equation, eigenvalue equation. Okay. Eigenvector, so this eigenvector and what is the, is it stable direction or unstable direction? Eigenvalue is 2.

So it is a unstable direction. It is going to flow outward. See if I start from here, it will grow. So my, if I, so here Ul is 0 by the way, right? Ul is 0. So any Rl I start with, Rl will keep growing. Okay. So that means if Rl is positive, that means temperature is greater than TC, then temperature will keep growing.

So it becomes more and more paramagnet. Okay. I mean more and more disorder type. Okay. If Rl is negative, it will become more and more negative. That means it will become more and more ordered. Because this is what you can interpret.

So this is the direction. Other direction is epsilon which is negative. So this is a stable direction. These are eigenvector is this. Okay.

So I did it by kind of visualization and the eigendirection is here. In fact, it is correct. Epsilon is negative. So y is positive, but x is negative. Okay. Here, so if I start from somewhere here, it is going to go towards 0.

So what is, what does it tell you? That Ul, whatever I start with, it is going to go to 0. That means the nonlinear term is getting more and more unimportant. Ul remember was a nonlinear coefficient of the nonlinear term.

Right? Ul phi 4. So nonlinear term we can ignore if you are near this fixed point. Okay. So what does it tell us? That for d greater than 0, if my dimension is 4, greater than 4, then I can work with a linear problem. Okay. And I can get the property from the linear problem.

Well, you need some nonlinearity to begin with. Otherwise, you cannot get a phase transition. Okay. Phase transition requires nonlinearity. But it becomes less and less important.

Okay. So that is what it tells you. And it turns out that this is a Landau theory. Landau, basically what does this idea tell us? That using nonlinearity you can get the, the magnetization mean field theory. Okay. You might have done mean field theory instead of my mean. The value of the mean field theory will remain the same, will remain the value for the magnetization.

There will no be, there will not be any fluctuation. Fluctuations will become unimportant. They will become very small in d greater than 4. Okay. So we use a nonlinearity to get basically the magnetization mean value.

And we did that mean field theory much before. Okay. And you can just look at standard Landau theory for mean, for magnetization.

That is what we solve. Okay. In fact, it is just phi minus phi cube, right, phi. So this was, we set it to 0. And that is our, that is solution phi bar equal to 0 and phi bar equal to plus minus 1.

So it comes from Landau theory. And that will be, it is the correct theory. And it has certain properties. Okay. We will, I will tell you the some properties bit later. Okay. So this is a trivial fixed point. It is called Gaussian fixed point.

Here the fluctuations are Gaussian, like in thermodynamics you say, you know, the fluctuations are Gaussian. So basically we get temperature as the mean temperature and the fluctuation around the mean temperature is negligible. Well, it is Gaussian so it decays very quickly.

Okay. So this is Gaussian approximation. Now let us go to non-Gaussian fixed point.

Okay. So this is a non-trivial one for which Wilson got the Nobel Prize. Okay. So this is, and this is approximate solution. Okay. This is not exact solution, but I think that is what they did in their work and they got the, so I will tell you how to get this.

So we again get the fixed point. So this is non-Gaussian fixed point. But we will assume d less than 4. But I am going to again use the same variable d epsilon is 4 minus d. But now since d is less than 4, epsilon is positive. Okay. So as a result, what happens to the fixed point which is at the origin? So the linear equation around origin will give you the exactly same fixed point 0, 0.

0 is the fixed point. And what are the eigenvalues? The solution remains the same 2n epsilon. But epsilon has become positive. So the two eigenvalues, both are positive 2n epsilon. So this fixed point becomes unstable. That means wherever I start around origin, I am just going to be pushed away.

So I am not going to go towards this fixed point. Basically I just go away. So that is why this fixed point is uninteresting actually, unimportant for d less than 4. For d dimension, 3 dimension, you cannot use that fixed point property. So the new property, new fixed point is important and the new fixed point I had to just again obtain by setting this to 0. Now this equation looks pretty complicated if I want to do analytically.

Of course in a computer you can do it easily in Mathematica or numerical Python. So one idea you can look at the first equation.

Okay. So I need to solve it the first equation. So I need to drop the nonlinear term. Okay. So here it starts with second equation.

Second equation is better. Okay. So Ul is not 0, Ul bar, Ul star actually. The fixed point is need not be 0. I am going away from the fixed point. So second equation I see 4 minus epsilon Ul bar equal to 3 by 2 Ul bar squared 1 plus Rl bar squared.

Okay Ul is not 0. So that goes away. So that gives us Ul bar is 2 by 2 epsilon multiplied by 1 plus Rl bar squared. Okay. So I just did it from here.

So now I am going to say well basically Rl bar is not too large. Okay. Not close to 1. So I will basically drop this term. So Ul bar is roughly two-third epsilon.

So Rl bar is much less than 1. That is assumption we make to solve it. That is what Wilson did. I am not doing it.

I recommend that you can do it more carefully and numerically. Okay. So this is this one solution.

It has become non-zero now. Right. But I should put a star here. So these are fixed points. So you should put stars. Okay. Two-third epsilon. Now you can plug that in in the first equation. The first equation gives us what? 4 Rl bar.

Well I am just putting 4 here equal to minus Ul bar divided by 1 plus Rl bar star. Okay. So again Rl bar is much less than 1. So I can drop this. So Rl bar is minus Ul bar star by 4.

So put this one-quarter into two-third epsilon is minus 1 to epsilon. Okay. So this is we do it analytically.

Of course this has done in this paper I think appeared sometime in 1970s. Okay. Late 70s. 1975 or something. Okay. So that is what they did. Now I need to study the properties of the eigenvalues and eigenvector near this fixed point. So these are the these are fixed point is one point in the phase space. My phase space is two-dimension Rl bar and Ul bar.

Okay. So let us so basically it turns out the nonlinear term becomes important for this problem because Ul bar is not 0 anymore and it is important. And I think we argued it before the fluctuations are important for d less than 3, d less than 4. So since fluctuations are important and nonlinear term is significant, what do you do for perturbation theory? So perturbation theory says that nonlinear that perturbation parameter must be small.

Okay. So that is another argument which is made by them is that I am expanding near d equal to 4. Okay. So my so this is called dimensional, dimensional expansion. Okay.

So basically I am varying dimensions but I am close to d equal to 4. Okay. We will see what the tricks, how it is used in in couple of slides later. Okay. So the eigenvectors for the two eigenvalues, well this is a fixed point.

I can linearize it. It is bit of work but you can do it. So this is my vector, this is matrix, sorry this is matrix. It has two eigenvalues, 2 minus epsilon by 3 and minus epsilon. So epsilon is positive.

So what happens to minus epsilon? Negative. So one eigenvalue is negative. And what about this? Epsilon is 4 minus d and d is near 4.

So epsilon is small. Okay. It is like standard epsilon being small. So my epsilon is small. So my eigenvalue is close to 2. Okay. So and my eigenvectors are these.

Okay. So I made a sketch only. I did not do it but I would recommend that some of you can can do this in Python or Mathematica. Okay. One this picture I want to, I already told you about. The 0 0 point has become unstable.

Okay. So it is going to go to the new fixed point and the flow will go to the new fixed point and the new fixed point is here, new fixed point. Okay. It turns out that Ul bar is positive and Rl bar is negative. Okay.

2 minus epsilon by 3. So it, okay, so the, sorry, so that is, that stuff. So epsilon by 6, what was it? Epsilon by 6 and one of them was 2 epsilon by 3. So this 2 epsilon by 3 y direction and this minus epsilon by 6. Okay. It turns out, so the eigen vectors are this line 1 0.

In fact it is similar to what we had before. And the eigen, other eigen vector is along that direction, this one. Okay. Now it turns out 2 minus epsilon by 3 is greater than 1, right, greater than 0.

So this is unstable direction. And this is stable direction. Okay. So what happens to u L bar? This guy is again going to go towards this thing, but u L bar is finite. u L bar is not 0, it is finite. So I basically come to finite u L bar for the new fixed point. And this fixed point will determine the property of phase transition. Okay. I am going to show you in, in a while. Thank you.