Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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So, my this is action S phi. So, how do I compute S phi is so, this e to the power minus beta S sorry beta S, S is integral of free energy for statistical field theory. I need to do the integral d d d d phi. So, you have to go to field theory. So, we have let us say spin variables magnetization. So, spin variables then I will construct the spin Hamiltonian, Hamiltonian will be we constructed Hamiltonian for some well a 5 4 theory. So, it is phi squared in front is let us say curve r then c grad phi squared Hamiltonian this is a density Hamiltonian density.

Then we have a non-linear term which I will write as u phi 4 by factorial 4. So, factorial 4 will I will tell you why we do it. So, that is a notation people use it. So, 4 will cancel with 4 and you get u by 6, the half will cancel with this 2 when I take the derivative and these are normal convention of Hamiltonian density.

If I integrate this Hamiltonian density or free energy, free energy in Hamiltonian density will keep it same interchangeably in this course. So, if I take the sum over all full space then I will get a number Hamiltonian total energy right or if it is free energy F then I take free energy and integrate over this then I will get H. This is a effective action. So, this effective action is coming here. So, when you say e to power minus beta H this H is the total Hamiltonian or total energy now and that is what you studied know and if you are putting in a heat bath and basically total free energy will goes in there.

Had it been isolated system then you put energy otherwise you put a free energy. So, this word I will S 5 will be there. Now, please keep in mind that exponential of a number of an object this must be a non-dimensional number, action is a non-dimensional number. So, we have H but I will divide this by beta right H by k B T, H by k B T is a non-dimensional number exponential of 1 centimeter does not make any sense, but we will make beta is 1. So, all the dimensions are absorbed in C.

So, I am going to put beta is 1 here. So, I will say S 5 is dimensionless. We will need this idea in coming discussion. Now, you may recall that these are the linear terms phi squared

and grad phi squared is a linear term. If I take the derivative to get equation of state or not equation equation of motion then I will get terms linear in phi and linear in derivatives of phi.

So, in fact, we can write down equation for this we did that d by dt in fact, it was double derivative you know double derivative of phi minus Laplacian of phi right. The right hand side was phi this was my equation and r is coming here is in fact, that was m squared phi r is same like r is m squared I wrote m squared mass squared in the previous discussion r is like m r is in fact, m squared I am not sure about the sign is a minus or plus, but we will ignore for time being. This is a linear equation in phi, but what happens to phi 4? If I take derivative of this with phi then I will get phi cube. So, this term will give us plus u by 6 phi cube. Yes, this is a term.

Now, is it linear or non-linear? Phi cube is non-linear and that is going to cause trouble that is what we need to compute. I have total magnetization when phi cube term is present and that is where the problem will start coming in. So, I am going to put this S phi a sum of two parts the free part or linear part and non-linear part. So, we call it this is called free part and this is called interacting part. Now, why is it called interaction? This is so, without phi cube this is a linear this is a free field.

Why is it called free field? Because if I take the Fourier transforms then I can write down equation for each Fourier mode. So, there will be a equation for each Fourier mode which will be d dt minus k squared equal plus phi k equal to R phi k. In fact, I can solve it analytically. Now, what so, this is each phi k is evolving separately there is no interaction each phi k is a wave they are just moving along. But as soon as I put phi cube then what will happen? Did we discuss it in the class? I am not sure.

See if I put a phi cube then it will so, I am doing a Fourier transform of this object left hand side and right hand side. So, Fourier transform of left hand side will give you this. So, Laplacian gives you k squared. What happens Fourier transform of R phi is going to be R phi k. Fourier transform of a product what is Fourier transform of a product? Is a convolution.

So, what will this convolution will be? U is a number. So, is u by 6. So, phi cube will be product of convolution of 3 3 phi k's. So, we write this as phi k 1 phi k 2 phi k 3 such that k 1 plus k 2 plus k 3 equal to k. So, this is the term these are all Fourier.

Now, you can see that this Fourier modes are interacting with each other right phi k 1 phi k 2 phi k 3 is interacting with each other and that is why this is called interaction among the Fourier modes and so, it is a called quartic interaction. So, if I represent this in terms

of waves what will I get? So, this is a wave number k and it is equal to k 1 plus k 2 plus k 3. I am not writing in a vector stuff is k 1 k 2 k 3. So, the four waves are interacting that is called quartic interaction. So, this is interaction term.

Now, connecting these two spins I have one slide. So, right now let us well let me just say this. So, we have spins know sitting on a lattice sides Ising spins like that up down or it could be vector you know. So, that is a Heisenberg spin if it is rotating. Now, we can write down Hamiltonian as h times spin sigma dot B right.

So, h dot sigma minus h dot sigma that is well B dot sigma let us write B dot sigma that is Hamiltonian. Hamiltonian is a so, I do not want to confuse with h which is Hamiltonian and the magnetic field h ok. In our discussion we will involve magnetic field a small h ok. So, B dot sigma let us call it B dot sigma dot product will it be linear or non-linear? If my Hamiltonian is B dot sigma and is it linear term or non-linear term? So, linear term we can ok. So, this is a linear in sigma.

So, there is no interaction among the spins. This is not looking like that this is for fields this is for like spin at the sides. Now, we can bring in interaction among the spins which is J sigma i sigma j sum over i sum over j right that is what you write interactions. Well, it could be local interaction mostly local interactions. Now, this is called interaction term which is between neighbors.

Now, it turns out which we will not do in this course that these interactions lead to term of phi 4 type. So, this corresponds to these guys and these guys have connections ok. Now, which we will not discuss, but non-linearity phi cube is in fact or phi 4 in Hamiltonian is coming from this sigma i sigma j. This is done by lot of stalwarts like the name which will come soon is Ken Wilson and the people who derive kind of phi 4 from here. So, that discussion we will not do.

So, I hope this part is clear that we need interactions for bringing in more complicated or more interesting phenomena. Now, how do I compute partition function with s interaction inside? Now, the linear part is quadratic phi phi k phi minus k right. In fact, when you do this partition function integral d phi exponential minus beta by 2 sum e k phi k phi minus k. This is can be written as a matrix know A A x x transpose A x. This can be written as a thing which I showed in the previous slides.

But when I have interactions which is not just products of 2 variables, but product of 4 variables. Now, this is for phi cube, but phi 4 I did not write it what is for phi 4 I if I want to Fourier transform of phi 4. I have 4 phi's sitting here in Hamiltonian. In the equation of motion I have 1 phi k, but corresponding term in the Hamiltonian is product phi k phi minus

k right phi squared is. So, we can write down the energy Hamiltonian in Fourier space like that.

So, this will be r phi k phi minus k this is the first term. Second term will be this half sitting here. Second term will be c k squared phi k phi minus k and the phi 4 will be involved what plus 4 sums. Well I have I should put a sum here total energy know sum over all. So, it is a here it will be phi k 1 phi k 2 phi k 3 phi k 4 and these are u sitting there and such that k 1 plus k 2 plus k 3 plus k 4 is 0 and I have to sum over all possible case.

So, this involves 4 phi's so, e to the power minus beta well I mean not u the u sitting here u u by da da da.. and there is sum of 4 of this and that cannot be written as x transpose A x right because 4 quadratic will be only 2 phi's these object cannot be written in the form x transpose A x right this cannot be written and that is a big problem. So, computing this partition function with interaction s interaction is like that and so, that is a that is a issue I hope everybody is clear. So, I cannot compute it analytically. So, bringing in interaction is a problem. So, how do I do it and this where the perturbative expansion will come.

So, this e exponential minus S interaction I start putting them as a series sum ok. So, this is Taylor series these are minus sign so, minus 1 to the power nu by nu factorial S interaction to the power nu, nu is going from 0 to infinity it is just Taylor series ok. S interaction is a set these are number of which function of phi, but it is a number. Now, what is this object by the way? Well I am not dividing by z naught, but this is expectation value well if I do the normalization part. So, remember what was d phi any function phi exponential minus S naught phi what is this? This is this n sitting there normalization this is average value of phi ok.

So, this basically average value of s interaction with normalization I have to dump in the normalization. So, now, normalization is basically the inverse of the free partition function everybody is with me or not right. So, this normalization is so, do you was inverse of this thing. So, that is the free partition function this is with no interaction this is the z naught is a partition function with no interaction and this partition function with interaction ok, partition function this full partition function ok. Now, so, z by z naught so, in fact, I was telling you here is a I need to divide by z naught to get this stuff.

So, this thing will keep appearing z by z naught and I am do some bit of hand waving because to put all derivation in the PPT is with lengthy. So, I just wrote that. So, in fact, this n is 1 by Z naught that is what is coming here.