Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

Week - 05

Lecture – 30

We make it non Gaussian, but we assume that it is still in equilibrium and the fields are still Gaussian. So, that is what we will assume that our x variables are Gaussian, which is by the way which is does not work when the system is in non-equilibrium. So, the assumptions would not work, but we are assuming in equilibrium, so that is one major point. If number of x are odd numbers, so x 1, x 2, x 3 then it is 0, for all odd correlations or correlation of odd number of variables is 0. I will quickly go through the proof again. There was one question what if x are same, like for example, if I how do I compute x 4.

So, it is easy assume that these are product of 4 x variables. So, 4 x, so you can label them x 1, x 2, x 3, x 4 then apply this. So, what will I get? So, I get x x, x x, but now I have to do all permutations, I am assuming all of them to be unequal and it turns out you will get 3, 3 combination. In fact, I showed you the 3 combination.

So, basically this is 3 x square. So, you can complete well, you can just do it manually, but there are ways to get this number 3 from mathematical formula, but I think we do not want to get too detailed into derivation. So, I showed you all this, I think we will skip this part. So, we follow the same standard technique for derivation that we put a source term, B is a source term, it could be magnetic field. So, apply magnetic field Hamiltonian is magnetic field time spin sigma dot B.

So, this is a source term acting at each for each variable. And then we can, in fact, I told I showed how to compute this analytically and this is a formula, this integral. So, we have A is a matrix, x r is a column, this x is a column vector and x is the x transpose is a row vector. So, this is a number, this is basically Hamiltonian, what we put inside is Hamiltonian. So, e to power minus beta H, H is Hamiltonian.

So, we are going to look at systems where we can write down Hamiltonian and they will be either in the quadratic form or some non-linearity. So, I am going to show you today how to take care of terms which are for example, it could be plus x x x x, four of them, x. So, this quadratic, so we need to do this perturbatively addition terms. So, but this integral

I, in fact, I showed you how to do this, this is the formula we derived it. And if you want to compute the correlation, then we take the derivative with B.

So, this I have done it, this row, this matrix is called density matrix with n the normalization in front. And why normalization is required? Because we want row x dx to be y. If I do not have n, then I would not get it. And this is because, so n is defined that is in front here, inverse of that. And I did not write it here, n is here, inverse of this.

And given density matrix I can compute any quantity, average value of any quantity f x is there. So, I have this density matrix and multiply by f x and integrate. Now, right now I am not doing quantum field theory, so my x is not an operator, x are just variable, classical variable. Now, a derivation of Wick's theorem, I sketched it in the last time. So, this is B transpose x, so this one.

And so I want to compute this quantity. Well, first Z(b), Z(b) I want to average of, this Z(b) is average of exponential b i x i. So, b i x i is a number, because this is sum over all i's. So, high strength convection b i x i is a number, b dot x. Now, this exponential I expand it using Taylor series and I will get all this combination b i x i x i like that.

So, and this object is the correlation here. But to compute this correlation x i, this object the easiest trick which I told several times is to take the derivative of this object and with b i's. Whatever x i, so this is a trick know the source term why do we keep it? We want to take the derivative with the source term and then set source term to 0 or b to 0, this we did it before. So, if b i 1 will bring I x i 1, x i 1 will come out when I take the derivative of this object, x i 2 will come from the next derivative and so on. So, this is what I want to compute.

And of course, I want to show this to be sum of two point correlations and that is the proof of Wick's theorem. Wick's theorem is I can always represent even number of correlations for even number of variables with a is a sum of two point correlation. Now, Z(b) this Z(b) this object, this Z(b) is a this object. So, in fact this is the previous slide know. So, I did show you that exponential minus x transpose A x plus b transpose x.

So, this is what I wrote as Hamiltonian, I integrated this put n out there and this is this object. So, I need to expand basically I need to work with this. I need to take derivative and exponential. Exponential is bit complicated it has many many variables, but that is the proof. Now, let us just to get familiar with this, let us do derivative twice b i 1, b i 2.

I want to compute the second order correlation. So, I denote all of it this object as x. So, including exponential, so this is called x. Instead of writing it again and again is easier to call it x. So, let us do the first derivative first partial derivative x.

So, what will come? So, this is a b i b j know A is a constant matrix A does not have any b. So, if I take the derivative b i 1 there are lot of b i combination right b 1, b 2, b 3, b 4. So, always quadratic two b's, but there are many many b's. So, n C 2 that many terms will be there. So, if I take the derivative here b i 1, so the b i 1 will be taken off because that will be d by d alpha by d alpha is 1 right.

So, that will go away and what will remain is b k, but this is a sum over all case. This basically means sum over all case. I am using Einstein convention right repeated index means sum. Is it clear to everyone? This is repeated index.

So, this is b k. Now, if I take another derivative with b i 2 what will I get? So, it will act on, so x is a function of b and this b is already sitting there. So, first is b i 2 acting on b k will give you the sum. So, you will filter out know this will filter out and I get i 1, i 2. This is trivial. Further option is that I operate this derivative on x and leave the this part as is.

So, this part is as is this one and then I do derivative of x with related to b i 2. So, I will get b i 2 L. So, this is L is a dummy variable b L x. Now, this involves two sums. One is k sum and one is L sum.

It is a very big, I mean this is a number of course, but it is a long series many, many terms. It is n square terms b k b L all case and all else. But now, it is set to b e to 0. So, I want to compute this. So, I need to set b equal to 0.

If I set b equal to 0, then what happens to the second term? 0. So, this term is gone and what happens to x? What happens to x when b equal to 0? Set exponential of minus b b A. So, set b 0 as exponential of 0 is 1. So, this object becomes 1. So, my second order correlation function, this object is nothing but A inverse i 1 i 2 straight forward.

So, this is the second point, two point correlation. The matrix A has all this thing and we just pull that, take the inverse of the matrix A and one of the entries is i 1 i 2. There are many, many numbers and i 1, well we just took i 1 row and i 2 column and that will be the two point correlation. Now, that is what I wrote here. Now, what about three point correlation? How do I show the three point correlation is 0? So, look I already have second derivative here, take another derivative of this function.

If I take a derivative of this, I will get some more b's know, but if I take a derivative of this, I get another b. For each time I take a derivative of x, but this is a constant know, this is a constant no problem, but if you derivative x, I get a b and I need to set b to 0. So, that gives you the triple point correlation to be 0. So, this is 0. So, this all there will be at least one b, there could be more b's, but does not matter more b's is a barrier, you just set b to 0 and you get 0 for odd well triple point correlation.

Now, let us go to higher order correlation. Now, we can just do it for i 1 to i m, m is even, let us assume m to be even. Now, follow the same idea, I want to compute this. So, let us first take one derivative with i 1. If I take i 1 derivative, I showed you in the last slide, this m is A inverse.

So, i 1 x b k x. Now, I want to take the more derivatives. So, let us take with i 2, which I told you already. So, i 2 to i m. So, now, here I take with i 2, then I take with i 3, but so, this is I am going to do my induction.

So, first take with i 2. So, if I take with i 2, then I will basically get rid of this b k and I will get this. In fact, I did in the just previous slide. Of course, there is a term with products b k b l I wrote, but I do not want to compute, I do not want that term to be coming. I am going to basically take the derivative of this with i 3. This is derivative with i 3 and do not will i 2 you push it for later.

So, I take this function and I take derivative of i 2, then i 3 here, with i 4 and da da da da wait till I m. So, I will basically well I will get this two point correlations as the factor as a factor. If I start doing the derivative of the other term, then I am going to do double counting. So, this will avoid double counting. So, this is the term because I am going to set b to 0 for at the end.

So, the term which is I am leaving out will anyway go to 0 that is what is the logic. Now, after this, so, we got this two point correlations out. So, i 1, i 2, i 1, i 3 there m of them already out. Now, this object is very similar to what we started with this object except this has m minus 2 numbers or m minus 2 operation not numbers m minus 2 operations and this is m operations.

So, apply the same idea exactly same idea. So, what will this involve? We can figure out easily that it involve A inverse i 3, i 4 and then m minus 4 thing, then i 3, i 5 then da da da da. So, it will which is go on like that. So, keep getting this two point correlation that each time and until it is over. So, you can imagine that is a sum of products of two point correlation that is idea. So, well the proof just goes on in a inductive manner and that is what it is.

So, I need to of course, keep all possible products for 4 or 6 you can do it by manually, but for higher funds you have to use computers. So, we can write as all pairs p 1, p 2, p 3, p 4 da da da. So, it just you need to do it for all pairs and for odd ones it is 0. I showed you for three point you can argue from here had it been odd then one b will be sitting here and we need to set to 0, at least one b will be there. So, this is for classical fields is a very powerful theorem which we will need it in in maybe not today's lecture, but we will need it in the next lecture.

Now, let us look at quantum fields. So, quantum fields is more complicated I am just going to give a simpler version is indeed very complicated wick's theorem, but I will just give a simpler version which is correct, but things are more I mean I am just going to give a slide bit later where I will. So, we need a time ordered product and a normal ordered product and to explain all that will require more time. So, we will going to skip it the details of it, but I am just going to show you very simpler version. So, we have four operators A, B, C, D and I want to compute the expectation value with vacuum state. Now, this the statement is that this is product vacuum is like that 0 0 0 C D 0 and all combinations.

So, that is a corresponding version and these are all operators that is what I will prove, but I have not made any time ordered or normal ordered stuff here if you have done a field theory then you know that that complication is there which I am not doing it right now. So, let us see how to prove it. So, operator A I write is a combination of creation and analysis operator A and A dagger and we know that A acting on vacuum state is 0, destruction on vacuum is 0 know when there is no state so it is 0 and if I invert it or transpose if I take a transpose A becomes A dagger and vacuum comes to the left so I get 0. So, this basically doing the adjoint.

So, I am going to use this these objects. So, substitute for A, a dagger plus a B C D. Now, so a dagger a this one is in the left one so a dagger acting on this is 0 acting on vacuum this is 0 when this is coming from here so this is nothing but this alright. Now, we write A B C D in terms of commutators. So, A commutator with B C D these are B C D are operators so I did not put this hats well the hats are there but so this is sum of three objects. So, these are theorem you can show it that is [a,b] C D, B[a,c] D and B C [a,d].

Now, what is the vacuum expectation value of this? So, I just put it there. Now, we assume that this object so I will assume that A B 0 0 is a C number is a complex number. So, a and b are commutator these are complex number this assumption I make here. So, this will come out is C number so it comes out and I get C D then here a c is a C number so it will come out and I get B D ad is a C number so it will come out I get this. Now, what is aB in terms of expectation values? So, let us relate a B expectation value.

Now, if I do the expectation value with vacuum aB you will be get aB minus Ba so this one has two term know aB minus Ba 0 0 what happens to this object 0 A A 0 is 0 so this is nothing but aB 0. Your C number and this C number is same as this C number. Now, I already showed you that a B I can replace a because from this this statement know this one well if I put A as a plus a dagger I can always add a dagger in there know the dagger bra 0 is 0 right. So, I can just I use this and add that so basically, I get this. Now, you can see the proof know so this one this a B C number is replaced by this.

So, that is that is about it and that gives us this proof. Now, it is a simpler version of Wick's theorem I just want to kind of cover it right now. I do not think we will have a time to get deeper into Wick's theorem for quantum field theory. The real Wick's theorem is this one. So, the time ordered product of many many field operators A B C D this one is time order product T A B T C D like that and all combination. This is from book QFT for gifted amateurs I just proved from there.

So, this proof will be more complicated because time order product I need to bring in and that will be more terms and you might have seen these symbols. These also involves normal product normal product yeah we keep the a dagger. So, you need to keep the dagger all the creation operators to the right. So, normal product will we are not I did not discuss it so far I discuss only the time ordered product normal product is some more discussion. So, I am just skipping this part and more details we will not do it in this course.

I do not think we will need to compute this stuff in this course. Thank you.