Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

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Lecture – 29

Okay, so we are going to now calculations, diagrammatic field theory. So, these are Feynman diagrams and we will calculate for 5 4 theory. The one, so Klein Gordon I am going to put a perturbation which is 5 to the power 4. So, we have 5 squared, but we put 5 4, but before we go to that we need to get some more theorems. So, there are lot of theorems and the one which I am going to discuss today in 15 minutes is Wick's theorem, same Wick's rotation. So, if you have, if system is equilibrium, then we have m point correlation.

Make m even. So, x 1, x 2, so that, so these are fields or some variables at m points. Now, we can break it up into products of what 2 point correlation. This is a 2 point correlation.

So, in fact, these property of Gaussian variables with variables with probability distribution which is exponential this type thing. So, all these variables satisfy this property. To give an example, let us say I have random variable is 0 mean, mean is 0. So, let us keep that for simplicity 0 mean and I want to compute x, y, z, w. All of them are following Gaussian random probability distribution.

This can be written as sum of products of 2 point correlation. This you may know already. If you have done probability theory course, this is standard thing for in probability theory is written as x, y, z, w, this one. What else? x, z, y, w, then x, w, y, z. There are 3 possibilities.

There is only 3 possibilities and these are classical. No, I am not doing this. I mean, there are no operators which can commutation I am not worrying about. So, that is why statistical field theory is a good point to discuss and quantum is more complicated. We will postpone it for future.

Now these are 4, but we can also do it for 8. So, but you just have to write down all possible way all possible combinations. These are even number of variables, odd number of variables x, y, z. What happens? 0. All odds are 0.

So, there is a property of this kind of random variables and how to prove it? In fact, this is called Wick's theorem and which is used heavily in field theory. Statistical field theory we just need them as a classical variables, but in quantum field theory these are operators x, y, z, w. We need to treat them as a time ordered products and then it is more complicated. So, I have proved it for classical and I am going to state the result for quantum. I need to state it because this is a field theory course, but it has complicated aspects.

A proof is nice. I again saw the proof. This is a book by Kopietz. This is nice book. So, I will tell you the title, but this is a book where I took this derivation from statistical field theory.

So, we had this formula before. So, we have source term here and we wrote like that. We need the source term. For correlation source term is required, but I am going to take derivatives and pull out x standard thing.

If you do not remember much in the course, at least you should remember that part. Now, we put an n normalization which is inverse of this. I basically take it to the left. Of course, I need to invert it and this two will left like that. This we call it Z(b) partition function normalized or is a symmetric matrix and we wrote that this is a density matrix.

So, any function f x, I can compute using density matrix. This is a side remark, but now let us get to Wick's theorem. So, I have this B transpose x. So, it is B i x i. I just expand the exponential like this.

So, we just expand it. Now, I am looking at this Z(b) which is the expectation value of e to the power B i x i which is computed by this rho is the density matrix, probability density matrix. So, this we need that e to the power minus x transpose A x that complicated looking stuff. Remember that was the density matrix. So, this is a n dimensional integral for different x i.

So, there n x i, x 1, x 2, x 3, x n. I want of course, to get this correlation x 1, x 2, x 3 like that. So, I just write this as a, I mean this part is substituted here, exponential I expand it. So, there are infinite series, this infinite series product. So, this m going to from 0 to infinity.

I just substitute it here, no problem. Now, this part is this expectation value of x i, x 1, … like that. So, Z(b) has lot of information. Now, so, Z(b), so, here how do I get this quantity x i 1 dot x i m. So, we have Z(b) which is this quantity with the source term.

So, I think I should write Z(b). Z(b) was exponential integral d n x minus x i A i j x j plus

B i x i. So, I want this x i 1 dot like that. So, Z(b) is I have it here. So, actually you should focus on this one.

So, I want, by the way this is infinite sum. Right now, I have to pull out, pull out this one. So, as I said I take this Z(b). In fact, you can look at it here. I take the derivatives partial with b i 1, partial with b i, b i m.

I have m of them. I just take the derivative of this. So, each time I x i 1, x i 2, x i 3 will come out and we will sit here x i, x i 1, x i 2, x i m. So, that is what we would like and that is what I have it here and we set b equal to 0. So, then I will get the expectation only of x. Now, I need to connect it to the 2-point correlation, product of 2-point correlation.

That is a trick, I mean that is a complicated part. This Z(b), I said it is this, this integral we should said is this. Well, by the way I need to divide this by integral exponential minus x i A ij x j. So, but this part does not do anything to the derivative. So, this part is that this, the integral of this is that.

We we discussed so much, I mean half b i A inverse i j b j and you set b to 0. After taking the derivative, you set B equal to 0. Everyone is with me, it is a bit complicated maths but I said if I want the correlation, then you take the partial derivative with B source terms and then set b to 0. And this has the answer. Now, of course, we need to do it carefully and pull out the 2-point correlations and that is here.

So, this part is, this is not difficult but just you have to follow my set of arguments. We call this as X, capital X. This is the only function of B's and A inverse and B. A is a matrix, A inverse is a matrix. So, I need to take derivative is with that.

So, first I want derivative, I take a derivative of X with this one. So, what will I get? So, this A inverse. So, A inverse, so we have b i and b j. So, I am doing with I, well I should part, so b b i 1, so this is, if I take the derivative with b i 1, so b i 1 will go away, no? So, b is a variable. If I take the derivative, that will give you 1.

So, I get b i 1 k b k because b i 1 in this, in that, in that expansion will go away and of course, this exponential will come here as well. But when I take the derivative, I get A inverse i 1 k b k and this i 1 is sitting here. So, this is a number and I multiply by X. This is basically sum, right? For this, there are n terms in fact, in this. It is n by n matrix, I multiply by B k and I get this.

First part is done. I done only one of them. So, now let us take the other derivatives. Now, I just rewrote this part here. Now, take the other derivatives. So, this is the product, you know, this one and X has a B k sitting there inside.

This X is the exponential part. So, there b sitting inside X and the b sitting here. That is how I am going to get lot of terms. So, this one will involve. So, let us do with i 2. If I do with i 2, I am going to get A inverse i 1 i 2.

So, k will, so in this sum, I put k equal to i 2, then i, the b i, b i 2 will be there. That will get cancelled and I will get A inverse B k into X. Now, right now let us keep non-diagonal. I think it is possible, but no, no, no, you are not, we have no diagonals.

It is a independent variable. So, example I give X, Y, Z, W, they are independent variables. So, product of two different variables, same is not allowe. So, they are all different. So, when k equal to i 2, I will get this and now set b equal to 0.

So, X at b equal to 0 is 1. So, this is the first term. Well, I am sorry. I am not setting X equal to 0 because I need to take the other derivatives. Do not do it right now. I have taken the i 2 derivative and I will get A inverse this one and then this guy will act on X.

But there is one more term which is set to 0. So, here I acted this one here and X was outside. Well, I mean I think this is a slightly complicated, my head is also spinning. A inverse i j b k multiplied by X, I do b i 2. So, b i 2 acts on this. So, this is A inverse i j b k and B k is gone and X will be there plus then b i b i 2 X will come and then A inverse b k will come.

So, when I take the B going to 0, this will go away. So, basically I get this one and the derivative will act on X. So, this product sometimes go away and first time when I do i 2, I get this one and the remaining m minus 2 ones will act on X. Next operation we do with b i 3, I get A inverse i 1 i 3 and then this guys will act on X. So, basically by I reach line 3, I have done two derivatives operations and you can see that we get all the two point correlations are coming out i 1 i 2, i 1 i 3, i 1 i m.

Now, we have m minus 1 derivatives to be computed for other product other objects. So, follow the same thing recursively. So, you just follow the same steps and we keep taking b going to 0 limit where we do not need to do derivatives by anymore. For example, I was saying here that this this guy is going away and so and then you just follow this, then we will get all the products. So, this is a recursive proof that if you do it all this derivatives, then I will get all possible products of two point correlation and is written as sum of all pairs A inverse i p 1 p 2 p 3 p 4 p m minus 1 p m.

So, it is a pretty complicated well it is bit tedious, but if you follow the idea you will you will understand the proof. M derivatives, but then you do two of them then remaining m minus 2 then keep doing it this for all odd even ones odd one is 0. So, we will use this in future work, but this is Wick's theorem for statistical field theory and quantum field theory we will start in the next class. Thank you.