**Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives**

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**Week - 05**

**Lecture – 28**

So, we are going to get deeper into this. So, we need to compute the integrals. And again listen you see that you know quite a bit of what is being done, but it is done in nice way for many variables, these integrals are the standard way to do this integrals and you can get pretty nice interesting results. So, we will go further into how to compute the integrals. So, so far was only formalism I said well the partition function is there, but we need to compute the partition function for many for example, there are many spins either in real space or Fourier space and I need to compute the integrals and that can be quite tricky. So, let us try to use it and also Feynman Path Integral.

Though the formula is pretty nice that it sum over all paths e to power minus s by h bar I sum it up, but summing up it is pretty complicated know. So, but we need to do it if you want to get some quantitative numbers. So, let us do some of these integrals. And as I said field theory involves lots of integrals.

So, that was my remark in very early times and is true there will be lot of integrals. So, let us look review quantum Feynman path integral. So, classically for classical particles we compute action know which is Lagrangian of this particle well q q dot I should have put q q dot. So, position and velocity and we do the compute the action this called action s is action and which is internal L D T from A to B I go from here to there. And what classical physics tells us that the trajectory chosen by classical physics is the one which extremizes s mostly minimum, but sometimes is a maximum as well.

But in quantum mechanics particle does not follow classical trajectory. In fact, there is no particle, but we can get a pretty nice picture of the probability amplitude by following this analogy. So, electron is going from A to B, but we can compute we cannot say which path it goes through, but we can say probability amplitude to go from A at time t 1 to B at time t 2 we can get the amplitude. And this is what is this well what Feynman said that it follows all paths, but each path has certain weight and you sum it up and that will give you the probability amplitude which is the Green's function our connection is the Green's function. So, it is a time dependent Green's function.

So, q 1 t 1 which is the starting time in a starting position to q 2 t 2 and this is the Green's function it is a starting position q A to q B. So, the starting position at a time t A and final position is Q B at time t B, but it follows all possible paths. And here it has been discretized you take various snapshots this trajectory seems pretty vague right I mean a particle is non it is a derivatives are not defined non-differential right I mean this is like Caspier. But these allowed in Feynman's calculation all paths are allowed it should be continuous that is about it there cannot be a jump. And there are ways to think of antiparticle as well, but we are not doing it just think of particles and whether sometimes particle is going back in time well here is all going forward in time, but things can go back in time that is also allowed in path integral.

So this is a so e to power i s by h bar this is the amplitude for a given path probability amplitude and we sum up over all paths. So, this is integral know I mean this this integral and this d q is a functional integral right. So, q is a function well q is basically here this g is function of q and, but q itself is varying. So, remember we had started with the function. So, i is function of f.

So, my integral is function of f which is which can be any function. So, my q is a arbitrary function which can be non-differentiable as well. Anyway, so this is a prescription now point is how to compute this, this requires a pretty complicated integrals it is possible to do it analytically for some standard potentials I am going to show one potential well I am not going to do it, but it is we can find references and you can do that. I am just going to show you how to compute for free particle and mention about the procedure for oscillator. Again you can look at this book Riester and Blundell some more are passed that is a difficult part you need to compute.

So, some integrals now these integrals will come quite often. So, this is a Gaussian integral it is a x squared by 2 a is a real number positive real number and the answer is square root 2 pi by a minus infinity to plus infinity. Now, b x the b is a source term. So, this is a nice so for this 1 d and one variable x. So, b is a source term.

So, I had a integral e to the power minus beta h, h could be x squared plus source term J x that is what I had in the past few slide. So, and this is some more all possible x. So, this is d x. So, for a particle in a thermodynamic particle in a chamber. So, this would be the v x could be velocity half v square and J times v.

Now, we can think of that is a some force connected with velocity. So, x is a variable now. So, what is the integral for this? We can well this is you can bring it to quadratic form you need to bring it this form to change the variable. So, I say x minus B by a whole squared.

So, this part is and of course, I need to correct it because I am adding something here.

So, correction is this add subtract. I am sure you done this, but if you add subtract and bring it to exponential form. So, this part I can use this formula and this part is constant it is not function of x. So, this is straight forward e to power b squared by 2 a square root 2 pi by a. So, this is coming straight from this one this middle part.

Now, let us make it more complicated well I am just rewritten it now x square by 2. So, bx now what we will do is we will make this a is a matrix. So, I need to bring in many variables. So, what should I do? So, let us just get the idea here dx1 dx2. So, we have x1 and x2.

So, here instead of x squared I will have various combination what combinations will I have? Exponential. So, we will have x squared x1 x squared x2 squared plus it is all minus plus x1 x2 and there will be coefficients in front. So, a b c it looks pretty complicated, but you can write this in terms of matrix. It is a quadratic know so quadratic can always be computed into the matrix. What is the trick? We put a matrix a x1 x2 and x1 x2.

Now, I am going to pull this coefficients of a now x1 x2 is symmetric. So, in fact, we know how to do this one what should be a? So, the first diagonal element will be small a this one. So, x1 x1 will become a x squared these guys should be b and what should be the off-diagonal terms? c by 2 c by 2. So, that is how we should write for n variables you can always construct this matrix a it is complicated no doubt, but it is possible to do it and this is going to be symmetric matrix. Now, that is how so if this is a matrix then how do I compute this integral.

So, that is the next slide. So, we write this in terms of matrix x i a i j x j I showed you for 2 by 2, but we can do it. Now, this is proven you can look at the standard probability theory book. This Gaussian is very important probability distribution know. Why is Gaussian important? It turns out if you have n independent variables and real independent variables and they are interacting well I mean, but they are inter independent.

So, well let us say thing independent variables and probability distribution n independent variables is Gaussian. So, that is why the particle distribution for in the thermodynamic gas it is Gaussian because they are not interacting and they are colliding, but they are not that is how it is randomizing, but they are not interacting per say there is no correlation between the particles. So, that is the central limit theorem. So, exponential is a Gaussian is a very important distribution which is comes in all places non interacting typically will be Gaussian. So, this is my these I will see it for n variables.

In fact, we see it soon in our course. Now, this integral well one thing is to diagonalize it right idea is to diagonalize it. So, once you diagonalize it is going to be e to power half x tilde 1 square in diagonal form it will be x 1 tilde square x2 tilde square and sorry lambda Eigen value lambda 1 right that is how diagonal things are I mean this linear comes into play here. So, lambda 1 x1 square plus lambda 2 x2 square and so on and each of them are independent Gaussians. So, we will get if I do the integral I will get lambda 1 lambda 2 lambda 3 all that square root below this A will become lambda 1 lambda 2 lambda 3 and so on.

So, this is what we are getting 2 pi n 2 to the 2 pi to the power n by 2 which is from here and this A is replaced by Eigen values which is determinant of A. So, interesting so n variables we can also compute the integral. Now, we will also encounter cases when we have source term whether this we can write in terms of x transpose A x, x is a vector and A is a matrix. Now, the source term we write as B transpose x. So, this is like this and like that and we just follow the same scheme what we did here then we will get this term is translated to this form is quite nice.

Now, B is B transpose 1 by A is A inverse and this B is B. So, and this determinant A remains as this. Now, we will also encounter situation when we have this integral divide this by minus infinity to infinity d x 1 and d x n exponential minus half x t A x. So, what will I get? I divide this divide by that. So, this this guy will get get cancelled and I will get just this.

So, we will see this kind of stuff. So, this is a full partition function with interaction. So, this guy will be like a source term and this will be free Green's function a free partition function without source. This one is with source and without source. If you do the ratio we will get only this exponential part.

Now, so once you know this formula at least have a reference you can keep it with you and then you can compute lot of integrals. Now, that was for x 1, x 2, x 3 and so on right, but and my A was a matrix. So, that is a discrete variables, but we can have a continuum ok. So, this this is some continuum operator A x y and here we need to find A inverse. So, this is very similar to what we did, but this for continuum A.

You cannot form a matrix, but you have a function A x y. It is possible especially when you have continuum, but it is possible, but we will use discrete for our future discussion. I am going to go to Fourier basis and I am just going to use discrete x 1, x 2, k 1, k 2, k 3 like that typically that is what I will do ok. Now, this is for path integral for oscillator. So, this some tricks which we need it well I am not going to do it, but this in the course I am also going to mention how to do this stuff.

So, this part is quadratic know x squared, but this part is not I mean q dot is a quadratic in dot, but I want to get rid of this dots d by dt. So, this one is solved by doing a integral by parts. So, if you do integral by parts then this is first function integral of second minus integral of second well derivative of the first function integral of second. So, this is standard trick to convert double to two derivative squared to function and double derivative and source term is 0. So, this is now q q right q q and this is an operator linear operator ok.

So, it is a kind of not so convenient looking operator, but it is a linear operator. So, I can use now well basically my operator is this ok. So, remember I was telling you that I cannot write this into the matrix, but is a function A x y that is what I wrote there ok. So, it is a function and there are ways to compute it you can look at this book. So, this derivation is definitely not easy, but if you follow this path well how to find a derivative of this operator ok.

So, now that is what is done there in terms of Eigen values and so on. This is the Green's function or probability amplitude for simple oscillator to go from origin at time t equal to 0 to origin at time t ok. Particle does stuff and comes back for a time t derivation I am not doing it here ok. Anyway this is some ideas about path integral. Now, let us get to some more complicated operators.

Now this is our Klein Gordon field right Lagrangian for Klein Gordon field. So, this my so I am again it is like a partition function ok. It is a action actually we will call it action in quantum mechanics called action. Partition function is integral of e to power i s by h bar plus i plus i s by h bar ok. So, this is the action I exponentiate the action this like ok no sorry this action I am apologies this action this is partition generating function I mean generating function is not called partition function generating function and d phi exponential I am doing this, but with the source g phi.

Now this square again we have derivative square no. So, this is converted to Laplacian to to to derivative operators. Same trick which I did in the earlier slide. So, phi this one phi and this is source term g phi ok. Now, so what we did for oscillator we can follow the same procedure well after this more complicated integral.

And so this Z naught J this is a Mathcal. Mathcal it is written as this divided by I am dividing the so this source term sitting here and I divide the one without source term. And this is the integral which in fact I mentioned to you how to do it by this formulas. So, this part I bring it down and basically B divide by A we are doing B divide by A that a 0 will be more complicated, but basically you do the B divide by A and that is what it is right this exponential B transpose A inverse B that is what we got here. So, this B is this is B B transpose and this is field operator phi.

So, this is B and the inverse of this operator is Green's function right g inverse was. So, we had L g equal to 1 or delta function depends. So, basically g inverse is 1 by L ok. So, 1 by Laplacian this A inverse is coming here which is nothing but the Green's function and delta is the Green's function. Feynman propagator which I derived in the last class ok.

These are Feynman Propagators ok. So, you see that these are underlying math is very similar and it is all in here ok. Now, once we have this then I can compute the correlation functions or Green's function by taking derivative with J I mean discuss this before. So, but the correlation function is the most important thing spin correlations, field correlations and this is for in real space well this 2 point correlation x y and this is n point correlation. Well I am well I am just cut paste here and if you write down this of course, you need to remember this stuff, but ok.

Now, this is a slightly tricky part. Now, in quantum field theory we have integral in space time right we have d t d x cube. So, space time t and x are different, but I want to convert it to Euclidean space where all of them are in the same footing. So, we had discussed before that t square minus vector this square is invariant, Lorentz invariant quantity, but I do not like the minus sign I want a plus sign. So, what should I do? I replace t by minus i tau. So, this change of variable is called weak rotation this weak rotation weak is a person this is a so weak's rotation and so idea is that we are doing the integral in time in the real line like that.

So, this is this time, but I want to do it in the complex line. So, I do a rotation I go down like that that and that. So, if the integral over this quarter circle is negligible then this integral and this integral will be the same value ok. Because for analytical function for any loop integral must be 0 in a in complex plane ok.

So, this this is a weak rotation. It turns out that we can convert the Feynman path integrals to a Euclidean integral in where all the 4 coordinates or 4 variables are in the same footing ok. And in statistical field theory we do not have time ok. We are basically doing equilibrium and we are not thinking about time playing a role. So, get rid of time and it turns out that lot of formulas which you derive for quantum field theory can be ported here. And specially for statistical quantum well quantum statistical mechanics ok that these are important framework.

So, I am just giving a brief introduction because we are doing all this. So, for completeness I am mentioning this. And as a result I just replace this t by minus i tau then it becomes you see this is the same all of them are added plus plus plus ok. And this integral this one is minus i d 4 x e. So, it is like dx dy dz dw standard integral we do in classical mechanics or classical physics.

So, this is Minkowski space ok. And my this dt square minus Laplacian becomes plus right because this i square will give you minus. So, it becomes plus. So, the Euclidean Laplacian and this is interesting. So, this i h bar by i h t by h bar this is a final path integral action.

And so my i t is minus i tau. So, basically I am replacing i tau by h bar as beta well these are tricks. So, basically I am just briefly discuss in the next slide we are getting rid of the time dimension by some some further trick. So, we do not want to do 4 dimensional integral we want to do only 3 dimensional integral and there are complications there. Space time is 4 dimension these also 4 dimension, but my statistical physics integrals are in 3 dimensions.

So, that is the next slide and there is a minus integral. So, this is the trick. So, this is a picture. In time is assume that so you make a box the bottom plane and top plane is periodic in time. So, I want to get rid of this time part. So, this is a space part which is nice I can correct it is the minus beta h Hamiltonian is integrated in 3 space.

And this part is periodic so with the frequency will come into play. So, string in a periodic domain that will be oscillating with certain frequencies and that is how we get rid of this part. And so I am just mentioning it here tau is finite 0 to beta and so basically the time part is eliminated by some tricks. So, d plus 1 space dimension is converted to d dimensional space with time eliminated and this possible under certain limits and certain algebra. And you can again look at this book QFT for gifted dimension. So, I think I will end about the Feynman-Path Integral at this point may be, but we need some of these formulas for our future.

Any questions on this? Well of course, this was only this was mostly math except the last part which is not easy. So, the connecting field theory with statistical field theory that part is tricky which I do not understand fully either. So, we need to do the computation then only we will understand it. So, I have not done it. So, but I think you need to go back and look at your notes.