Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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So, in earlier last class we discussed field theory of a scalar particle, scalar I just said scalar well it will become clearer later as we go along. So, it was a one field which was a real field and we quantize that. Now, we will less continue the story in the same lines, but we will bring in more, but remember that was a boson. The number of particles for a given wave number or given state could be 0, 1, 2, 3 like that. Now, we will also introduce fermion in today's lecture. So, this called Many body Physics.

I will I am not doing relativistic right now, I am kind of skirting it out because it has pretty tedious algebra ok. So, we will we will do that later. So, let us start with Schrodinger equation again start with classical Schrodinger equation. So, this is the Lagrangian which we derived or well we started from this and derived like Green's function and so on.

So, from this Lagrangian we can derive equation of motion know that is straight forward Euler Lagrange equation. For time being we say we will discussing free particles we set V equal to 0. So, this potential is 0. So, this is basically free particle know this potential this is coming from momentum squared V squared by 2 L, V squared operator. So, I am setting V x equal to 0.

So, it is a basically free particle. So, in your standard particle course is called free particle Schrodinger equation single particle. Now, we will quantize it that is a idea today we are going to quantize it and we will get many particles ok. So, Hamiltonian for this system is this which can derive and if I derived it in the class from this Lagrangian I can derive Hamiltonian and total this is Hamiltonian density this Hamiltonian density. So, this total Hamiltonian over whole space and a Green's function is this.

So, Green's function we can derive from this equation. For me I kind of derive Green's function from the equation is straight forward from here for potential 0 this potential 0. We will introduce potential later in fact, today's class also we will see when potential will be introduced we will do it is a perturbation. So, a free particle so, I am going to now start going to Fourier space and starting to quantize that is what we will do. So, this Hamiltonian so, now this phi this is only field phi it turns out for this we can write down momentum, momentum density but we do not need it.

So, some of it is like art how to do quantization. So, for the earlier lecture for oscillating field we had both momentum and phi. So, we quantized both pi and phi and the commutation relation for them today's class will be different. So, we have phi x so, this phi x becomes an operator. So, it is written in terms of phi hat k it is a Fourier amplitude, but this is an operator this is not a number is it is not a real number or complex number it is a operator.

You can invert it phi hat k is phi hat it is just invert. So, this we defined it but I do not need pi and I can substitute it there. So, the grad will give us k square grad square will give us k square and this phi square will become phi dagger k phi k. So, this Hamiltonian so, you see that we have become basically pretty similar structure but I do not have momentum pi dagger pi. So, in some sense I already quantized it now I mean so, my Hamiltonian this is an operator and I have this written as a product of phi dagger phi and for wave number k and do not interpret this is a oscillator now.

Earlier we interpreted them as oscillators but this is sum of smaller well sum of Hamiltonians which are basically phi k dagger phi k and this 1 by 2 m. Now, dagger so, interpretation of phi dagger is same as before it is transpose conjugate and so, this sum I write as integral sum is so, we are doing with infinite box so, the sum is basically integral so, commutation relation. So, this is what I was mentioning I mean I started discussing this in the last class. So, in quantum mechanics there of course, lots of particles but 2 electrons are indistinguishable. In classical physics we have stones I mean 2 stones or 2 balls it could be identical looking balls red ball but they will look identical but we can distinguish it from the trajectory.

So, there is a trajectory of particles and we know the trajectory is well defined trajectory is well defined so, from the trajectory I can distinguish the particles. So, classically the particles are well defined and they are distinguishable so, that has the consequence I mean the distribution in statistical physics this called Boltzmann distribution for classical particles but in quantum mechanics this particles are indistinguishable. So, the experiment which you can think of scatter and I shoot 2 electrons from far 1 and 2 but you see electrons are quantum particles so, I cannot track it this position and momentum of this particles are not well defined. In fact, I do the experiment I shoot something and then I observe it sometime later. So, this particle will come out come out here so this particle A and particle B.

So, there is no wave you can figure out from this is a postulate of quantum mechanics in fact experiments also do not tell you whether particle A is 1 or 2. Here I am not destroying the particle there is no creation destruction this is just pure quantum mechanics of particles. So, this particle A and particle because in between I cannot track it how the trajectory went like is it went like that or did it go like this. So, there is a in fact there is no trajectory so, in fact there is a wave function which was here and then it came out and when I measured it I got A and B so, this is called indistinguishable. So, photons 2 photons are indistinguishable quantum particles are indistinguishable that is one major postulate which happens to be true in all experiments.

It is a consequence so, fermions obey statistics which is different than Bose-Mann statistics. So, classical particles obey different statistics in statistical physics and quantum particles obey differently. So, bosons in statistics and fermions fermi statistics I will not derive it in this course but they are different. So, now let us see how to use it in field theory in fact field theory makes it very very clear that is beauty of this formalism. So, think of you have two particles you create particle 1 and then particle 2.

So, I create A dagger 1 then A dagger 2, dagger is a creation operator know. So, we have two particles it happened in but the field theory so, let just make in field theory you can create particles and destroy particles. So, just imagine that in some experiments in nature these two particles were created. It is or you can do so, you can create two particles in different wave you can create two particle two first and particle one later these are operators. End result is the same.

So, we cannot distinguish which was be done before which was done later this two particles what we see. In experiment we will just see two particles. So, this two particles this product of the operators is identical in terms of observation. It turns out you can so this one can change by a phase in fact by a number plus 1 or minus 1. So, this could be 1 or minus 1 this product.

The difference can be 1 or minus 1. It turns out it can also have some e to power i phi kind of stuff which is also new set of particles but we focus only 1 and minus 1. If it is 1 then that is a boson. We discussed that in the last class. A dagger commutes A dagger k 1 A dagger k 2 commutes.

But if it is minus 1 then they are not commuting. So, if I multiply two operator A B. I get minus B A. So, this is called that does not commute but this is called anti commute. So, commutation from standard quantum mechanics we write A B two operators A and B we write as A B minus B A. If they commute then this A B commutator A B is 0.

If does not commute then I get some finite number or it could be also an operator. But if anti commute then we have different symbol. We write as a curly braces A B which is written as A B plus B A. So, if A and B anti commute these are operators. If A B anti commute then this will be 0.

So, Fermi was anti commute. Fermi operator has anti commute. This is new I believe if you are not done field theory then this is the new thing which you learn in field theory. So, this is what I have stated it here. So, A dagger we create two particles with momentum p 1 and p 2. We can change the order it turns out this product A dagger p 2 then A dagger p 1 is equal to A dagger p 2 A dagger p 1 with factor lambda in front and lambda can take value plus 1 and minus 1.

n 1 is boson and minus 1 is fermion. So, this is how we construct the full fermionic field theory and bosonic field theory. So, boson we already studied that. So, but now let us just revise it, it does not hurt.

Let me make one more remark. Some people use this commutation A B commute is plus and anti commute is A B minus. So, this is another wave to write it, but I will follow braces. We will not follow this notation. So, boson we saw that A dagger k A dagger k prime commute no, the creation operators commute, destruction operators also commute, but creation and destruction operator together they do not commute. So, A k A k dagger is 2 pi cube delta k minus k prime.

So, that we studied we derived it in fact. Well either you have to use as real space commutation as a postulate or Fourier space commutation as a postulate. This is a postulate like this Heisenberg and Uncertain relation generalized to fields. So, you cannot ask why this is this how nature is. So, we said there well in field theory we have a state psi and a state psi is specified by number of particles in each state each sub states.

So, this is corresponds to wave number k 1, k 2, wave number k like that. So, this I defined it earlier. So, we have various states of the system and you see how many particles are there in different different states. For bosons we can have more than one particle or 0 particles and it has a very important consequence you will you see it very soon. So, this is state psi and we have two operators very important operators creation operator and destruction operator.

So, destruction operator with k is a state you can call it k is a state or wave number k state. So, a k acting on psi will work on this number of particles nk. It will just decrease it destroy one particle. So, this is kth substrate and then there are three particles will just destroy one and bring it to 2. So, nk minus 1 is a pre factor the square root nk.

So, this property of destruction operator. Now creation operator ak dagger if you act on this psi this state this state then it will increase the number of particle in that state is nk plus 1 and the pre factor is square root nk plus 1. So, if I act ak dagger on this one, this on the full state, but this k will tell you which state which substrate to act on I hope this part is clear. So, in Fourier space we have many many wave numbers.

So, like the photons in this room. So, I look at particular wave number and I want to create or destroy photon in that wave number. So, or hydrogen atom we have many many levels know. So, for given state whether we can create a electron or destroy electron in that state in that Lm cells. So, that is what it means. And then we have number operator Ak dagger Ak which is acting on psi will give nk for that given state k and Hamiltonian is Ak dagger Ak plus half omega k.

For Schrodinger equation omega k is defined. We know that is h bar square k square by 2 m. So, for Schrodinger equation we will use k as a state level or substrate level if we call it. If we call this is a state then these are substate know when the levels of the state. So, this I did it in the last class. So, we are doing it bit fast and in real space their corresponding commutation relations are these.

Right now for Schrodinger equation we do not have pi. So, we will have only this and well we can construct pi, but less you know this part. We will just deal with for Schrodinger we will just deal with Ak dagger Ak with Fourier number as a state level. Now, fermions this something very important and we will see how the properties of fermions come out in field theory. So, again state psi is same as what we describe n1, n2, nk, but we will soon realize because of anti-commutation relation that n1 or n2 or all this can be values 0 or 1.

It cannot be 2 just because of anti-commutation property. So, as I said Ak dagger Ak is 0 anti-commutator not commutator and same with creation operators, but Ak dagger and Ak this one Ak Ak dagger is not 0, it is in fact delta function. For, alright so, particle numbers I am going to derive it in the next slide it is either 0 and 1. So, you will just see the derivation in the next slide. In the real space you replace this commutator by anti-commutator.

This square brackets are replaced by this relation. So, I will give some homework you just work out using this property you can basically can get some simplify operator. So, we will see one example here, let us see. So, I will focus I will basically drop symbol k, I am dropping symbol k.

So, I am focusing on one state. So, I am just do not want to carry this k, but just imagine that we are given a given wave number Ak. And so, anti-commutation is 1. So, i h bar so h bar h bar is there not i h bar h bar is there and delta function. So, since the same wave number delta has become 1. So, I you can think of instead of infinite box you can think of a finite box with the Kronecker delta.

So, from this so this is clear know this is anti-commutator so I have to add not subtract. So, I get a plus 1, a dagger plus a dagger a. Now what so I just rewrote that is a a dagger equal to 1 minus a dagger a. So, I just take this to here. Now a dagger a is what is a number operator for that given state.

So, aa dagger is 1 minus n number operator. Now we know that aa is 0. So, this anti-commute so aa anti-commute is 0. So, aa plus aa is 0 so that means aa is 0. So, this is 0 and these are also 0. But the product of two operators is not 0 a a dagger sorry creation in destruction operator you multiply then is not 0.

Now so I am just going to look at this operator n into 1 minus n hat look at this. So, what will that be n n number operator is a dagger a this one. What is 1 minus n n operator? This 1 minus n operator this is aa dagger 1 minus n is I just derived it so you got this. Now you can see the in between there is a a sitting there.

So, a will give you 0 you just apply any vector psi. So, psi will act on a dagger so that will give you phi but a acting on phi will give you 0. So, a is 0 from here so this is 0. So, what does it mean? So, if I act this on any state psi eigen state psi then I will get either n to be 0 I if psi is the eigen state of this operator then n will be 0 or n will be 1.

So, this implies that n is either 0 or 1. So, this coming from the property of anti-commutation of these operators. So, either the state is empty or is only one one particle in that state. It just follows from this operator this is pretty strange thing but it just follows. So, this Pauli's exclusion principle for fermions. So, these are called fermions these particles which obey this come anti-commutation relation are called fermions.

These are different bosons, bosons can have any number of particles in a given state but this cannot 0 or 1. Now, you can easily see that a acting on so this is a vacuum state. So, in a vacuum state there are 0 particle I cannot destroy well if I just act a destruction operator then I will get 0. But a acting on 1 will give you 0.

So, there are one particle I destroy 1 I will get 0. What about a dagger acting on 1? 2, but 2 is not possible so this is 0. So, a dagger 1 is 0. So, it just gives you the empty state. So, a dagger 1 is 0, a dagger 0, a dagger acting on empty state if you can create and you can get 1. So, this is a yeah you have to be careful this is a difference in bosons.

But if you bring in spin no problem you have to count the down spin of spin separately.

No, no, no the n will not become different so n up operator 1 minus n operator is 0. So, so as a given state so you say given wave number and spin if you like you can add it or n also same with n down. So, this is Pauli's exclusion in hydrogen atom you know I can have in one state 1 up and 1 down, but you cannot have 2 up. So, so this where we will take a break and if any a question you can let ask me any questions on this? Yeah there is only 0 and 1.

Number of particles is 0 or 1. No, no, no wait, wait. So, in a system there will be many, many states and each state there can be either 0 particle or 1 particle. So, in hydrogen atom we have many, many states no. So, in each state we can have only 1 or 0 spin up particle and 1 or 0 spin down particle. So, you remember that there are states I just put horizontal lines and you can fill according to the rules of bosons or fermions. So, we see one example free fermi liquid that is what we will do well simplified free fermi liquid any other question? Thank you.