Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Second quantization, now we are going to quantize a classical field and we will see what complications come with classical field. So, let us look at a one classical field there will be lots of classical fields we will encounter well we did encounter many of them tons of them in fact, we will focus well let us look at several of them in today's class. So, this is Lagrangian for a field which has this is Lagrangian given Lagrangian field theory is fixed we are not assuming boundaries, boundaries infinite for simplicity. So, this is our in fact, we know this is for string we derived this for string this was not there v phi now we are bringing in this v phi this potential. So, this string is in a potential. So, we will assume a simple potential which is half m square v phi square m is called mass parameter of the field.

Now, without this is a pure wave equation for a string rho is mass density kappa is a constant. So, we can derive Hamiltonian which I did in the previous classes and equation of motion. So, equation of motion coming from Lagrange equation of motion this is definite classical fields phi is a field which is function of x and t. So, we get this is equation now you might you have encountered this part, but not m square you may have encountered I am not sure, but m square is a new term which is coming is mass of the field mass parameter without m it is a wave which is propagating know when in fact, we computed the greens function and so on.

In fact, with m also I have computed greens function. So, the greens function for this field in Fourier space k omega space it is that. So, k square coming from Laplacian d by dt gives you omega and m comes from here. So, this is the greens function now this I am sure you can derive it from what I did now from the Hamiltonian now these are Hamiltonian I can write go to Fourier space I already wrote the greens function, but we can just recap. So, phi of x is written as Fourier term well basically phi is written in terms of Fourier coefficient phi k phi k is a Fourier amplitude it is amplitude of the waves.

Now we assume this normalization and we are assuming the box to be infinite now as I said you will always encounter not dk, but dk by 2 pi to the power d d is a dimensionless space which could be 2 3 4 as well 4 in fact, higher is it. Now we can invert it when invert it this phi k is written in terms of phi x space integral and so, this is my plus sign here. So, from phi k to phi x I put a plus sign and phi x to phi k I put a minus sign that is my convention. Now this for phi real it is a real field so far you can do the same thing for momentum phi is a momentum of momentum density. So, exactly the same thing these are real fields classical fields or we can do the Fourier transform no problem.

Now you can substitute it here now this Hamiltonian density so, total Hamiltonian is integral of Hamiltonian density or d to the dx in d dimension. So, that is so, I can write down Hamiltonian in the real space total Hamiltonian or in Fourier space before that one more thing the phi is a real field so, this is a reality condition so, phi of k and phi of minus k are related. Now this I will not derive it I will leave it as a homework for you can derive that phi of k star is phi of minus k. So, k and minus k are not independent and we will see some reflection of it in quantum fields as well same thing for phi k phi is a real momentum no I mean so that also follows this. And please note that this phi amplitude is arbitrary for classical fields it can be any number like for a string it could be 1 centimeter height 0.1

There is no quantization of the field in classical physics, but in quantum that would not be the case and Hamiltonian this one which is in real space we can write down in Fourier space this is half coming from this half and the pi squared becomes integral pi squared becomes pi k pi minus k sum over all k this sum is basically I write this sum for convenience, but this is really this integral e k this is what you will come always. So, this is quadratic in real space becomes quadratic in Fourier space, but pi k multiplied by pi minus k or we write as pi k mod square this is a Parseval's theorem. Now similarly, for this one this plus this together this guy grad square will become k squared no. So, that is why you get k squared phi k phi minus k. So, this is coming from the grad phi square and the third one is v phi which is m squared phi k square.

So, this Hamiltonian you need to keep this in mind I am not going to derive it this I think it is good to derive it yourself. So, this you should do it yourself. So, classical field so far now let us go to quantization. Now this field functions will become operators and now in quantum mechanics we do not have fields. So, quantization of particle motion is called first quantization electron in a hydrogen atom when you quantize it, it become a wave function.

Now we already have a wave then when when you quantize what will we get. So, imagine thus we have a this electron wave function is already the psi is there now you need to quantize the psi or electromagnetic field coming from the from the tube light. So, you quantize that in fact, we get back particles in fact, we get lots of particles. So, the second thing is to quantize fields for example, QED or like the light coming from the tube light. So, here fields in fact, are you will see that the formalism is not so difficult to grasp the fields will be become fields give lot of particles.

In the field function phi which we wrote phi of x t these guys will become operators which will change with time, but these are operators these will be matrices or well these are basically operators like position operator in oscillators. So, this cause second quantization first quantization is quantize particle motion second quantization is quantize the fields. Field could be Schrodinger wave function itself we which we we need to quantize. So, many fields like in fact, the phi square field which I have a wrote few may slides back we can quantize that as well in fact, we will quantize that field. So, here we have two types of particles bosons and fermions scalar quantum field.

So, presently we will consider scalar particles for simplicity photon in fact, is not a scalar particle is a vector particle it has component. So, photon has spin one it has component. So, for simplicity we are going to keep very simple Lagrangian for quantum operator and I am going to consider bosons first. Now, you see the difference what happen I mean this is the formalism is the proofs are difficult, but formalism is not not so that difficult. So, Fourier amplitudes become operators and this operators will evolve in time and so this is a Heisenberg picture.

In fact, we will not normally deal with a wave function. So, we will lots of particles n particles in fact, the particle number keeps changing right if you destroy a particle then wave function I wrote a wave function for n particle imagine like in your usual course write n particle wave function. And suddenly one particle has been destroyed or two particles have been destroyed then what happens to the wave function it is a problem know. So, dealing with wave function where number of particles are variable is not a good framework to work with. So, we work with operators and the number of particles can change in this framework by creation and annihilation operators.

So, this is the thing. So, you make this wave function Fourier amplitudes operators. So, we have phi know phi I showed you for the string like field. So, we will say phi xt and phi x prime t.

Now, this commute. So, this is a commutation same thing what we do it for quantum mechanics phi xt operator phi x prime t minus phi x prime t. So, this is different operator I mean this is this same operator and this a and b and b minus a b minus b a. So, this phi hat xt. So, it turns out the phi and phi commute these are two different operators right this at x and this is x prime. So, this is 0 the commute is a postulate.

Just like commutation relation x p for particles is i h bar is a is a postulate. pi pi commute right position position commute for particles momentum momentum commute for particles, but x and p for particles do not commute. So, similarly phi. So, this is a particle wave function or field and momentum operator for the field. So, that is why momentum operator is very important and we can derive from Lagrangian this momentum operator.

Phi and pi do not commute and the right hand side is i h bar in 3 d it is delta cube x minus x prime. If they are at the same position they will become infinite if they are at different position they will commute. So, it is like particle you know particle position and momentum at the same place do not commute. So, that kind of is being captured here. Particle if you have something is there at the same position momentum and position do not commute, but if you have the different position then they commute, but this is not a really particle and this is an operator.

So, these are postulates for our field. Now, this is a real space. Now we can go to Fourier space. I think I need to go slow down a bit. So, is that clear I mean we postulate this field operators to the position position operators commute momentum operators commute, but position and momentum operator do not commute similar to what you do it for particle quantum mechanics.

But since this phi is function of x and time we have phi x and pi x prime. It was at same time it was same time we are not doing two different time. Now look at Fourier space. So, like exactly what we did for classical fields we can do it for quantum fields. Now these are operators, but no problem I am just going to take these operators and multiply by e to the i k dot x.

This is some kind of matrix which is function of k for every k we have phi k this one I multiply by e to i k dot x and integrate. So, I get phi hat x. So, you should think of this operator as abstract operator you do not need to really think what that is it is an abstract operator it does something to the states. And we will define those relations what does it do to states and what are the states. So, this is going from Fourier space phi k to real space.

You can invert it exactly well by say if you know classical field theory transition is reasonably smooth. You just have to remember this commutation relation and we will find we can work around quite easily. So, from phi hat x I can go to phi hat k same thing you just replace this fields by operators. Same thing you do it for pi no problem like that I just replace those things by these operators which are not not very well reproduced in PPT, but these are what they are. So, we are Hamiltonian well commutation relation let us do the commutation relation for the Fourier operators.

So, we will work with Fourier space very very often. Now, Fourier space has shown intrinsic beauty which I think I am going to mention it bit later, but let us just do the formalism. So, it turns out phi k n so there are like we have x and x prime different positions. Similarly, we have different wave numbers k and k prime. So, phi k and phi k prime well irrespective k can be equal to k prime here field operators commute with each other for different wave numbers.

So, this is well this postulate you can derive from the phi x. So, either you take the real space postulate or Fourier space postulate you do not need to assume both we can derive one from the other I am going to derive one of them just give me one minute. And so similarly one thing is not it should have been here phi hat k phi hat k prime sorry not phi k phi k prime is also commute they also commute. Momentum operators also commute in Fourier space, but what about position in Momentum operators in Fourier space.

So, let us derive it. So, phi k here I write as phi hat x t e to power plus did I make a mistake no sorry minus minus from x to k is minus. So, this is that this guy this guy and integral gives you phi hat k. Now, I cannot change the color this one this one and this one gives you phi k prime hat. Now, commutation here product this is product AB minus BA. So, I can also transfer the commutation here correct.

So, you have to keep this order intact in quantum mechanics you cannot switch it classically we can switch it, but not quantum right I mean this you know from your particle course. So, this commutator I keep it here now I know what this commutator is phi x phi x phi x prime is a delta function. So, we put the delta function. So, this one I put it here I have got rid of h bar h bar is 1 for me and because you need to carry h bar. So, that is produced here and two exponentials.

In fact, this is a totally real real thing there is no operator in this delta function is not an operator it is a function. Now, what what will it give us x equal to x prime I just replace x and x x prime equal to x then what will I get e to power i minus k plus k prime dot x integrate this what do I get i sitting here I will get a delta function. And this 2 pi will also come 2 pi times delta function and in 3 d is going to be 2 pi cube i k plus k prime and I forgot this i here this should be i this square root of minus 1. So, this is you can from real space I can derive Fourier space commutation relations and vice versa. If you had if you had given these in Fourier space you can derive the real space straight forward derivation.