Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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So, we will do some conservation laws for field theory and this plays a very important role some is obvious, but one non obvious thing will come in today's lecture. So, this is lady Noether, she struggled quite a bit in earlier times they were not women are not allowed in universities. So, she was working as assistant of Hilbert then so, but this theorem is very very important it appears in all areas of physics. So, we will just I will show you the how it appears, but we did it for particles in the beginning right I mean in the last lecture and one of the last lectures, but we will do it now for fields. So, the derivation so, let us just look at what is the essential aspects of Noether's theorem in fact, the derivation in fact, is a simple derivation, but some derivation are more detail, but you will get the essential idea here. So, is a change in Lagrangian we are looking at change in Lagrangian and of course, when derivation of Euler Lagrange we make a constraint that we vary the phi vary the field in such a way that at the end points the field should not change right for particles we make sure that the end points of positions are same.

For fields you have to make sure as I said initial condition and final condition the field must be the same in between you can vary the field you can apply perturbation and so on, but there is a field configuration. So, please remember Euler Lagrange equation tells you that there is a configuration of the field which goes from initial to final which minimizes action. So, that is a and that is what we have seen classical solution that is a classical solution, but here I am going to not constraint that phi to be variation in phi to be 0 at both the ends we will we will say that well you can choose whatever phi you like I am not going to constraint it, but under that constraint what is delta L. So, we look for the delta L.

So, that constraint is not imposed now. So, this is a well I mean delta L change in the Lagrangian density let us look at the Lagrangian density. So, delta L delta phi plus delta L delta mu phi multiplied by delta mu so this is a derivative. So, the Lagrangian can change due to derivative in phi in a change in phi as well as due to change in the derivative these are two contribution. We rewrite it this part remains as is the first part the second part I am going to so this is phi mu phi mu of x this is the definition of the momentum density.

So, this is that and this is I am basically commute the derivative at delta it is allowed. So, change in the derivative of delta phi. So, now this term is a derivative of the second function it is of that form. So, now we are going to use the product rule. So, this part is rewritten as derivative of both the f g minus derivative of the first function g.

I mean I am basically nothing doing nothing more complicated g equal to del mu f g minus g del mu f now I mean this is the product rule. So, that is what is being done here now combine these two delta phi is outside great. So, delta phi is arbitrary. So, this is coming here and this term is coming here now Euler Lagrange equation is still valid. So, what happens to the first term is 0.

So, whatever delta phi you choose it does not matter the first term is 0 and this is second term. Now, typically we want delta L to be 0 delta L I want well no sorry I do not want let me sorry I made a mistake I want to write this this is a del mu I just wrote that. So, we write delta phi. So, basically delta phi there could be many component of phi and so on. So, we want to parameterize phi in terms of parameter lambda.

Right now phi is only one component, but I may have more components. So, we write delta phi is some parameter delta lambda multiplied by d phi this is a derivative phi. So, then we write. So, this is the definition and then we say that this is a current. In fact, I am going to set that to 0 indeed delta L.

So, these are defined as current N is for Noether g mu is a current is del mu a pi mu. So, this is pi. So, this is this part I am going to write as delta lambda is outside keep delta lambda outside. So, this is a function this is a function we call it current Noether current and it has a interesting consequence just hold for a minute. Now, we want delta L to be 0 we set it to 0.

So, I do not want change in Lagrangian under the perturbation Lagrangian should not change, but my perturbation is arbitrary perturbation on phi. So, delta l is 0 then that means, this quantity must be 0. So, this divergence of the current is 0 is a divergence no del mu means divergence, but it could be four dimensional divergence in 1 d is going to be 1 space 1 time is going to be dt J 0 this is J 0 plus dx J 1 is 0 in 1 d, but in 3 dimension 3 space dimension in 1 time dimension J 0 plus divergence space divergence is 0. So, all the sum of these derivatives. So, that is why you need to kind of do it as an example I will probably give you some homework I am going to show you some real examples, but you need to do more.

So, this is the content of Noether's theorem well one more slide I am going to show you what you can derive from Noether's theorem after this. So, divergence of a current is 0 and that current is a Noether's current. So, we write rewrite. So, let us do the space integral of that Noether's current well del mu J mu is 0. So, that means, there integral also will be 0, but I am doing space integral not the space time integral this 3d space integral.

So, now, this divergence I write in two components these are time derivative and this is space derivative. Now, what happens to the space derivative? What is space derivative integral space derivative of divergence of something Gauss theorem will tell you 0 of course, the current at infinity should be 0 that is what we will assume the field is basically vanishing and there is no current. So, the second term is 0 by Gauss's theorem second term. So, what does the first term give you? So, this derivative with 0 is d by dt. So, d by dt of J 0 integral d cube x is 0, but d by dt can be taken out you can take the d by dt out.

So, this quantity is constant and this called Noether charge. So, there is a conserved quantity and that is the that is the derivation. So, this charge Q n is constant. So, for every conserved every well every symmetry will get a conservation every symmetry operation. So, let me just make this comment.

So, delta L will not be 0 for arbitrary change in phi under some symmetry delta L will become 0. So, if I change phi in arbitrary way I will not get delta L to be 0 for some symmetry operation delta L will be 0 that will imply. So, let me write this symmetry of delta phi is some symmetric variation of delta phi will lead to delta L to be 0 and this will imply some charge conservation some quantity will be conserved. So, we need to look for variations of phi in such a way that may Lagrangian is Lagrangian density is unchanged and that immediately gives me a conservation law. So, that is a line of logic.

So, let us do some examples. So, I will give four examples. So, time translation for Schrodinger equation we can do it for Dirac equation and we can do it for Kalklein Gordon, but let us illustrate by Schrodinger equation. So, I give you the Lagrangian of Schrodinger equation that is what then that was the motivation why we are deriving Lagrangian for Schrodinger equation. So, this is my so, I am going to make change in psi by translation in time.

So, I have this psi, but I so, this psi x t, but I say well I am going to change time to t plus delta t. So, psi is exponential i E t some stuff phi x. So, now, I make this psi I mean I am just writing some jumbled up some stuff. So, I make change. So, time transition means what? The wave function right now or wave function sometime later.

So, I look at the wave function sometime later that will be different function, but everybody understand what is meant by time translation. So, let me just go back little bit to motivate. So, if I do an experiment in a lab, let us do an experiment in the lab, this particles are coming and colliding and they are just getting scattered off. I do today, I do the experiment tomorrow, assume that the room is got no change like there is no electromagnetic stuff for some. So, room is same environment I do the experiment tomorrow, I should get the same observation right my the particle will scatter exactly like that in tomorrow as well.

So, that is laws of physics are invariant under time translation and that has a very important consequence, which I am going to come here. What is the consequence of this time translation invariance of energy is conserved. So, that is what we will derive it here. So, of course, well today and tomorrow we will do a new set up, but imagine that I look at this trajectory now and one second later. So, that is a translation I am translation in time.

So, my variables will change right my fields will change. So, that is what we get this field change. So, what is the change in psi this is the change in psi. Now, d t is t is lambda for me. So, I remember I wrote this as d psi time d lambda.

So, this will be d psi and this is d lambda. So, d psi this d operator psi I want some smooth function I do not want to carry this delta as well. So, this d psi is psi dot. So, that is change in the change in the function wave function. So, this is my Lagrangian I wrote like that term now Lagrangian I am using the old Lagrangian.

So, without the d by dt of f. So, the current we can write down. So, ignore this. So, this is something else this is the current I wrote right now another current pi mu del d is d psi and pi mu we know how to compute pi mu pi mu is this one d partial l partial psi dot that is pi mu that is a definition. Now, right now I had two components of the field psi and psi star we need to sum because otherwise I will not get a real energy I mean the energy must be real quantity you know.

So, I need both psi and psi star. So, I am doing it for both psi and psi star that is why I am summing over components. So, momentum is partial l partial psi dot for psi and there is a partial l partial psi star for psi star multiply this by this del psi d psi. So, for psi this is going to be psi dot if a psi star is going to be psi star dot correct. So, this is going to be I by 2 psi star psi dot and other one is just complex conjugate this component is just complex conjugate.

So, that is why we get real thing. So, I psi star H psi psi dot is by no actually no I know I psi dot this is a I H bar psi dot is Schrodinger equation H psi know I am skipping gradient of H bar. So, I H bar psi dot is H psi. So, there is no I here. So, psi star H psi plus well if you like you can put half a psi H psi star, but there we give the same this is a real quantity.

So, this should be basically. So, this is my stuff current this is not energy yet this is a current J 0. Now, of course, I will the integral of J 0 is constant this other component of J which is let me write. So, this other component of J. So, we need we will get 4 components of J right J is I mean remember the J was this this one I do not want to go back. So, J has 4 components I wrote del mu J mu is 0.

So, this is the first component and these are the other 3 components and other 3 components I need to compute pi k the space space momentum and psi dot, but I do not need to worry too much about it because I am looking for conserved quantity. So, conserved quantity is this quantity integral of this with space. So, if I integrate this. So, I integrate this with space what will I get is energy know is not it this energy psi H psi which is integral this integral.

So, this energy. So, we are able to derive conservation of energy by time transition of the wave function. So, of course, there is one constraint that if I apply some magnetic field in Hamiltonian then delta L will not be unchanged. So, there are some systems for example, if you for not every time your time translation symmetry is applicable. Imagine that my in the set up I put in a current which is changing my time external current then electromagnetic experiment energy will vary with time energy will not be remain the same, but take a proton electron and there is no other electric field outside then energy is conserved during the during the during the experiment, but you put as soon as you put some external electric field energy will not be conserved energy will be electric field will pushing energy into the charge particles. So, here we have to make sure that this external fields are not present in H if that is present in H then psi will this set up will not be invariant.

So, delta L will change for those wave functions. So, that is a implicit assumption made here. So, I am I am saying that delta L is 0 for these variation of psi that is a major assumption which is being made here and that will not be that will not hold if the potential is introduced which will change the energy. So, in classical you can immediately understand I mean I hope this part is clear.

So, symmetry is not always valid. Now, let us look at space translation I want to move fast. So, it is basically follow the same logic, but I am going to shift not in time, but in space. So, you can think that I am change I am shifting the setup I am shifting that by space. So, my delta psi will be d psi.

So, here I apply the same logic I need to rush it. So, 5 more minutes. So, no actually we got the 630 know 630. So, we have 15 minutes. So, I am fine with that. So, let us look at the J 0 again that will give me conserve quantity.

So, let us assume that for these variation of psi under space transition delta L is unchanged L is unchanged. So, delta L is 0. So, this is my Lagrangian. So, I can get the derivatives. So, pi 0 now my d psi is grad psi right I mean this state obvious from here it could be 3 it is a 3 dimensional field.

So, there is a my variation can be in x y z. So, that is why we get gradient. So, pi 0 grad psi and pi 0 we derived it earlier is the partial L partial psi dot. So, that is that and grad psi and I put complex conjugate. So, I by 2 psi star grad psi minus complex conjugate right.

So, this is straight away follows from there. Now, you may identify this this object. This is looks like the Schrodinger equation current know for Schrodinger equation we can derive current and that is exactly the current which you might have done in your quantum course J Schrodinger equation. So, there is a factor half is coming here this is a half is not there in the definition this is half here. So, now, if I integrate this. So, if I integrate this what will I get integrate this object.

So, what is the momentum operator ih by grad know the momentum operator. So, this momentum operator P. So, what is the average expectation of P psi P psi know. So, that is nothing, but that well I need to add otherwise it will become complex.

So, I need to add the take the real part of that. So, this is this give us total momentum. So, this variation space transition in psi which conserves well which does not change L Lagrangian density will give me conservation in momentum. It is nice know it is beautiful we just this symmetry immediately implies the conservation law. So, we done two conservation of energy and conservation momentum and linear momentum well angular momentum follows from rotation. So, that can also be done I will not do it right now, but we can do it by rotation which is there in I mean you can find well you can do it yourself that is a good exercise to do.

Now, third one which is slightly more complicated. So, you can change both space and time simultaneously. So, this is in relativity we change both space and time. So, that is why we have this four space time variation. So, this delta x nu is a space time transition.

So, my wave function will go like that. Now, here I am using the notation for relativity. So, this is a dot product, but it is a fourth space time dot product. So, covariant this covariant and this well this contravariant and covariant derivatives. I have slightly messed up in my no I think I am yeah.

So, this is delta L. So, this is a Noether current. Now we write delta psi from here, delta psi I just get it from there and this is I just substitute here. Now we can also so, now take the derivative of this actually this bit of I have taken this guy out delta x nu. Well I think I am doing approximate derivation I mean there may be some algebraic error, but let us just take. So, this is from here it follows that partial L partial x nu this is partial L partial x nu will be this object. I am slightly doubtful how how can we take out this this object outside that is what is my doubt.

I have taken this outside the derivative, but let us ignore that complication. So, this is a partial L partial x nu. x nu could be it can be time variation, space variation all all together. It turns out with this variation I can get both energy conservation and momentum conservation in one shot. In earlier derivation I was doing separately space translation and time translation it turns out you can do it simultaneously.

Now by definition we can write this I am just changing the change of variable well not change of variable I am just introducing a delta function here. And if I now these two must be equal sorry subtract subtract this a minus b. So, this a and this b a minus b subtract it then I will get that. I do not right now I am not equating this to 0, but I am just subtracting these two guys and I get by definition 0. And this object is called space time energy momentum tensor energy momentum tensor.

And so that is a derivation of energy momentum tensor. Now this t mu nu and the 0 component will be t 0 0 which is nothing, but Hamiltonian pi phi dot minus L is Hamiltonian and the 0 k mu component is a linear momentum. So, that is so both this quantity if I integrate in with three space dimension I will get conservation of energy and conservation of linear momentum. So, this is a combined transformation. So, it is more algebra, but it is same content what I did two slides earlier.

Now the next one is non trivial is called gauge symmetry. So, please pay attention to this and this is you have not seen it before I believe. So, in psi I can multiply E to y theta and this can be function of x y z. So, let us assume that under local by multiplying this my physics remains unchanged. So, let us assume that. So, mod psi square will remain unchanged with this, but I am doing a very non trivial operation I am multiplying the wave function by E to power i theta and theta is varying in space.

So, with this what is the change in phi change in psi this is change in psi for small theta let us assume theta is small. So, this is change in psi. So, by the way remember I was making a small change in wave function not arbitrary change. So, the d psi is i psi theta is of my parameter lambda theta is parameter lambda. So, my current g mu will be pi mu del d phi which is i psi dot i psi not psi dot i psi.

So, this coming from here. So, my Lagrangian is that. So, I need to basically use

Lagrangian to compute pi mu momentum component. So, momentum I know already that it is psi star and psi right when that is the stuff and this is d psi d psi is i psi. So, psi star i psi and complex conjugate, but i i because minus 1. So, I get minus half here and minus half here. So, I get minus mod psi square which is minus density right wave function density is a probability density.

What about the other component J k 1 2 3 pi k i psi, but pi k is a momentum in the space dimension which is nothing, but grad psi star by 2 m. This is the momentum know mean momentum. So, which is gradient operator for Schrodinger equation. So, that and complex conjugate, but this is nothing, but the minus sign by definition I checked the minus sign minus of the Schrodinger equation the current in the Schrodinger equation.

You seen this know this is called current. Now, what does a divergence of J give you? So, take the divergence. So, there is a minus sign there is a only change of minus minus we can ignore that. So, dt rho plus divergence of Schrodinger equation current is 0. So, this you seen it before know this follows I mean this you can derive from straight from Schrodinger equation, but it follows from some symmetry principle that if I change the wave function phase of a wave function arbitrarily small then that does not change the physics and that in fact, this conservation law follows from that that symmetry operation current conservation. And well of course, the first component again should be give you a conserved quantity and if I integrate this one I will get well this trivial know the probability density integral over space is 1.

So, that gives me constant. So, this is called non-linear Schrodinger equation which is we use it for super conductor super fluids. So, it is same equation plus this quantity. So, this non-linear psi, but we write this as V psi psi. So, V psi is mod psi square know is it correct? V psi is function of psi itself V psi is normally we think of potential as function of x in a typical Schrodinger equation know coulomb potential or delta function or, but here we make V psi psi itself.

So, this is a non-linear everybody agrees with that is a non-linear equation. So, this V psi now whatever I showed you in the earlier slide. So, we have several conservation laws for Schrodinger equation. One is mod psi square integral d x is constant d cube x is conservation of probability density. This is obvious I mean that is, but the other ones are energy conservation which is psi H psi we can also derive energy for this equation. Now energy is slightly non-trivial we can derive energy which is this quantity mod psi square coming from here plus well g is 1 right now for I put a g here.

So, g psi 4 Hamiltonian you have to derive for this quantity and this Hamiltonian for non-linear Schrodinger equation and this will be conserved under evolution. And linear momentum is in fact, this is somewhat straight forward, but this is also conserved.