Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Lecture – 15

So, now, Relativistic Field I will just write down one Lagrangian, but we will not dig into the details because the relativity is I mean some of you are expert of it, but let us not get dragged into that details. So, this will be the equation of motion I mean I did not derived it. Now, I need to worry about plus sign, minus sign and this basically metric. So, we write the Lagrangian for a for a particle of mass m like that and now field of mass m. So, m is called mass of the field we will worry for it later, but this is the term which is for a scalar field of with mass m is a parameter. Now, this one means del del phi del mu we write d t phi square I do not.

So, there is a minus sign now. So, if you recall the invariant measure normally comes like that delta t square minus delta x square this is invariant measure del s square right this is a minus sign. So, contra variant covariant you multiply. So, it is not a plus sign, a relativity the invariant quantity there is a minus sign for the time and space this separation this is minus sign.

So, same thing will reflect here this is a product of contra and co. So, I will get d x phi square minus m n square phi square. So, this is a expansion in t n x if you have y then there will be y will also be there. Now, if I just substitute it here I will get this equation I will leave it for you to do it just straight forward. So, we will get this is the fourth derivative which is written as that d t square minus Laplacian square.

So, this is in 1 d is going to be d by d x this like that, but in 3 d is going to be Laplacian all of you know right with double derivative with y double derivative with z they are all added up. So, this is the equation of motion for scalar field with a mass parameter m this m has come there and this is what we get. In fact, we derived Green's function for well we derived Green's function for such fields in the past. Now, so this Lagrangian so for any field we will we are given Lagrangian from that I can derive equation of motion and I can do other things as well I mean once I have equation of motion I can derive the Green's function like what I did in the class. So, we need to know that part and we quantize it that will be quantum field theory.

Now for fields we also write down Hamiltonian the way write down Hamiltonian for particles. So, particles Hamiltonian is function of momentum and position. Lagrangian is function of like for fields Lagrangian is function of phi this and d phi x, but momentum the Hamiltonian function of momentum field momentum of the field well momentum density actually and and phi. So, this will be replaced by the momentum this will be replaced by momentum and this is called Legendre transformation. So, we will have this momentum phi and it could be also function of this gradient of phi.

So, this Hamiltonian will will be function of this quantities. So, let me just briefly tell you how we construct Hamiltonian and we need that for our course. So, first we define momentum density like we have phi is function of x and t similarly, momentum will be function of x and t is a momentum density is not momentum. So, every position will be have momentum density. So, that is define as delta L delta phi dot well right now we have field has only one component, but there could be field with several components like velocity field has three components know.

So, right now we only one component. So, it is this object basically I made a mistake this is density is this quantity. Lagrangian density take the partial derivative with little phi dot just focus on equate these two this is connected with the total momentum, but we we ignore that. So, then we define Hamiltonian pretty similar to what we do for particles momentum multiplied by phi dot minus Lagrangian density this Hamiltonian density we recall for particles what do you how do you write Hamiltonian H is P i x i dot minus L sum over all i's right or that is generalized coordinate we we do it. So, this look P is replaced by momentum density x i dot is replaced by phi i dot.

So, i are components I mean as I said the right now we have one component in illustration, but I could have more components minus Lagrangian density straight forward. So and total Hamiltonian is integral of Hamiltonian density we also write typically as d cube x H in 3D in 1D it will be just d x. Now, equation of motion for particle we know how to write that right x dot is P for 1D and delta H by delta P and delta H and P dot is minus delta H by delta x. So, we do follow exactly same procedure now here H is a function of phi other stuff. So, we write the this variation derivative functional derivative.

So, inspired by that and phi dot is functional derivative of H with phi. So, this what how we define it. So, let us work it out is that clear the formalism everyone is just if you know for particles and we just translate it I am doing it with fast, but you just translate that. So, I derived this now for string phi dot square this time derivative minus that minus that one minus V phi in general like we had m square phi square for for one example I could have whether V phi I could write as phi square minus phi 4. So, these are potential.

So, potential we write as x square minus x 4 for particles or x 4 x square minus x cube. So, here we encounter potentials of this sort. Now, you have seen this potential somewhere. So, the free energy in phase transition Landau theory if you done start make Landau theory the free energy as this form and that is where we get this phase transition. So, we see this is called phi 4 theory this is a phi 4 phi to the power 4 and it plays a very very important role in many many phenomena.

So, potential is function of phi. So, now, let us write down the Hamiltonian for it this is a given Lagrangian I can come note to it Hamiltonian. So, what is phi? Momentum density partial L partial phi dot well actually I am carrying this i is 1 I is nothing. So, what is a partial L partial phi dot I will get from here right derivative of with phi dot is just phi dot right square the 2 will cancel with half this phi dot. Now, once I get phi then I can write down the Hamiltonian by this formula pi phi dot minus L there is only one field.

So, I do not need to do a sum. So, pi phi dot will be pi is phi dot another phi dot phi dot square minus this Lagrangian. So, Lagrangian has half phi dot square the minus grad phi square half minus V phi. So, phi dot square minus half phi dot square is this object this minus minus becomes plus is the potential energy well not we make a change this grad phi is like a some kind of tension coming from that. So, this is I had told first string we call this is a potential, but in field theory this kind of terms are called potential this is potential grad phi is intuitively you can think of that is potential also, but this notation keeps changing the definition.

So, this is coming from here. So, I hope it is clear, but this is Hamiltonian not yet because Hamiltonian should be function of phi this one and potential sorry momentum density. So, this is not function of this should not be here, but I know what phi dot is phi dot is nothing, but pi. So, this one becomes that and remaining two terms remain as is. So, that is our Hamiltonian a given Hamiltonian I can derive the equation of motion.

So, I wrote that know d phi dt is function of dt of phi with H with phi. So, here is straight forward is this going to be the right hand side will be just pi. So, phi dot is pi straight forward. Next pi dot is functional derivative of H with phi functional derivative and is not the simple derivative. So, functional derivative recall I had to perturb the function by delta function and subtract and do all that operation which.

So, it is a reasonably complex operation, but you need to do that. If you do partial derivative I will get only this term. So, that would have given you only this term, but because it is a functional derivative there will be a contribution from here as well. If I verify then grad phi also will vary. I put a delta function to phi.

So, the gradient also will get some kind of perturbation and that contribution comes here and recall what I did when I do the by parts I will get a minus sign and divergence. So, these all these things are done by parts and the surface term vanishes and we let it go to infinity when I do the integral and I get that term. So, this part will give us minus d V by d phi this one. So, I can take the total derivative.

I do not need to write. So, V is only function of phi and this one gives us V and this one is what partial H partial grad phi is just grad phi. So, that. So, I get this object. Now this is pi dot. So, I can use this one replace pi by phi dot.

So, this becomes phi double dot or this is dt d phi. So, that will give me the equation of motion for the field it is this. Earlier we did not have V and basically we had this equal to that, but now with V this is what we get new term. For relativistic field this was it came as minus m square phi. So, given Hamiltonian I can also derive the equation of motion.

So, this is simple example, but we will go to slightly more complicate example Schrodinger equation. So, Schrodinger equation is a psi is a field. So, we can also write down the Lagrangian for Schrodinger equation or Lagrangian for psi. Schrodinger equation will be derived as equation of motion. So, for this the Lagrangian is written like that.

You may this may be guessed it is coming from the Laplacian term no this will give you Laplacian this will give us V psi, but this term is give you what is a i h bar d by dt of psi. So, this looks this will be complicated i by 2 psi star psi dot minus i by 2 psi star dot this is a time derivative this is dt is psi. Now, let us try to derive the equation of motion. Now, whether this form is required well I am going to do that next set of slides on Noether theorem this is a good form for Noether theorem, but this is not good form for equation of motion. So, I need to convert this for equation of motion.

So, I can always add and subtract d by dt of a some function f. You know that know for particle dynamics you can always add total derivative of a function total time derivative of a function that because my action does not change because I have d by dt d f by dt and so f at the end points is this will give you 0. So, you can always add and subtract. So, here I can add in basically subtract no add d by dt of psi star psi if I do that then I will get in this you just add if I do that then you can see that well half put half here. So, if I do that one term will cancel psi psi star dot and I will well this is add is plus half of that.

So, one term will cancel with that and you the first term will get amplified and we basically get that half plus half. So, i psi star psi dot minus this term and minus that term. So, this is Lagrangian for field that will give me Schrodinger equation. So, let us start doing the stuff now this is Schrodinger equation non non relativistic. So, the first term partial derivative of Lagrangian density by psi star.

So, what will that give me this one. So, take the derivative psi star appears only it appears here as well as here two places. So, the first term will give us what? I psi dot here and the other term will give us minus V psi this is this term. Now, this is nothing called d by dt of psi star in this equation no no this is no this is a term here dt psi star no no there is no there is no term dt psi star is not there. This is psi dot I remove that dt psi star dot there is no term which is psi star dot I remove that by this trick.

So, I will get contribution from here. So, what will that give me? It will come from here no d partial derivative L with partial psi star x. So, that will give me grad psi or d by dx psi by this this object and I take another derivative I will get that. So, this term gives me minus half 2 m grad square psi. So, this equal to that and that is this equation. So, this is Schrodinger equation for field psi Lagrangian is this and we need this Lagrangian in future.

You can also derive Hamiltonian. So, this I will leave it for you to work it out. So, we just follow the same procedure and this homework you should do that homework it Hamiltonian. Now, from Hamiltonian can I derive Schrodinger equation? This Hamiltonian I should derive Schrodinger equation. So, let us do it. So, this is my Lagrangian from that I can get pi.

So, partial L partial psi dot which will give me i psi star right I mean this will act here and I get that. So, momentum density is i psi star. Then next I apply pi dot is partial H partial psi. So, partial H partial psi will be coming from this term which is that and I will get from here. So, in fact, it should be not partial this functional derivative.

So, that will give me this from this one, but pi dot I take a pi equal to i psi star. So, this is going to be i psi star dt like that coming from here. Now, they of course, this function of psi star. So, take a complex conjugate of the equation what will I get if I take the complex conjugate i becomes minus i this will become change sign. So, let me do it here minus i d t psi equal to V is a real function.

So, V does not get complex conjugate the minus sign no this correct here minus i star L minus i plus half 2 m grad psi. Now, this is a minus is half by minus no. So, just put minus 1 multiply both sides i H bar H H bar is 1 for me H bar I am not carrying around that is minus 2 m square psi plus V psi. So, this equation for the Schrodinger equation.

So, we can derive this for both Hamiltonian well this is a minus sign I made a mistake.

So, we can do it from Hamiltonian as well as from Lagrangian.