Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Lecture – 11: Part 1

So, now we will talk about Lagrangian, Hamiltonian, so these are called functionals. Why it is called functional? This is the next slide. So, Lagrangian is function of functions, right. When you say Lagrangian of a particle, it is function of x, position of the particle which is function of time and x dot t, right. This is you done in mechanics. So, Lagrangian is function of position which is function of time.

So, x of t is a function, right. This is standard school text, school function, what you studied in schools, but Lagrangian is not a function. Lagrangian is function of x which itself is a function of time. So, these are called function of functions or is called functional and it is in the standard notation is f not as, not a small bracket, but a big large bracket like that.

So, this f is function of functions which is f and it is not one function, it is kind of any arbitrary function we can choose. So, one example, several examples I will just illustrate. So, we have a functional, this is functional, f, capital F is the functional which is integral 0 to 1 fx dx. So, whatever fx I insert, I will get a number. So, normally for functions we insert like value of time, but here you have to insert the full function.

Now, insert f of x and I get a number. So, f of x let us say I put x to the power 4, this function, then what will I get? So, I put x4 dx which will be x5 by 5 and answer is 0 to 1 which will be 1 to f. So, I get a number of course, but its value depends on the function I insert. So, this guy is accepting a function as an input. So, function is, so this is a functional, it inserts, it takes the function and also a argument x, is a machine which is taking a function, but it says, well it has some parameter a knob and knob is x and it gives you value of the function f at x.

So, the definition of this functional is this function f, f of y delta y minus x dy. So, I need to get a, it is a filter function, this is a filter function. So, it gives you value at x which is a knob, you can change of course knob to x, but its input is a small f function and you can easily see that this integral will give you f of x, because this is property of delta function, so this is f of x. So, there are huge number of algebra we can do with this, this new thing called functional. Now, I again, I will not dig deep into this, but I will just use it for whatever we need.

So, I need to define variation principle or I need to state it and that is what we are going to use functional for that. Functional derivative, so we know how to do the derivative of a function dF by dx or dF by dt, it is tangent, but here there is no, I mean, it is a functional, so I cannot define a tangent. So, how do I define a tangent of a function? So, the idea, the definition, I think you have to be careful is this definition, it is a rigorous definition. So, this is standard Newton's idea or Leibniz idea of function derivative, derivative of a function fx plus epsilon minus fx by epsilon. For any function f of x, at this point I want to take a derivative, I just take two points which are close by and do it.

For a functional derivative, so let me put the definition. So, these are functional f, it takes this argument f. So, I take the difference of two functions. So, because the function, this capital F takes function as an argument. So, it takes g is the argument and small f is an argument, g is a function and f is a function, but g is a slightly disturbed function, slightly function which is altered from f by small amount and how much is that deviation? It is this, epsilon delta x minus x prime.

So, g and f are very close, but at x, I have a delta function and is multiplied by epsilon, epsilon is small. So, delta is a big, it goes to infinity in fact. So, I multiply by this by epsilon and this is the difference. Now, I am going to use, subtract capital F g minus capital F f. So, now let us substitute it here, f will be involving f x prime plus epsilon delta x minus x prime, this minus f of f x prime and this divided by epsilon limit epsilon goes to 0.

So, this guy will give you a number and this guy will give a number and of course, it will be proportional to epsilon and you divide by epsilon and that is the answer. So, let us illustrate by some examples. So, my one first function is integral 0 to 1. Now, I want to compute this functional derivative at x naught. So, I just substitute by definition.

So, I create a new function g of x which is this function f x plus epsilon delta x minus x naught. So, I am wanting to compute at x naught and subtract it with this guy. So, difference will be what? This will cancel with that, epsilon in fact will cancel and you get a delta function integral and delta function integral will be 1 not always when x naught is within 0 and 1. So, if x naught is within 0 and 1, I get 1 as 0. So, that is the answer of this derivative.

Of course, the value of the function I told you how to compute it is just integral, but if I want to take a derivative this is what it is. Normally, so far what you might have done is just well if I take a derivative of this with delta f by delta f by the way please keep in mind that this delta is not this d dx by dt like this d is a derivative this is called functional derivative delta this delta is a functional derivative. It has meaning which is given by this definition. Now, it worst, but there is a lazy way to do it if I want to take a derivative here then I just simply go inside and take a derivative then you will get basically you may guess it will be 1. Of course, this is integral.

So, that is a lazy way to do it, but this is the definition. Example 2 slightly more complicated example this is f x to the power n pi x prime dx prime this should be x prime here dx prime. Now, we get a number now with this, but this is a auxiliary function phi x prime. Now, the derivative you can I will not do the algebra, but same idea I add a epsilon delta function here do all this algebra, but finally, you will get this functional derivative of this functional is this you may make a guess that you will get n f x n minus 1 phi x you might make a guess. Now, when this is I need epsilon delta x minus x now I when I do this n f x prime plus I do the Taylor expansion when I do the Taylor expansion I will get the leading order term to be that the one cancellation in the next term which will be non zero will be this term and you do the delta integral and you will get this more example.

So, this is a math. So, I thought I will give this example, but I taken all these examples from this book called quantum field theory for gifted amateur Lancaster and Blundell. So, this is the book if my functional is like that function of function this is g f f of x. So, then the functional derivative is g prime f x now some of it you can just use as a formula you do not need to derive it all the time this is the formula a derivation you can refer to the book and g f x prime f prime. So, now, this g f f x f prime this is a minus d by d x terms it comes because of we do not have two integral parts. So, this for this functional the derivative is that I have an example.

So, imagine the functional is f prime x squared and I ask for functional derivative of this functional then which formula should I use I should use this formula function of a derivative. So, then minus d by d x d g a d g a prime by f prime now this is you take the derivative of this which is 2 f prime and d by d x of that will be this whether I am going to use it very soon for particle Lagrangian. So, this look like and uses math, but it is not we will just going to use it very soon for particle dynamics. Now, this is the kinetic energy now we are already getting into particles. So, kinetic energy of a particle what do we write half m v squared v is x dot.

So, half m x dot square by 2, but for functional derivative I need a integral. So, you better write in terms of integral. So, we make average kinetic energy in a small time domain. So, 0 to tau d t prime of half m x d t. So, I will create a new variable t prime and I integrate it in that domain this is exactly same to that same as that when you take the limit tau going to 0.

So, what is the functional derivative of this. So, I define by the way keep in mind that this kinetic energy is a functional this can be at least this one why it is functional because I can get a number for any x t in standard mechanics we write x dot we use x dot as a number, but x dot is not really a number x dot is a functional because x t well x is function of time, but when I say kinetic energy then I say this square. So, this is function of function. So, it is really a functional. So, t is a functional and I define a functional derivative of this with x and what is that.

So, remember the formula we had f prime g of f prime this is already a derivative squared. So, I last example I just did that. So, if you have g of f prime then the functional derivative is minus d by d t of delta g by delta f prime that was the sorry this to get it this is formula 2 f prime by d x. Now, here prime will be d by d t. So, in that formula I will get d by d t of 2 d f is x.

So, d x by d t is a minus sign. So, that will be minus 2 x double dot. So, that is a second derivative of time. So, this is what we got second derivative of time and this half will cancel 2 and this 1 by tau and mass is here. So, this is the functional derivative of kinetic energy with x m x double dot.

So, we can get Euler Lagrange equation by from Lagrangian of course, this will be one ingredient in that formula. So, let us look our particle in a potential. So, principle of Least Action, so we define Lagrange. It is a crash course on particle dynamics I will finish in 15 minutes, but we will use in terms of functional derivative L is T minus V and I am going to define T as a what I did in the last slide integral well I am not divide I should have divided by T, but well we will ignore right now. So, this is average kinetic energy minus average potential energy no sorry action I am defining action apologies action is integral L dt t1 to t2 that is action.

And what is the principle of least action that particle chooses a path for which action is extremum. So, that is what we are going to use. So, first we define the action and action is well is L integral d t. So, it is T minus V. Now, I am going to call this as S1 and this is called as S2.

This is extremum that is the principle of least action it could be maximum as well, but normally it is minimum. Now, we have done this. So, how will I write in terms of functional derivative S is a functional. So, I write delta S by delta x this should be 0 my function is x which I am varying particle trajectory it could be particle. So, if you plot x plot it is this path or it could be that path or could be that path and classical trajectory is one path for which action is minimum this you done it, but I am just stating it again.

So, I need to write. So, basically delta S by delta x is 0 that means delta S1 by delta x t plus delta S2 by delta x p is 0 these are functional derivative. So, I use delta not partial. So, this part we already done it delta s 1 this is minus m x double dot there is a limit tau into 0, but we do not need to worry this is m x double dot. Now, what is this functional derivative of this one here it is function of x only. So, the g of f integral d of x here the derivative is simply g of g prime of f we did that in a previous slide.

So, that is why I say v prime of f. I am using the formulas and now the sum of these two must be 0 because it is extremum. So, this is 0. So, that gives us m x minus m x double dot minus v prime is 0. So, that means this. So, this is for a particle in a potential this is standard derivation, but I think I am not this is a rigorous derivation in terms of functional I am not sure how it was done in 401, but this is more rigorous derivation.

Now I am going to also get to Noether's theorem. So, Noether's theorem is. So, I will also show you how we can exploit this functional derivatives to get to more complex physics. Now, this derivation is from this website Kevin Brown is a interesting website the math pages dot com and there are lot of topics are covered in a nice way. I am of course, we are going to change the trajectory.

So, I have right now I am only using particles we will go to fields after this discussion, but for particles I have x t no x of t. So, this is t and this is x for functional derivative we need to change the trajectory. So, I am following this website I am going to write this epsilon delta 1 t this is a general small function it is not a delta function, but I will just adopt this notation is a function, but epsilon is in front. So, it is small function it could be a blip. So, we have just imagine this to be a small delta 1 to be a function is a perturbation on x.

So, this is my perturbed trajectory and I will say my delta s by delta x this would be 0. So, delta I t. So, here we are we may have a more than one particle. So, we have particle level is i. So, I can change the trajectory for any particle.

So, I define the action dl by dt and take d by d epsilon. So, this action I want to minimize it. So, one way is to just take the derivative with epsilon and set it to 0. So, L is Lagrangian is function of x and x dot. So, we will have delta partial L partial xi and partial x dx by d epsilon plus partial L partial xi dot this dot sitting here and this dot here.

So, this function of x and x dot. So, I need to take two partial derivatives for every i and some more all i some more all particle. Now, this bit of algebra, but when I took the do the epsilon this what I get xi. Now, this is can remove this is this should be 0. Now, we want this action to be minimum.

So, this should be 0. In fact, this just this statement. Now, well there is one difference that this one is derivative what is delta xi by delta epsilon that will be delta i. Now, because if I take the derivative with epsilon i will get this delta i. So, I get delta i here and delta i dot this delta i dot. You have to just bear with this algebra, but I think you will get something interesting after this.

Now, we have one variable delta i here and here is delta i dot. So, what should I do? Idea is to convert delta i dot to delta i. So, that I can combine them and that is where I get Euler Lagrange equation. So, I use by parts this part I am going to use by parts.

So, this is done here. So, this first function integral of second plus function integral of second which is that minus derivative of the first function d by dt integral of second. In fact, this is standard derivation. I am just redoing it. This should be 0 when this statement is equivalent to that. Now, for derivation of Euler Lagrange equation what do we do? We are varying this xi with this delta.

What is the constraint on this delta for Euler Lagrange equation? Exactly, end point must be fixed. So, if we fix end point then what happens to this interval? It goes to 0 and we get this to be 0 and this 0 for all delta I's as long as the end points are fixed that means inside this guy should be 0. For arbitrary delta I should be 0 that means the integral must be 0. That is the argument which is given and that is the Euler Lagrange equation delta is end point and we get this.

This derivation all of you know and excellent. Now, what if delta is not 0 at the end point but is arbitrary? It is possible, no? And that is where Noether's theorem come. Noether is a famous lady in she wrote this paper in 1916 and to just to know the background women were highly discriminated and she was in she did not get a position in the university. She was doing as a assistant of a in fact Hilbert first and later she got some temporary position. Anyway, this is a very famous theorem which was which is used everywhere.

But for particle Lagrangian I am just going to state it. If delta is arbitrary but you need to keep the Lagrangian unchanged. So, I change my delta my x is changing but I have to constrain that my Lagrangian should be unchanged. So, for example, if your Lagrangian like half if I am going to give the example in a minute x dot square plus potential. Now, this Lagrangian is not function of time, explicit function of time. So, if I change time to time plus epsilon or time plus something is this Lagrangian unchanged.

So, for two particles potential which is 1 over r minus r prime this potential for two particles. So, we will have two kinetic energy terms particle one kinetic energy, particle two kinetic energy plus potential energy. This Lagrangian well I said minus now minus vx this Lagrangian is unchanged if I change the coordinate shift the coordinate axis. Because r and r prime both are shifted by constant number so it is unchanged.

So, Lagrangian is unchanged by many many various combinations. So, imagine that I do some change in my coordinates. So, that Lagrangian is unchanged then what can you say about the which quantities are conserved that is what is non zero. It turns out we have delta I's which are arbitrary, but they are not changing the Lagrangian. So, we go back again to this equation which we derived this should be 0 because Lagrangian is not changed.

Now, of course, delta is not 0 at the end. So, this function is not 0 by Euler Lagrange equation of course, assume that this Euler Lagrange equation is of course, respected followed. So, this is 0 by Euler Lagrange equation. So, I had to get I will get this. So, delta L by delta xi dot multiplied by delta I is constant.

So, this is a conservation law. This is going to be constant during the evolution of the system. Now, I will illustrate by examples that is where I think will become clear and this is called Noether's conservation law. Now, of course, we want to get some you want to again simplify that that equation. So, idea is so let us look at some examples. So, again we will work with 1D and 1D with n particles, but I assume that potential is function of x minus xi minus xj is a separation.

That means, if I shift my coordinate system by a constant value, then my Lagrangian will not change because this will also shift by a and this also will shift by a. So, my Lagrangian is unchanged by coordinate shift. So, this is called space translation.

So, here I am shifting all of them. So, this delta I will be is a shift. So, that will be delta sigma. I am making a change. So, xi prime actually is xi plus delta sigma. So, this is delta sigma and the same for all in the sum.

So, that will come out and it comes out is basically that. So, when it comes out, then I will get this sum. This sum is constant xi dot. I am sorry apologies. This sum is constant not for individual because each particle is shifted by a constant amount and this is the total momentum of the system. This is mi xi dot where Lagrangian is x square minus potential v xi minus xj sum over this is sum over all ij this sum over all xi i's.

These are Lagrangian. I hope you can see my handwriting is probably not too good, but Lagrangian is kinetic energy of all the particles plus potential energy which will involve two sums. Well, you can put a factor half not to double count. Now, this even I do the derivative partial derivative with xi dot, I will get contribution only from that which is this is a total linear momentum. This is total linear momentum and that is constant not individual momentum because individual momentum will change with because of interaction and this is what you get from Noether's theorem and this is one example.

I mean of course, you can derive more things. So, this for 1D, you can do it for 3D. Now, 3D, this is I do not need for this course, just for completeness. We define a generalized coordinate qi which has xi xj xi yi zi is a thing. So, the qi is 3 times n when n is the number of particles. Now, it is very easy you can do it just follow the same logic, but now I can vary xi independently yi independently zi independently and then we basically put a coefficient in front kx, ky, kz, they are not equal now, these are semicolon.

And then for so basically from this derivation which you can do it easily, space transition along x, y along z all are independent. So, you get three conservation laws, linear momentum along x, linear momentum along y, linear momentum along z. And this kx and ky basically are arbitrary in fact, it follows that this kx multiplied by linear momentum along x, this is along x direction along y direction along z direction that is constant. So, it is a nice norm and you can derive this formally, you do not need to apply any physical argument, it is mathematical and you can derive it. Space rotation is very similar, so I my shift of the coordinates, idea is to model this.

So, I have n particles when I rotate about z axis, rotate about z axis, z cap then what happens to x and y? So, you can just verify that dxi is d sigma yi and dyi is minus d sigma xi. So, this is the d sigma is d phi azimuthal angle, for small angle this is what you will get. So, cos theta i sin theta and so you will basically get this one. So, my d sigma is the parameter and d zi is 0.

So, I just substitute it here and what I get is here. So, this one will give us mi xi dot mi yi dot and so xi dot is multiplied by yi and yi dot is multiplied by minus xi minus xi. And what is this guy? xi dot yi minus xi yi dot angular momentum, so this is Lz. So, Lz is constant, multiplied by mass, so r cross p, so this is constant. Third one which is slightly more complicated is time translation. It is not complicated, but we are just you know it is not following the same logic, but slightly different.

So, this is the equation dL by d sigma. So, this is change in L, the delta L we had written and we set this to 0. But I will say well I do not set it to 0, but when I change by parameter sigma, then I get dL by d sigma, right hand side is dL by d sigma. Do not set it to 0 and then I rewrite that as a like that and well basically choose sigma as time. So, we are rewriting this equation and d by dt L minus the sum and what is this object? This is Hamiltonian, this is Hamiltonian. So, Hamiltonian is conserved and this is in fact very important assumption in this, that L is not function of explicit function of time, right.

This is not coming in our equation. So, L is independent of time, is not explicit function of time. So, the potential has no time dependence. So, potential between two particles which is gravitational potential, it is not function of time. So, for those systems energy is conserved, but if you have oscillator which is forced by electric field which is function of time, then energy is not conserved.

So, this is the assumption. So, that is why it is called time translation. So, time now that system energy now or Lagrangian now or two hours later will be the same Lagrangian, that is a time translation. So, this is a conservation law in fact from symmetry. So, these are symmetries. So, symmetry, first symmetry was space translation, then I said space rotation, time translation and there could be other symmetry.

So, I just gave the obvious ones. For fields we will do some more symmetries, but for particles we will just do the three main symmetries. You can derive conservation laws and mechanics of Landau is a good reference. There are many many references, but you can look at the website as well, the one which I cited and that is a good derivation. Thank you.