Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 09
Derivation of the moment equations II

So, in this lecture, we will derive formally the different order of moment equations for the velocity and we will see that finally, how it can be related to something which we already know. Maybe after this lecture that will not be 100 percent clear, but gradually it will be clear.

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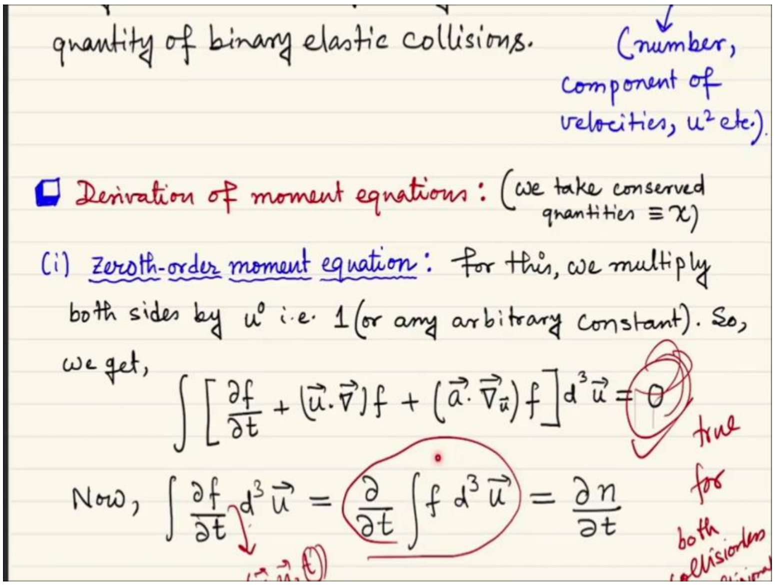

So, first you see that we start by deriving the zeroth order moment equation. That is the quite automatic thing and zeroth order means velocity to the power 0 and this is nothing, but one or I mean simply this is the I mean what is the conserved quantity for that because we know like only if we just try to derive a moment equation for something which is conserved in a binary elastic collision, then only for collisional Boltzmann equation, we will get rid of the collision integral after integration.

So, what is that? This is nothing, but the number conservation, two particles remain two particles right. So, that is the thing.

So, in order to derive the corresponding moment equation, we have to multiply the whole equation by either one or any arbitrary constant and then we have to integrate it in $\boldsymbol{u}$ space. So, and that is exactly what we have done here and the right hand side as we know that, now here is the catch that.

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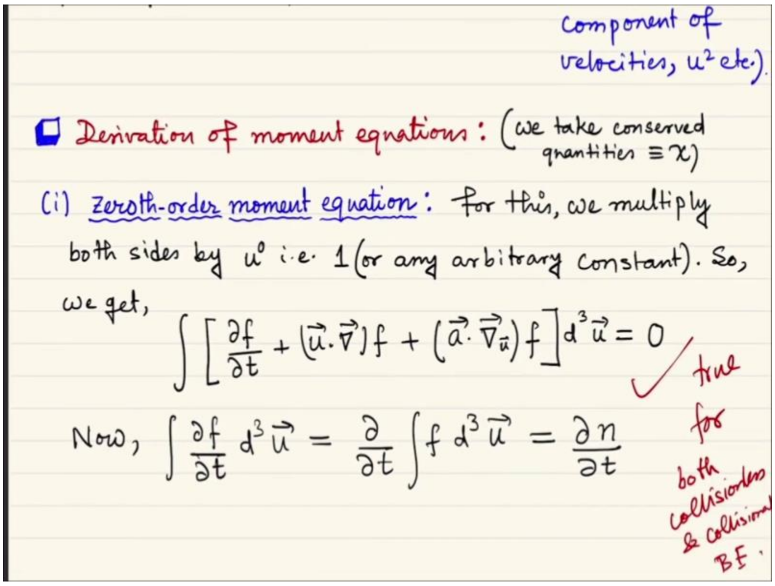

component of velocities, u^2 etc.)

Derivation of moment equations: (we take conserved quantities $\equiv \chi$)

(i) Zeroth-order moment equation: for this, we multiply both sides by u^0 i.e. 1 (or any arbitrary constant). So, we get,

$$\int \left[\frac{\partial f}{\partial t} + (\vec{u}\cdot\vec{\nabla})f + (\vec{a}\cdot\vec{\nabla}_{\vec{u}})f\right] d^3\vec{u} = 0$$

true for both collisionless & collisional BE.

Now, $\int \frac{\partial f}{\partial t} d^3\vec{u} = \frac{\partial}{\partial t}\int f\, d^3\vec{u} = \frac{\partial n}{\partial t}$

Now from this we cannot easily understand whether this is a collisionless Boltzmann equation or collisional Boltzmann equation because this is true for both collisionless and collisional Boltzmann equation. That is something you have to realize. Now, here what we do that, we just integrate everything in $\boldsymbol{u}$ space and we just say that irrespective of this is collisional or collisionless Boltzmann equation the right hand side will be 0.

Now, the first term, we try to evaluate term by term. So, now, the first term is $\int \frac{\partial f}{\partial t} d^3\boldsymbol{u}$ and here this is really important throughout $\boldsymbol{r}$ the course to understand how these reductions are happening.

So , $\frac{\partial}{\partial t}$ is activated when the systems, I mean the function has an explicit time dependence and this integration is over in $\boldsymbol{u}$ space. So $\frac{\partial}{\partial t}$, and the integration, they are totally under two different variables. So, $\boldsymbol{a}$, $\boldsymbol{u}$ and t, all of them are independent of each other. Although you know like both $\boldsymbol{r}$ and $\boldsymbol{u}$ they are implicitly depending on t, that means, when I am saying that f is depending on $\boldsymbol{r}$ and f is depending on $\boldsymbol{u}$. There is an implicit dependence on time, but explicit dependence of time is independent of its explicit dependence of and $\boldsymbol{u}$ and that is why this $\frac{\partial}{\partial t}$ and this integration can commute and $\frac{\partial}{\partial t}$ can come outside. And you can simply have $\frac{\partial}{\partial t}\int f d^3\boldsymbol{u}$,and what is $\int f d^3\boldsymbol{u}$? That is nothing, but the number density. So, that is nothing but

$$\frac{\partial n}{\partial t}.$$

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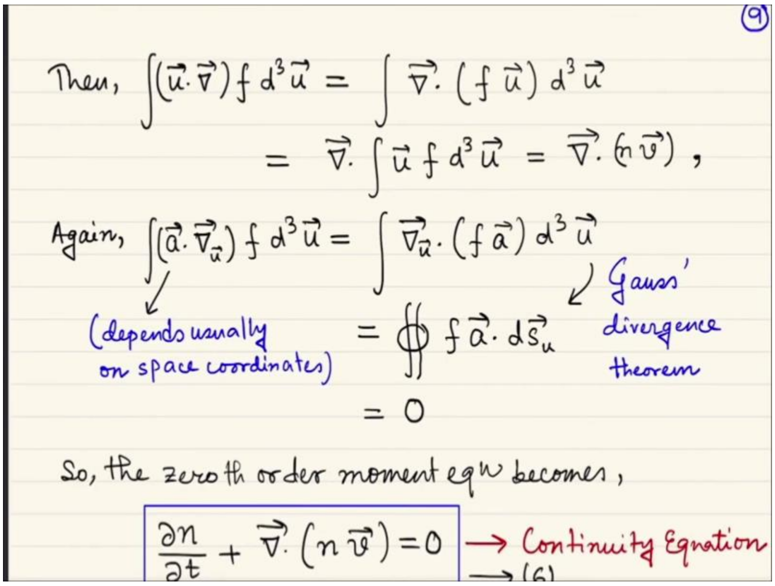

Now, what about the second term that was integration? $(\boldsymbol{u} \cdot \nabla) f d^3\boldsymbol{u}$. Nabla is a space operator. Now f has, space dependence as well as time dependence may have as well as velocity dependence, but $\boldsymbol{u}$ and nabla. Once again nabla is a pure space operator. So, it is the function of $\boldsymbol{r}$ in that sense and $\boldsymbol{u}$ is the kinetic velocity which is independent of space.

So, $\boldsymbol{u}$ and nabla they I mean $\boldsymbol{u}$ is independent of this space operator that is why I can write this one as $\nabla \cdot (f\boldsymbol{u})$. So, $\boldsymbol{u}$ will act as a constant vector in front of this nabla. So, it will simply be $\int \nabla \cdot (f\boldsymbol{u}) d^3\boldsymbol{u}$

Now, here is a common source of mistake. Now some students can tend towards an integration using Gauss's divergence theorem. Well, I have a divergence I have a volume integration, but this is not true because the volume integration is on velocity space and the divergence is in the for the pure space. So, this is not going to work. Now, what will be the proper reduction in this case?

This divergence is a space operator whereas, this integration is on the velocity space. So, this divergence will come out and what is inside? Inside you have $\int \boldsymbol{u} f d^3\boldsymbol{u}$, and what is that once again? This thing by n is equal to angular bracketed $\boldsymbol{u}$ which we call $\boldsymbol{v}$ that is the average velocity or I said like it is a bulk velocity or fluid velocity **v**, then I can easily say this thing is equal to $n\,\boldsymbol{v}$. So, this is nothing, but this is equal to $\nabla \cdot (n\boldsymbol{v})$. Again, the last term $\int (\boldsymbol{a} \cdot \nabla_{\boldsymbol{u}}) f d^3\boldsymbol{u}$. Now you see here I have an assumption a priori $\boldsymbol{a}$ can or cannot be a function of velocity, but most of the cases it usually depends on space coordinates.

For example, if it is a gravitational force, then most of the cases the I mean the conservative force fields. But in nature, we have several force fields which are very very relevant, but they are actually depending on velocity and that is why whether they are conservative or not that is a bit subtle for example, magnetic field which is a velocity a dependent force.

And for that also you can show that although this is I mean not independent of velocity, it is $\boldsymbol{a}$ dependent of velocity, but the divergence is 0. So, whether $\boldsymbol{a}$ is independent of velocity or whether $\boldsymbol{a}$ is a divergence less vector, I mean divergence less means in terms of the velocity divergence. In both cases you can write this. So, this is an expression which actually is true for both electrostatic, magnetic and also the very popular gravitational force. So, this reduction you can do.

Now, here we are just thinking of because we do not have charged particles here just the gravity in mind. So, $\boldsymbol{a}$ just can enter as a constant vector because $\boldsymbol{a}$ can be a function of time, $\boldsymbol{a}$ can be a function of space, but not a velocity function. So, it will come inside. So, now you have a velocity divergence of some quantity integrated in I mean volume integrated in velocity space.

Now I have the right to use Gauss's divergence theorem and that is why you have this thing and you can, then say that for this volume I can extend the containing surface to infinity. And so, that at infinity I do not have any matter or anything. So, on that surface, this flux is 0 and that is why I have a vanishing quantity.

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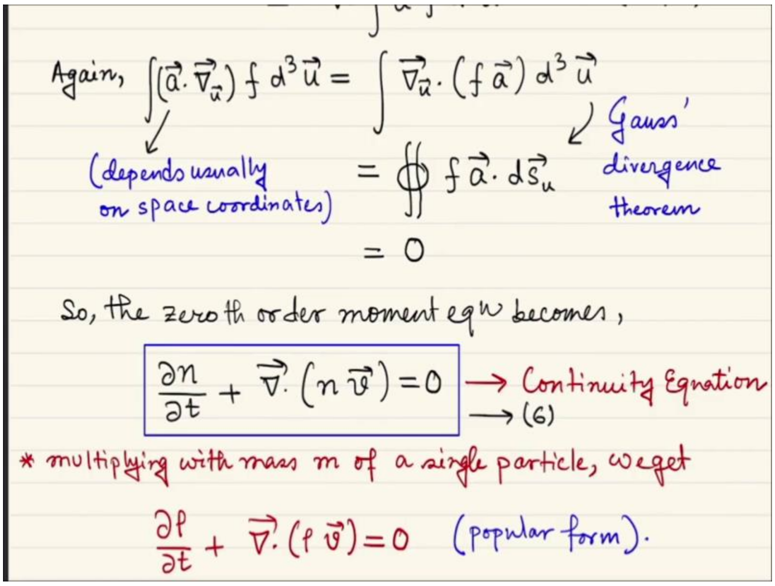

So, finally, this forcing term is not contributing at all, and the zeroth order moment equation simply becomes $\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{v}) = 0$. Now this is the most general form of continuity equation. Now of course, in very popular cases, like what we call for fluid dynamics, the continuity equation looks like $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$, and that is exactly can be immediately obtained if you just multiply this equation by the mass of one particle.

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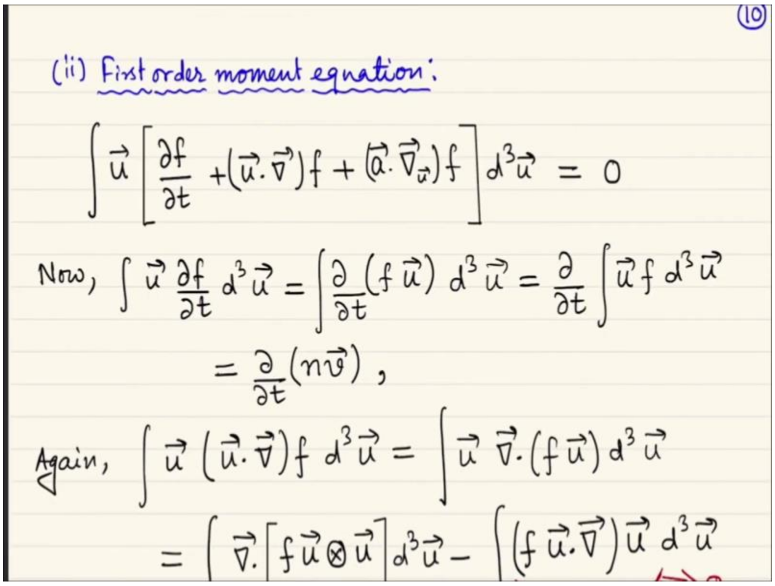

Now, what about the first order moment equation? So, you see that already something. So, the zeroth order moment equation gives us something which we already know in a different perspective that is the framework of fluids right. Now, here to be very honest, we have not considered till now any continuum or anything we have just done some integration in velocity space for the kinetic systems and we simply said that we are deriving moment equations. Do not confuse that at this step we are not even talking about the fluids. But we have this continuity equation. So, first order moment equation, this is the second I mean this is the next automatic step. So, then what we have to do?

We have to multiply the whole thing by vectorial u to the power 1 and which is $\boldsymbol{u}$ itself multiplied to this equation and the right hand side again irrespective of whether it is collisional or collisionless, it will be 0. Because $\boldsymbol{u}$ is something which is component wise and actually vectorially conserved in a binary elastic collision.

Now, the first term is $\int \boldsymbol{u}\frac{\partial f}{\partial t} d^3\boldsymbol{u}$, then what is happening? Again $\frac{\partial}{\partial t}$ and $\boldsymbol{u}$, they cannot see each other. So, $\boldsymbol{u}$ can enter inside $\frac{\partial}{\partial t}$. Again $\frac{\partial}{\partial t}$ and the integration, they are independent. So, $\frac{\partial}{\partial t}$ can come out of the integration. And what is inside? Inside this is nothing, but you all know that is n times $\boldsymbol{v}$ that we have already seen. So, this is $\frac{\partial(n\boldsymbol{v})}{\partial t}$.

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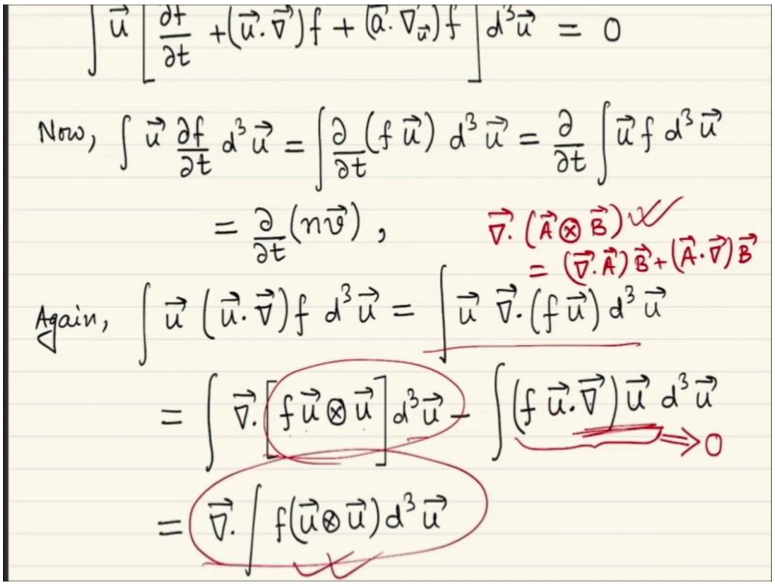

What about the second term? The second term is $\int \boldsymbol{u}(\boldsymbol{u}\cdot\nabla)f d^3\boldsymbol{u}$. Again, here you see that you have now this $\boldsymbol{u}$ and nabla, they are independent of each other. So, again $\boldsymbol{u}$ comes inside the nabla operator, nabla is the space operator. So, you have $\int \boldsymbol{u}\nabla\cdot(f\boldsymbol{u})d^3\boldsymbol{u}$.

Now, in the assignment 0, I asked you to do something a tensorial identity which is very very important, that is, if you have something like this divergence of two tensorial product of two vectors, then it should look like $(\nabla\cdot\boldsymbol{A})\boldsymbol{B} + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$. Although a priori at the first look it may look like some symmetric thing over $\boldsymbol{A}$ and $\boldsymbol{B}$, they are not. So, you see in the first case, you have a divergence of $\boldsymbol{A}$ which is multiplied over $\boldsymbol{B}$, in the second case it is the gradient is now operating over $\boldsymbol{B}$, but it is a different way. So, if you just interchange $\boldsymbol{B}$ and $\boldsymbol{A}$, this is not exactly the same. Interchanging $\boldsymbol{B}$ and $\boldsymbol{A}$, this is not exactly the same because $\boldsymbol{A}$ tensorial $\boldsymbol{B}$ has a symmetric part and also has a antisymmetric part. Just read about that, that is a very interesting thing to cultivate. Now, using this identity, you can write this thing as $\int \nabla\cdot[f\boldsymbol{u}\otimes\boldsymbol{u}]d^3\boldsymbol{u} - \int (f\boldsymbol{u}\cdot\nabla)\boldsymbol{u}d^3\boldsymbol{u}$.

Now the second one is 0, because nabla is acting on $\boldsymbol{u}$ and nabla is a space operator so, nothing to do. So, this is 0 and you have simply the divergence of this tensorial product.

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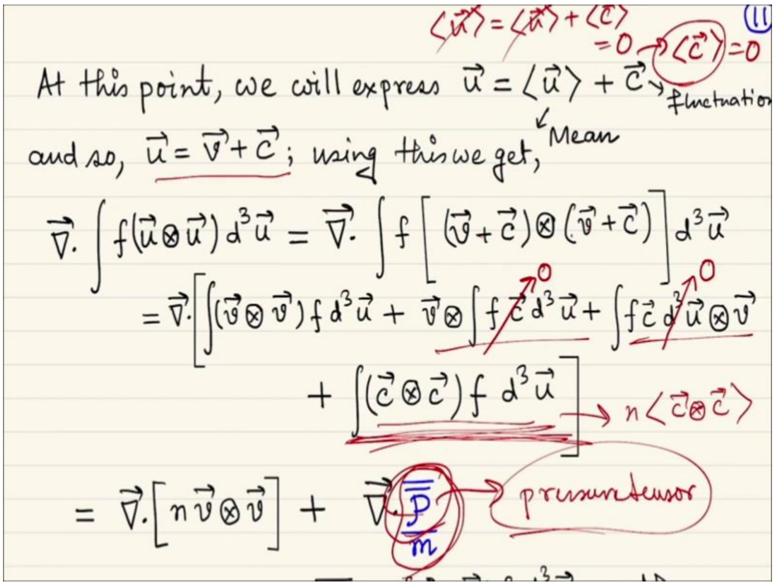

Now, at this point, we can do something much more interesting. We express the $\boldsymbol{u}$ at every point as the mean value of it and the fluctuation with respect to the mean. So, the mean value is nothing, but the angular bracketed $\boldsymbol{u}$ or $\boldsymbol{v}$ plus, we call the fluctuation as $\boldsymbol{c}$.

So, the $\boldsymbol{c}$ is fluctuation means that the statistical average of $\boldsymbol{c}$ will be 0. The statistical average of fluctuation is always 0 because if you do the average in both sides $\boldsymbol{u}$ average will be equal to $\boldsymbol{u}$ average. So, which is again $\boldsymbol{u}$ average plus $\boldsymbol{c}$ average and this will be canceling each other giving $\boldsymbol{c}$ average is equal to 0, that is easy to see.

Now, if we write in this way finally, $\nabla \cdot \int [f\boldsymbol{u}\otimes\boldsymbol{u}] d^3\boldsymbol{u}$ is equal to then every $\boldsymbol{u}$ can be written in terms of $\boldsymbol{v}$ and $\boldsymbol{c}$ and then you will have to do actually tensorial product four terms.

So, the 1st term will be including $\boldsymbol{v}$ and $\boldsymbol{v}$, 2nd and 3rd term will be a mixture of $\boldsymbol{c}$ and $\boldsymbol{v}$ where you can exactly show that both the terms will vanish because of the simple consideration that the statistical average of the fluctuations, they are vanishing and the final term will be containing only $\boldsymbol{c}$. And this is the tensorial product of the fluctuations. And this is actually is nothing, but it's another way of writing $n < \boldsymbol{c}\otimes\boldsymbol{c} >$. So, everything is under divergence. So, that thing is known as another tensor $\overline{\overline{\boldsymbol{P}}}$ which is the second rank tensor by the mass of the particle and this $\overline{\overline{\boldsymbol{P}}}$ is known as the pressure tensor. Now, the question is that how we know that is the pressure tensor. Well, we actually do not know. This is somehow from when we mix our previous knowledge of kinetic theory a bit and we also mix our previous knowledge of fluid mechanics, then only we see that this if we identify this tensor as $\overline{\overline{\boldsymbol{P}}}$ or the pressure, then everything looks actually matching, that is why it is the pressure tensor.

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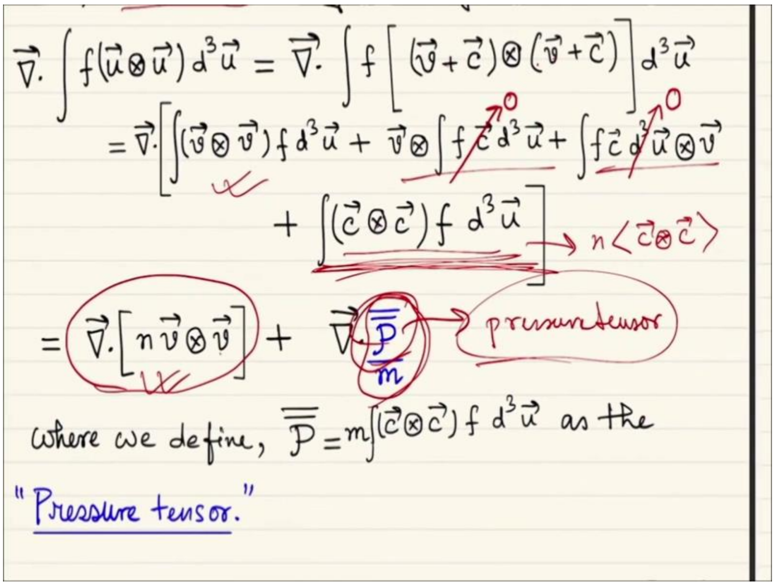

Now, let us try to understand these equations; what I am doing here? I am trying to derive the moment equations starting from kinetic equations. Now, finally, we will see that the target is to derive the fluid equations starting from the kinetic equations right, but, well before that the engineers, they already derive fluid equations sometimes by empirical things, sometimes by macroscopic considerations. We also do something using macroscopic considerations.

In normal fluid dynamics book, you will always see that derivation of the force equations or the continuity equations, is using the macroscopic considerations. So, finally, to be very very honest this is not something new what we are doing, this is we know something the results the final relations we know. Now we try to reach or recover those things starting from the microscopic theory or the kinetic theory.

Now, you see here this part is the part where the term is containing only $\boldsymbol{v}$. So, this is $\nabla\cdot[n\boldsymbol{v}\otimes\boldsymbol{v}] + \nabla\cdot\left(\frac{\bar{\bar{P}}}{m}\right)$.

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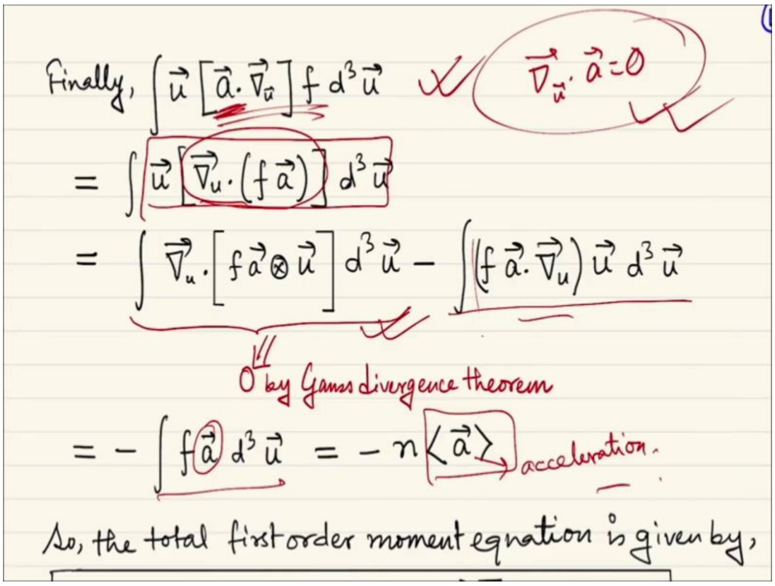

Finally, we have the last term of the forcing. So, we have $\int \boldsymbol{u}[\boldsymbol{a} \cdot \nabla_{\boldsymbol{u}}] f d^3\boldsymbol{u}$. Once again you see $\boldsymbol{a}$ is a function of space operators usually. So, $\boldsymbol{a}$ should go inside and that should give you divergence velocity divergence of $f\boldsymbol{a}$. This is also true if $\boldsymbol{a}$ has a velocity dependence, but $\boldsymbol{a}$ is divergence less, that means, $\nabla_{\boldsymbol{u}} \cdot \boldsymbol{a} = 0,$ that is also possible and that is exactly the case for magnetic field.

So, if you have this, then you can again say that using that tensorial identity this is nothing but, $\int \nabla_{\boldsymbol{u}}[f\boldsymbol{a} \otimes \boldsymbol{u}] d^3\boldsymbol{u} - \int (f\boldsymbol{a} \cdot \nabla_{\boldsymbol{u}})\boldsymbol{u} d^3\boldsymbol{u}$. Now, the 1st term is 0 because now, you can use here the Gauss's divergence theorem. Remember, previously we could not use Gauss's divergence theorem because that was spatial divergence with velocity integration.

Here you can and you can make it 0. What about the 2nd term? This is easy you can just expand and check this is nothing but is equal to $-\int f\boldsymbol{a} d^3\boldsymbol{u}$. And what is that? Any quantity when this is just integrated with a $f d^3\boldsymbol{u}$ that is nothing but n times the angular bracketed of that quantity. So, it will be $-n < \boldsymbol{a} >$, $\boldsymbol{a}$ is nothing but the acceleration once again.

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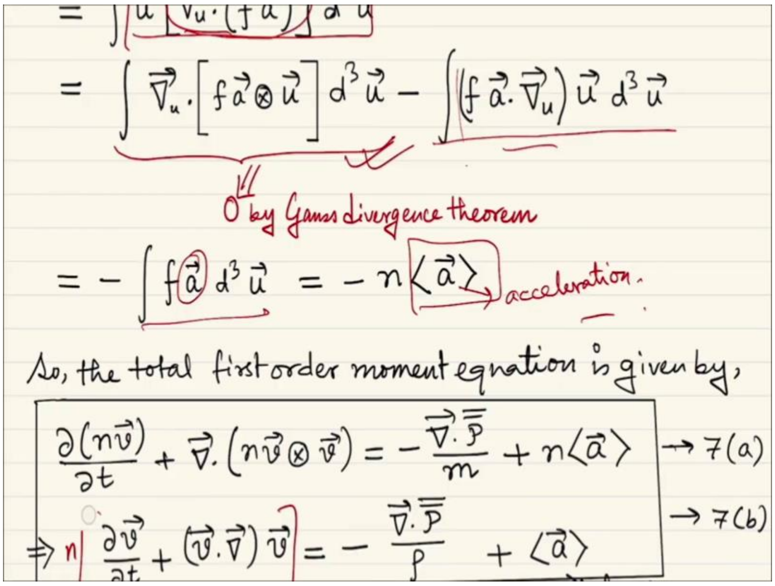

So, the total first order moment equation can now be written as

$$\frac{\partial (n\boldsymbol{v})}{\partial t} + \nabla \cdot (n\boldsymbol{v} \otimes \boldsymbol{v}) = -\frac{\nabla \cdot \bar{\bar{\boldsymbol{P}}}}{m} + n < \boldsymbol{a} >$$

Another alternative way of writing this, is if you expand everything, just use the tensorial notation, tensorial identity once again and use the continuity equation. Then finally, you will see there are two terms, there will be cancelling due to continuity equation and you finally will have

$$n\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right] = - -\frac{\nabla \cdot \bar{\bar{\boldsymbol{P}}}}{m} + n < \boldsymbol{a} >$$

Now, if you multiply everything by m, then it will be a ρ everywhere. So, you again divide everything by ρ. So, finally, you will have this form

$$\left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v\right] = --\frac{\nabla \cdot \bar{\bar{P}}}{\rho} + < \boldsymbol{a} >$$

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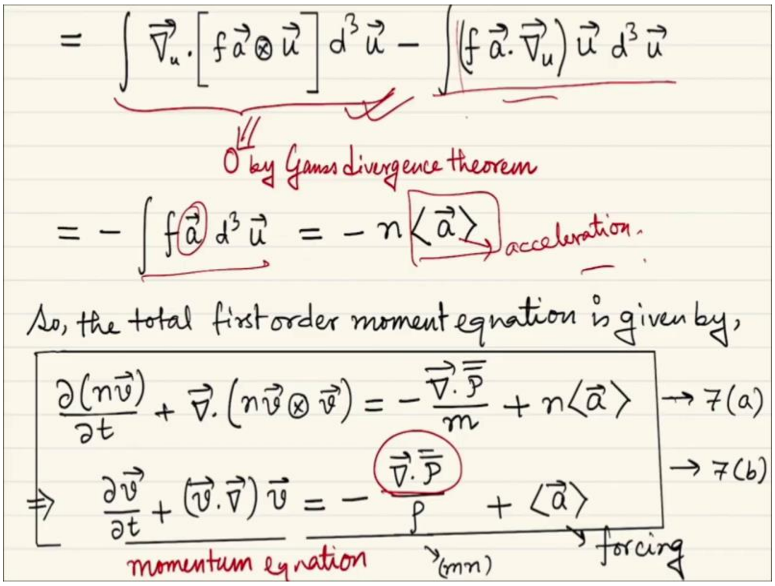

The best way is to do it by yourself. If you do it by yourself and you cannot reach there, please let me know.

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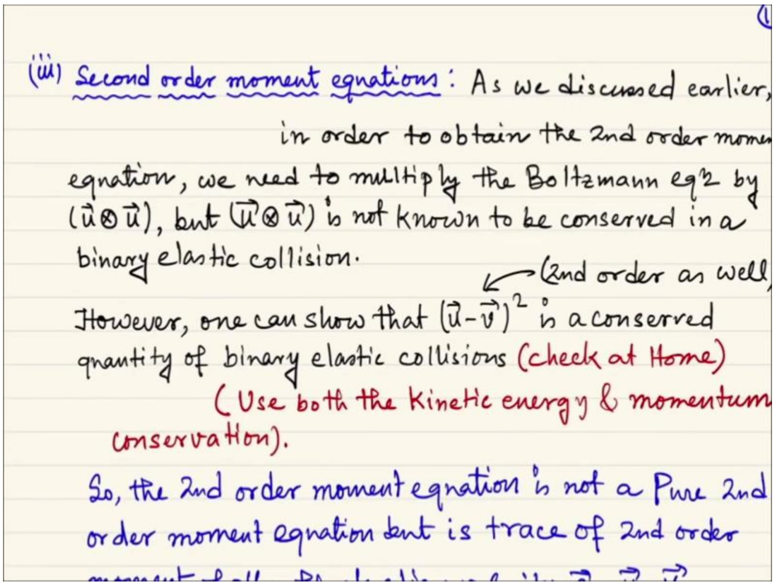

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$$= \int \vec{u}\,[\vec{\nabla}_u\cdot(f\vec{a})]\,d^3\vec{u}$$

$$= \underbrace{\int \vec{\nabla}_u\cdot[f\vec{a}\otimes\vec{u}]\,d^3\vec{u}}_{0 \text{ by Gauss divergence theorem}} - \int (f\vec{a}\cdot\vec{\nabla}_u)\,\vec{u}\,d^3\vec{u}$$

$$= -\int f\vec{a}\,d^3\vec{u} = -n\boxed{\langle\vec{a}\rangle}_{\text{acceleration.}}$$

So, the total first order moment equation is given by,

$$\frac{\partial(n\vec{v})}{\partial t} + \vec{\nabla}\cdot(n\vec{v}\otimes\vec{v}) = -\frac{\vec{\nabla}\cdot\overline{\overline{P}}}{m} + n\langle\vec{a}\rangle \rightarrow 7(a)$$

$$\Rightarrow \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}\cdot\overline{\overline{P}}}{\rho} + \langle\vec{a}\rangle \rightarrow 7(b)$$

Finally, one thing is that you see this is nothing but the momentum equation. So, of course, here it should look like some acceleration equation, but it is actually momentum equation in the sense that this is an evolution equation of momentum. So, which is roughly called the momentum equation. So, here for example, if you just multiply this by m so, that will be then nothing but $\frac{\partial}{\partial t}(mn\boldsymbol{v})$ and you multiply actually throughout.

So, I am just talking about this first term $\frac{\partial}{\partial t}(\rho\boldsymbol{v})$, but $\rho\boldsymbol{v}$**,** what is that? This is nothing but the momentum density. So, this is the evolution of the momentum density, that is why it is called a momentum equation. If you try to understand this is nothing but a Newton's law, the equation of force. So, acceleration is equal to some forcing.

So, this is actually you will see that in Newton's law, $\frac{\partial \boldsymbol{v}}{\partial t}$ and $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ gets combined to give $\frac{d\boldsymbol{v}}{dt}$ and that is equal to the whole source of force on the RHS. So, this is the acceleration equal to force. I mean well acceleration equal to force by mass of course. Here when we are talking about, now you will see when you are talking about macroscopic things or the continuum, we will not be talking about per unit mass, but per unit volume. So, there we will talk about the densities. So, here you have the force density which is equivalent to acceleration.

Now, let us talk about second order moment equations. Now, as we discussed earlier that in order to obtain the second order moment equation, we need to multiply the Boltzmann equation by $\boldsymbol{u}^2$, and that is nothing but the tensorial product of $\boldsymbol{u}$ and $\boldsymbol{u}$. But $\boldsymbol{u}\otimes\boldsymbol{u}$ is not known to be conserved in a binary elastic collision, that we do not know.

One thing we know that $\boldsymbol{u}^2$ is conserved, and one can actually show that, please do that at home that is a very very good homework. Please do that, that if all the components of $\boldsymbol{u}$, they are conserved because of the momentum conservation in a binary elastic collision and $\boldsymbol{u}^2$ is also conserved due to the kinetic energy conservation, then $(\boldsymbol{u}-\boldsymbol{v})^2$ is also conserved, $\boldsymbol{v}$ is a constant once again.

So, $\boldsymbol{v}$ is something which is a function of $\boldsymbol{r}$ and t that is true, but here in this case where you are talking about like for example, in kinetic level, $\boldsymbol{v}$ just is a constant because $\boldsymbol{v}$ has a unique value for the total system. So, this is an averaged quantity. It is a mean value.

So, just remember that when we are talking about collisions, we do not talk about the change in position, we are talking about a simply point position which does not change in a collision. So, if $\boldsymbol{v}$ is a I mean property of position only, then in a collision that should not change.

So, just using all these facts try to show that $(\boldsymbol{u}-\boldsymbol{v})^2$ is the conserved quantity. If its a conserved quantity, then we can derive a moment equation. So, this is a hybrid moment equation ok; if we multiply the collisional or collisionless Boltzmann equations by this and then integrate in $\boldsymbol{u}$ space.

So, once again as I just said that second order moment equation is not a pure second order moment equation, but is the trace of second order moment of the fluctuation velocity $\boldsymbol{c}$ which is equal to $(\boldsymbol{u}-\boldsymbol{v})$. So, this is nothing but $\boldsymbol{c}^2$.

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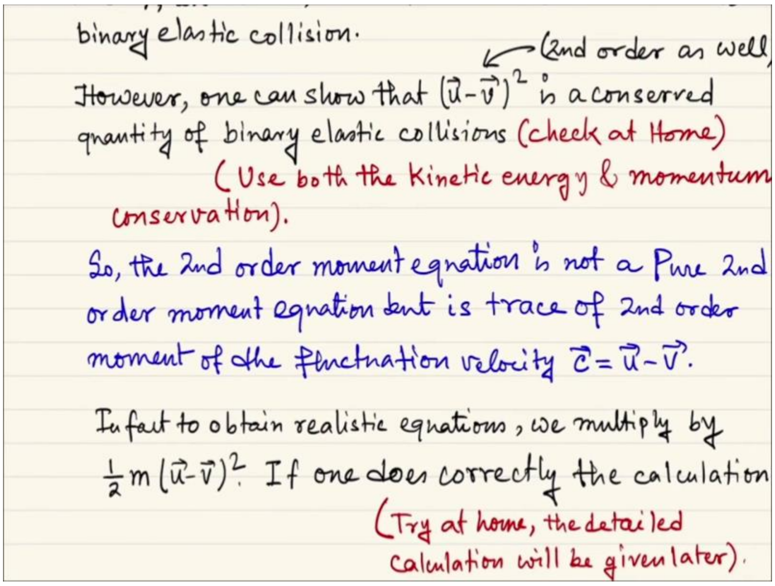

Now in fact, to obtain realistic equations that is something very important, we multiply by $\frac{m}{2}(\boldsymbol{u}-\boldsymbol{v})^2$ because then we recover something which we already know, we feel happy right. So, if one does correctly the calculation basically and this calculation is a bit longer. So, I will give you the solution after some weeks, but first try it at home.

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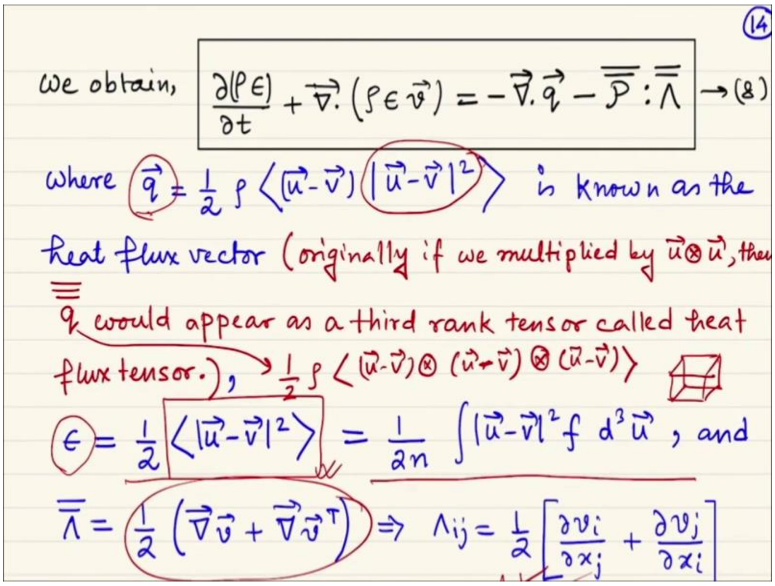

So, this is a very good exercise. If you do the algebra correctly using vector identities and the corresponding definitions correctly, then you will see the equation the moment equation should look like this

$$\frac{\partial(\rho\epsilon)}{\partial t} + \nabla\cdot(\rho\epsilon\boldsymbol{v}) = -\nabla\cdot\boldsymbol{q} - \bar{\bar{\boldsymbol{P}}}:\bar{\bar{\boldsymbol{\Lambda}}}$$

Where, the colon sign means $\bar{\bar{\boldsymbol{P}}}$ is doubly contracted with $\bar{\bar{\boldsymbol{\Lambda}}}$. Now what is ϵ, $\boldsymbol{q}$ and $\boldsymbol{\Lambda}$? They are new here. So, where $\boldsymbol{q}$ is nothing but $\frac{1}{2}\rho < (\boldsymbol{u}-\boldsymbol{v})|\boldsymbol{u}-\boldsymbol{v}|^2 >$ and this is known as the heat flux vector.

This is a special case or the trace of this third order tensor which is known as heat flux tensor, which looks like $\frac{1}{2}\rho < (\boldsymbol{u}-\boldsymbol{v})\otimes(\boldsymbol{u}-\boldsymbol{v})\otimes(\boldsymbol{u}-\boldsymbol{v}) >$. So, it should be represented by a cubic matrix.

So, in three dimensions, it will have 27 components ok. But we were not going into this because we are in a simplistic case where we are multiplying with $|\boldsymbol{u}-\boldsymbol{v}|^2$, that is why the vector out of this third rank tensor.

So, and what is ϵ ? ϵ is nothing but $\frac{1}{2} < |\boldsymbol{u}-\boldsymbol{v}|^2 >$. So, you see this is nothing, but something like a fluctuating kinetic energy type of thing. So, if you constitute a kinetic energy type of thing starting from the fluctuation vector velocities. And what is this capital lambda tensor? that is $\frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)$. So, this is a symmetric matrix by definition.

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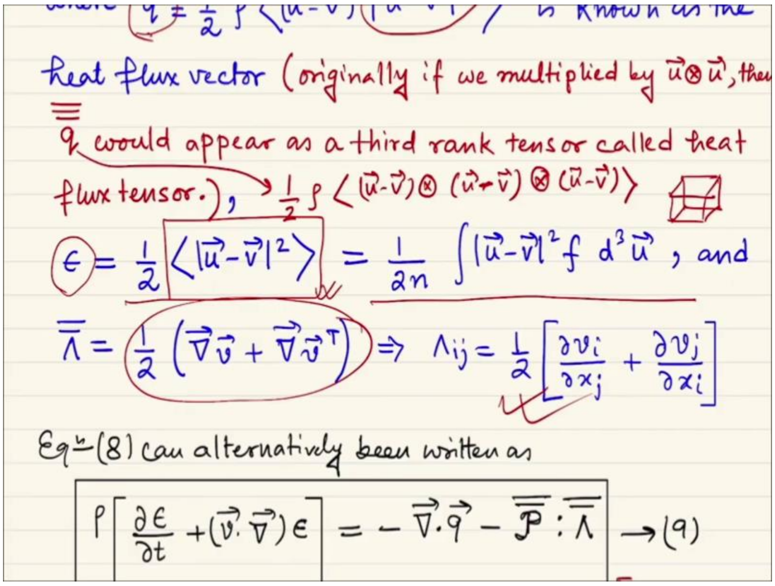

So, using all these things actually and this is I mean doing again the same trick what I just suggested for the first order equation, equation (8) can be written as

$$\rho\left[\frac{\partial \epsilon}{\partial t} + (\boldsymbol{v}\cdot\nabla)\epsilon\right] = -\nabla\cdot\boldsymbol{q} - \bar{\bar{\boldsymbol{P}}}:\bar{\boldsymbol{\Lambda}}$$

Here what I just did, I expanded and used the continuity or the zeroth order equation. This equation is known as the energy equation. Now just by putting an analogy to the momentum equation, you can understand why it is called energy equation.

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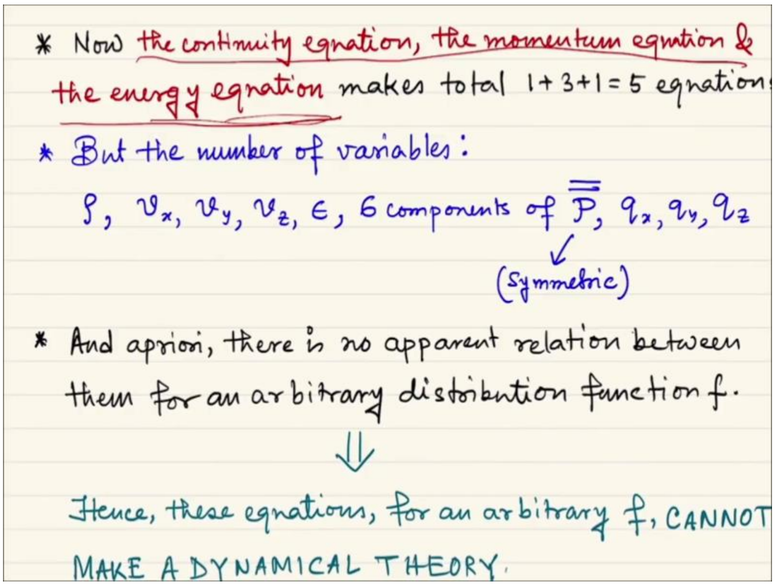

So, we have the continuity equation, the momentum equation and the energy equation. So, if we just I mean look back, they make total (1+3+1), total 5 equations. Continuity equation is one scalar equation, energy equation is one scalar equation, the momentum equation is one vectorial equation, so three component equations.

But what is the number of variables? The number of variables this is much larger, you have ρ , you have (v_x, v_y, v_z), ϵ, 6 components of $\overline{\overline{\boldsymbol{P}}}$. Why 6 components? Because this is a symmetric matrix.

So, in a symmetric matrix, you will have 3 diagonal components which can be different and 3 off-diagonal components which can be different. So, that is the 6 unknown components of pressure tensor, then (q_x, q_y, q_z), the 3 components of the heat flux vector. So, total (1+3+1+6+3)=14 unknowns and 5 equations.

And, there is no apparent relation between them for an arbitrary distribution function *f*. If we do not know what *f* is, then we do not know a priori how is there any relation between ϵ, $\overline{\overline{\boldsymbol{P}}}$, $\boldsymbol{v}$, q etc.

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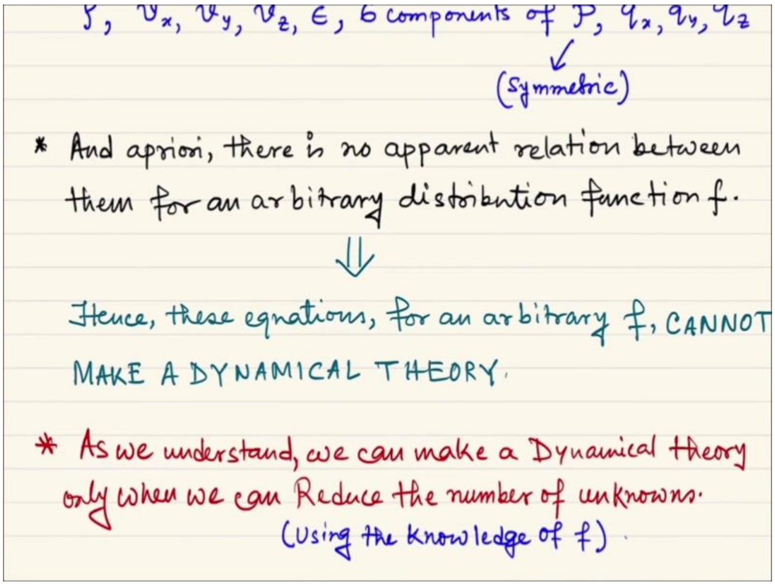

So, a priori 5 equations and 14 variables can never make a dynamical theory. So, as we understand, we can make a dynamical theory only when we can reduce the number of unknowns and that is possible only using the knowledge of *f* as we will see.

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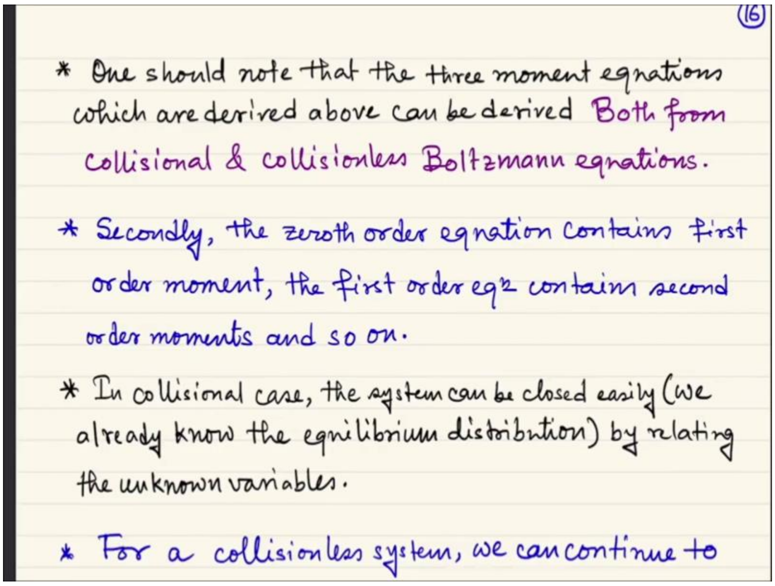

Now, one should note that the three moment equations which are derived above can be derived both from collisional and collisionless Boltzmann equation. This is I am repeating, but this is very interesting and important to understand this point.

Now secondly, the zeroth order equation contains first order moment, the first order equation contains second order moments and so on that is very very interesting. In the zeroth order equation if you see, attentively in the zeroth order equation you have $\boldsymbol{v}$ which is the first order moment.

In the first order equation in the first order equation, you have $\overline{\overline{\boldsymbol{P}}}$, which is the second order moment. I mean of course, I understand for fluctuating velocities, but this is second order somehow or at least you can also say from the term $\nabla \cdot (n\boldsymbol{v}\otimes\boldsymbol{v})$, but that is ok. In any case I mean you have some higher order tensorial things. For second order, you actually have a third order or the trace of the third order thing. So, in any case n-th order moment equation should contain the $(n+1)$-th order moment and that is exactly the problem of hierarchy, that is you can never close the system. So, in collisional case, the system can be closed easily. We already know the equilibrium distribution.

So, we can as I just said that by the knowledge of that if we can reduce and actually we will see that this is the case when we will derive the ideal fluid equations that by using the knowledge of this equilibrium distribution being the Maxwell Boltzmann distribution. We actually could relate some of the unknown variables to make finally a, Dynamical Theory.

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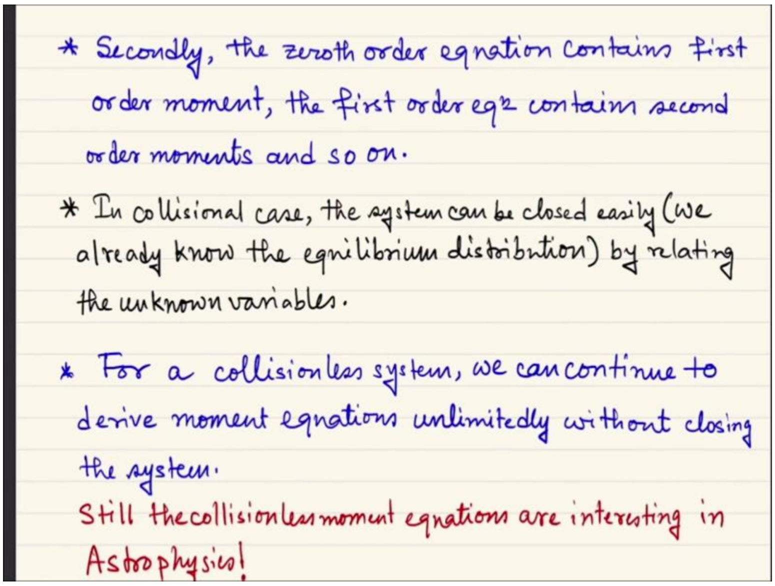

But for a collisionless system, we have no hope. We can continue to derive moment equations unlimitedly without closing the system ok. Still after all these things, there is a good news. And what is the good news? Even for the collisionless systems, the collisionless moment equations are of very very interest in astrophysics.

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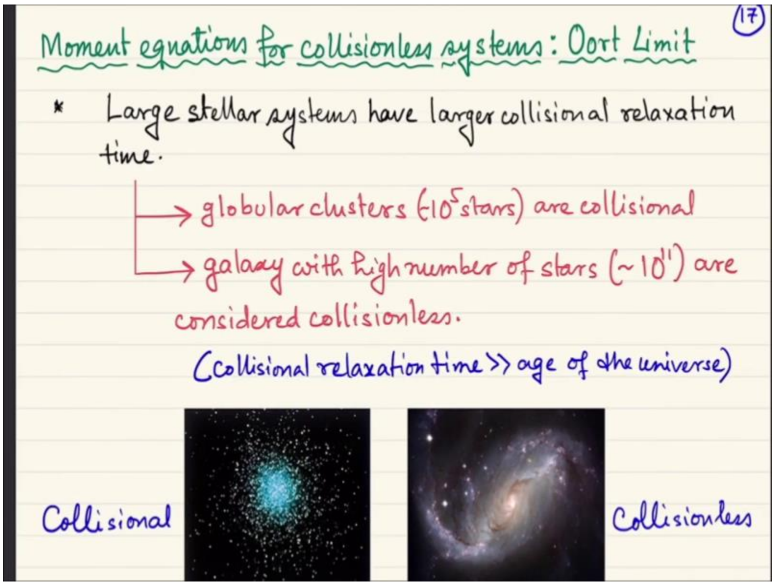

And we are talking in the framework of astrophysics. So, that is why we will discuss all these things and that is what we call the Oort limit, that we will do in the next lecture.

Thank you.