Introduction to Astrophysical Fluids

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Lecture - 08
Derivation of the moment equations I

Hello and welcome to another lecture of Introduction to Astrophysical Fluids. Previously we derived both collisionless and collisional Boltzmann equations. Today we will learn how to derive the moment equations starting from these equations.

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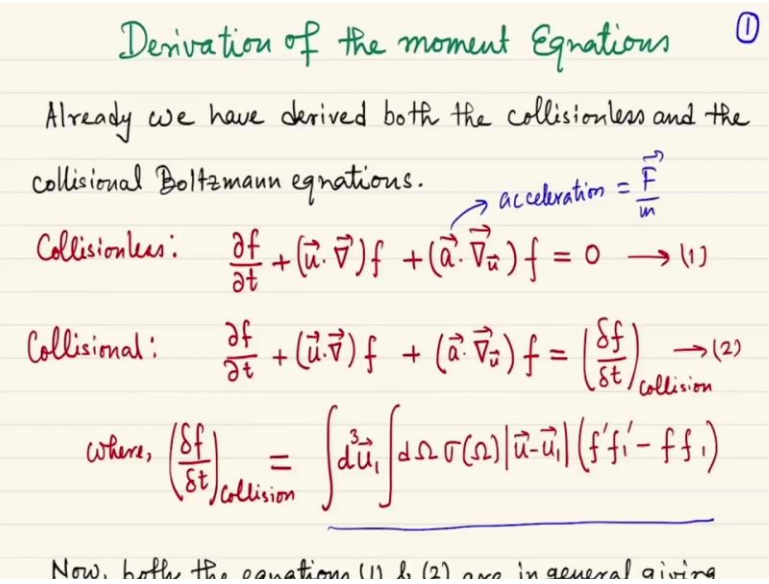

So, as I said that we have already derived the collisionless and the collisional Boltzmann equations. So, collisionless equation you have

$$\frac{\partial f}{\partial t} + (\boldsymbol{u}\cdot\nabla)f + (\boldsymbol{a}\cdot\nabla_{\boldsymbol{u}})f = 0$$

And in case of collisional systems we have the same thing in the left hand side, but the right hand side gives us a collision integral which symbolically we just write, $\left(\frac{\delta f}{\delta t}\right)_{collision}$ and we know that for normal elastic binary collision which is a very simplistic case, but which is somehow very reasonable for microscopic cases.

For example, the I mean kinetic systems of gas molecules. So, for this the collisional integral term should look like this one. That is, $\int d^3\boldsymbol{u}_1 \int d\Omega\sigma(\Omega)|\boldsymbol{u}-\boldsymbol{u}_1|(f'f_1' - ff_1)$.

Once again the prime coordinates are the distribution functions after the collision and the unprimed the distribution functions before the collision. So, here you can again, just for reminder, $\boldsymbol{a}$ is the acceleration and this is nothing but the body force by the mass of one particle.

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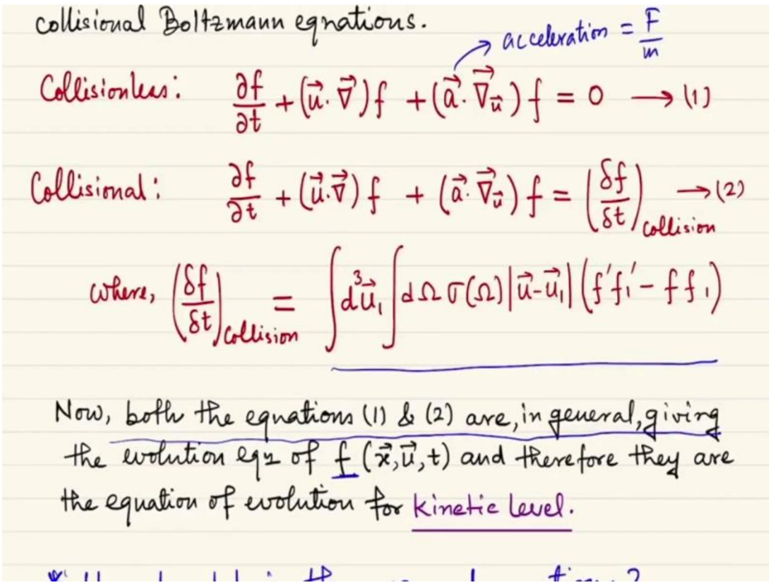

Now, as I just said that both these equations (1) and (2) are in general giving the evolution equation of f that you can see here. And therefore, they are the equation of evolution for the kinetic level right. As I said in one of the previous lectures actually that we are actually trying to develop dynamical theory at different level. So, it is the kinetic level dynamical equation of evolution.

So, we need for every dynamical system an equation of evolution and of course, before that we need a well defined state. So, this is the equation of evolution. Now, the question is that we all have this and as I just promised that this in this lecture we will basically learn how to obtain the moment equations. So, that is something we have to now learn.

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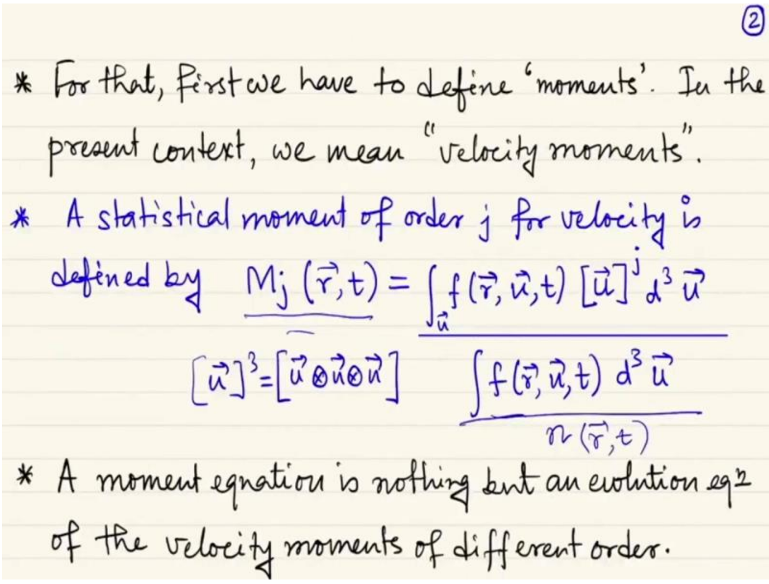

So, for that we first have to define what moments are. So, in the present context moment means simply velocity moments and velocity means kinetic velocity, that means, $\boldsymbol{u}$. So, a statistical moment of order j for velocity is simply defined by M_j which is a function of $\boldsymbol{r}$ and t; that means, only space and time that will be simply equal to $\int f[\boldsymbol{u}]^j d^3\boldsymbol{u}$ **.**

So, this is a vectorial j so; that means, if it is like, let us say let us say $[\boldsymbol{u}]^3$, then this is nothing, but $\boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{u}$. So, this is the thing and so, that is integrated over the velocity space which is normalized by $\int f d^3\boldsymbol{u}$.

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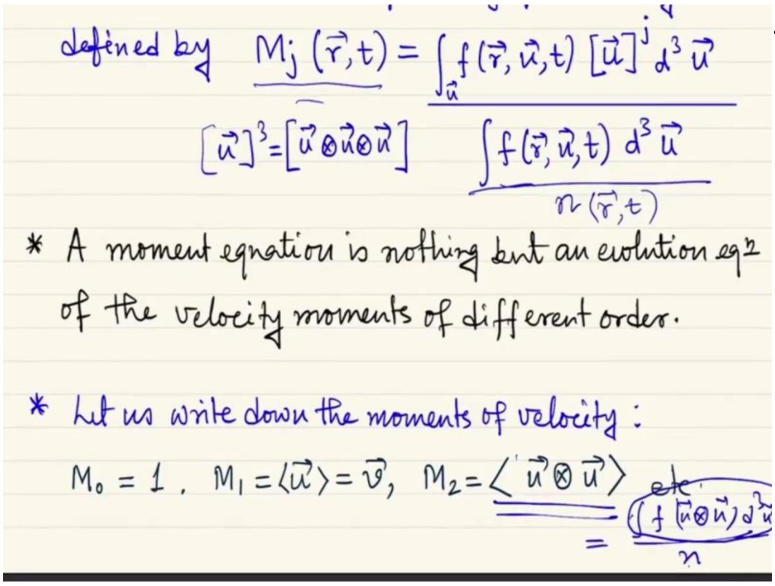

And already during the derivation of Liouville’s equation we have learnt that this is nothing, but the number density right. So, every time for the normalization, every averaged quantity should be normalized by or should be divided by the number density in order to get the normalized value of the statistical moment.

So, a moment equation is nothing but an evolution equation of the velocity moments of different order. So, this is the moment of order j for the kinetic, velocity now a moment equation is nothing but the equation which involves the $\frac{\partial M_j}{\partial t}$.

Now, let us again write down the proper definitions of different moments of the velocity. So, for M_0 you can easily understand that this is simply $\boldsymbol{u}^0$, so, then this will be the both the numerator and the denominator they will be the number density. So, this will be 1. So, when I just say 1; that means, you can actually instead of using one you can use any arbitrary constant.

M_1 will be obtained if we put j is equal to 1. So, it will be simply, $\frac{\int f u d^3\boldsymbol{u}}{n}$ and which is nothing but the average value of $\boldsymbol{u}$ and this is known as $\boldsymbol{v}$ okay. So, average value of $\boldsymbol{u}$ we will call it $\boldsymbol{v}$.

And actually you will understand later that **v** is something which is similar to the macroscopic velocity. Then what is the moment of the second order? As I just said that will be simply nothing, but the $< \boldsymbol{u}\otimes\boldsymbol{u} >$. Moment of third order for example, you know now that it will be then $< \boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{u} >$.

Once again when I use this angular bracket symbol that simply says that this for example, this says that this is $\frac{\int f(\boldsymbol{u}\otimes\boldsymbol{u})d^3\boldsymbol{u}}{n}$. So, if I just want to hint at this integral that will be $M_2\, n = \int f(\boldsymbol{u}\otimes\boldsymbol{u})d^3\boldsymbol{u}$. So, once again, the angular bracket means that the integration of that quantity along with the distribution function in the velocity space divided by the number density.

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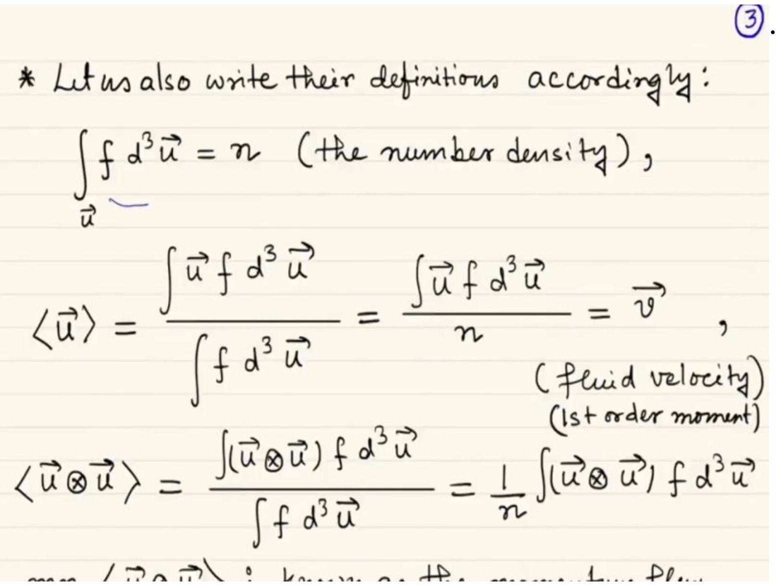

Now, let us write the definitions accordingly and for that for that we can easily say that as our also previous knowledge says that $\int f d^3\boldsymbol{u}$ is nothing but the number of particles per unit time per unit volume. So, only f is nothing but the number of particles per unit phase space volume.

Now, when you integrate this in the velocity space then it will simply be the number of particles per unit real volume and that is nothing but our particle density or number density n.

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$\vec{u}$

$$\langle \vec{u} \rangle = \frac{\int \vec{u} f \, d^3\vec{u}}{\int f \, d^3\vec{u}} = \frac{\int \vec{u} f \, d^3\vec{u}}{n} = \vec{v} \, ,$$

(fluid velocity)
(1st order moment)

$$\langle \vec{u} \otimes \vec{u} \rangle = \frac{\int (\vec{u} \otimes \vec{u}) f \, d^3\vec{u}}{\int f \, d^3\vec{u}} = \frac{1}{n} \int (\vec{u} \otimes \vec{u}) f \, d^3\vec{u}$$

$mn \langle \vec{u} \otimes \vec{u} \rangle$ is known as the momentum flux density tensor and is equal to $m \int (\vec{u} \otimes \vec{u}) f \, d^3\vec{u}$.

(2nd order moment)

Then what is $< \boldsymbol{u} >$? Already I defined this is $\boldsymbol{v}$. $< \boldsymbol{u} \otimes \boldsymbol{u} >$ so, this is one by , already I discussed about these things. Now, this thing $mn < \boldsymbol{u} \otimes \boldsymbol{u} >$. So, one thing you should that should be clear that $< \boldsymbol{u} \otimes \boldsymbol{u} >$ for example, should have a dimension of $\boldsymbol{u}^2$.

So, $mn < \boldsymbol{u} \otimes \boldsymbol{u} >$ is nothing but, so, n is the number density m is the mass of one particle. So, mn is the mass density. So, mass density times u^2. So, you can easily understand this is nothing but something like the volume density of energy and you can actually check in the standard literature this is known as the momentum flux density tensor.

So, basically if you see that the density of momentum is nothing but roughly $mn\boldsymbol{u}$. And when this is somehow multiplied of course, here it is tensorly multiplied, but somehow multiplied with another velocity it gives you the corresponding flux density type of thing. Once again this is the definition and this is the standard second order moment for our case.

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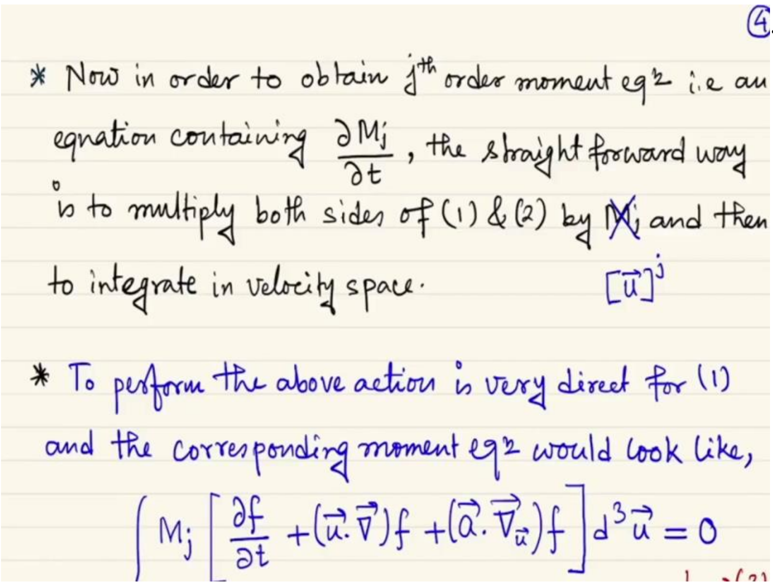

Now, in order to obtain a general j-th order moment equation that is an equation containing $\frac{\partial M_j}{\partial t}$. The straightforward way is to multiply both sides of (1) and (2), (1) and (2) just for recapitulation are nothing but the collisionless and the collisional Boltzmann equations. So, we just have to then multiply both sides of those two equations by this corresponding M_j and then we have to integrate that in velocity space.

First, we have to multiply by $[\boldsymbol{u}]^j$ and then to integrate in velocity space.

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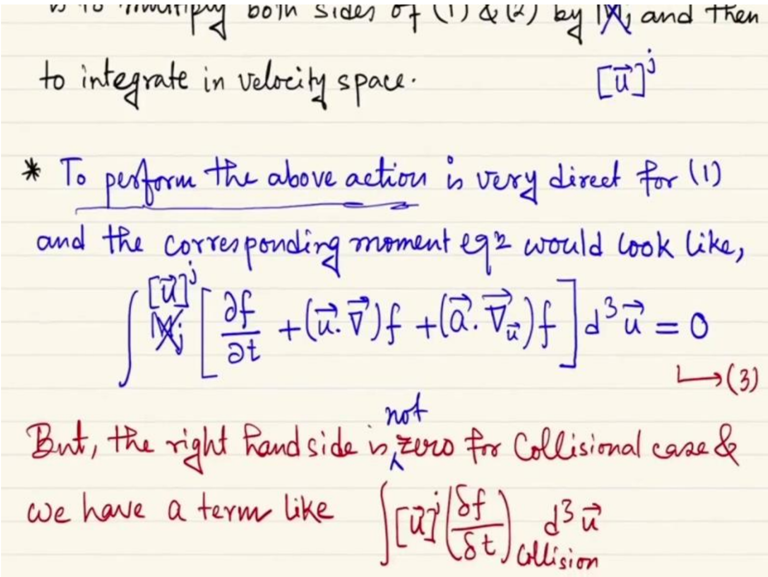

So, as you can easily understand that for collisionless case that is that is written here. So, to perform the above action which I prescribe here is very direct and very easy for one that is the collisionless Boltzmann equation, because the right hand side is 0. So, if you just multiply something with this that will be 0 and then you integrate over a closed volume, then that will also give you 0 and then, the corresponding moment equation should look like simply

$$\int [\boldsymbol{u}]^j \left[\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f + (\boldsymbol{a} \cdot \nabla_{\boldsymbol{u}}) f\right] d^3\boldsymbol{u} = 0.$$

So, for the collisionless case 0 is guaranteed, but for the collisional case the right hand side is 0 only for special cases. In general, once again the right hand side is not zero for collisional case and we have a term like $\int [\boldsymbol{u}]^j \left(\frac{\delta f}{\delta t}\right)_{collision} d^3\boldsymbol{u}$.

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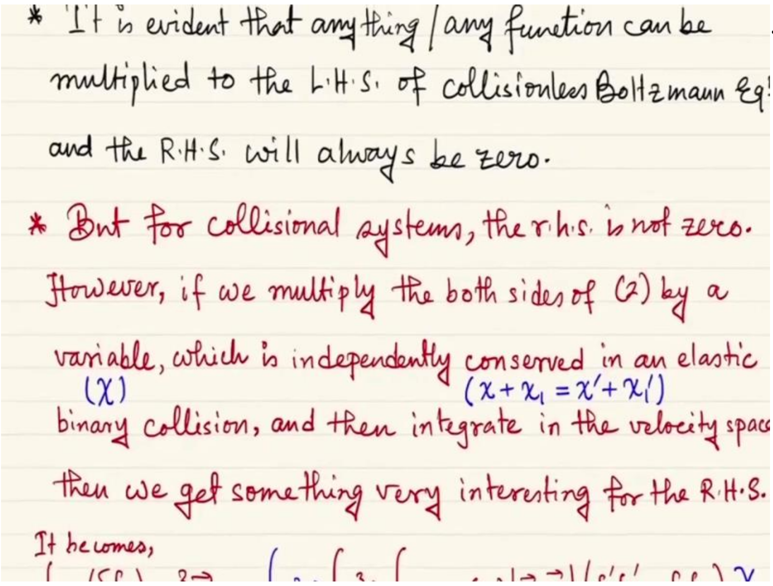

So, it is evident that anything or any function can be multiplied to the LHS of collisionless Boltzmann equation because the RHS will also always be zero. And then that will be easier to handle because you see that the collision integral itself is not a very simple quantity.

So, when you have the collision integral for something where you know the system; that means, you just have binary elastic collision that is or in any other simple case where you have load models. But in an arbitrary case it is not really a guaranteed that you can always model the collision integral sufficiently or sufficiently accurately. Then what happens that I mean the analysis is incomplete. So, you cannot proceed further.

So, that is why it is good that in some sense if the right hand side is already 0 because, so, the left hand side is somehow known, but the right hand side is not always known, the right hand side is known only for binary elastic collisions and some very limited cases.

So, in the next part you will see that for some cases the right hand side basically becomes 0. So, for collisional systems the RHS is not automatically 0; however, if we multiply both sides of the collisional Boltzmann equation by a variable χ just remember which is an independently conserved quantity in an elastic binary collision for example. So, even for elastic binary collision you know like I mean integrating the total collision integral is not trivial.

So, for example, at least the challenge is that at least for binary elastic collision can we do something. So, that we can get rid of the collision integral.

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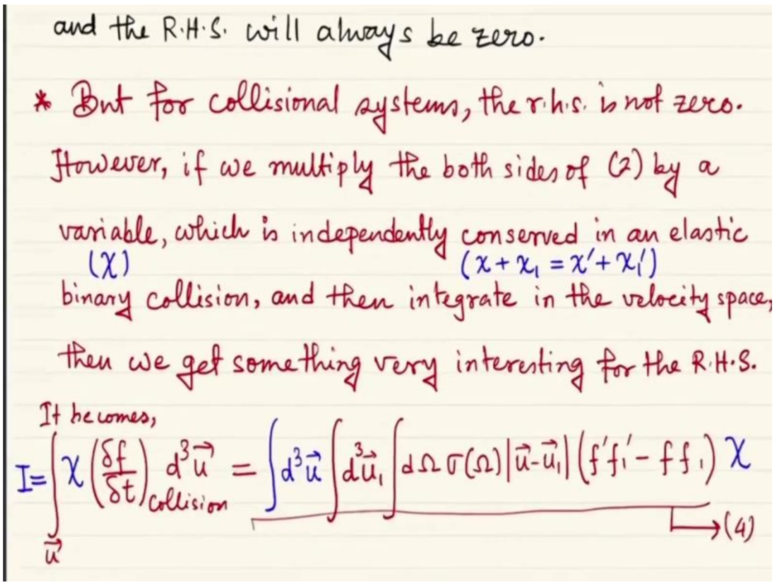

So, but once we know that, we can actually try to incorporate this for any arbitrary interaction as well. So, that we then always will search for some variables which are conserved in this type of interactions as well.

Now, whether this is always equally easy or not that is a question. Now, at least for binary collision how to handle the how to get rid of the collision integral, in the moments equation. Then the answer is that, if we multiply with some independently conserved quantity let us say χ. You know for example, the numbers, the components of velocity, the kinetic energy.

And then integrate in the velocity space, then we get something very interesting for the RHS. Why? Because then the RHS becomes simply that I call I simply which is nothing but

$$I = \int \chi \left(\frac{\delta f}{\delta t}\right) d^3\boldsymbol{u} = \int d^3\boldsymbol{u} \int d^3\boldsymbol{u}_1 \int d\Omega \sigma(\Omega) |\boldsymbol{u} - \boldsymbol{u}_1| (f' f_1' - f f_1) \chi$$

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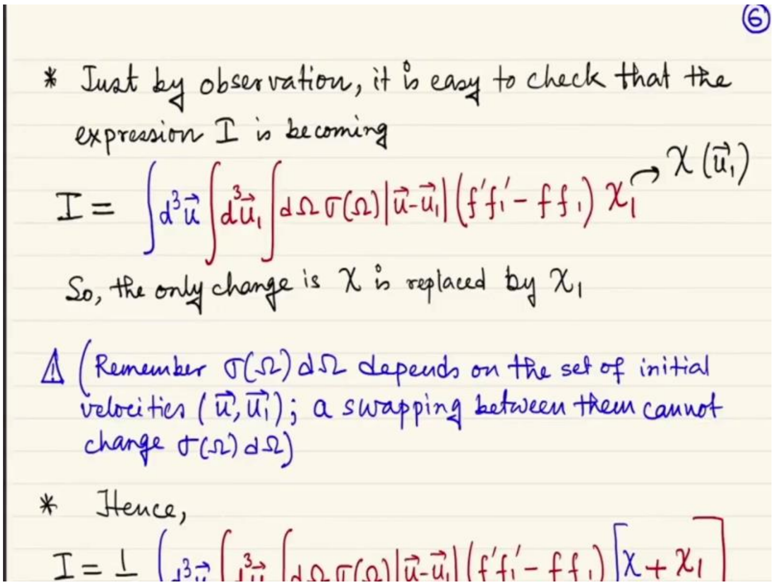

Now, just by observation you can easily check that this integral I , which I have written over here just becomes this; this one. If you simply swap between the $\boldsymbol{u}$ and $\boldsymbol{u}_1$. Then what happens? Of course, all the $\boldsymbol{u}$ will be $\boldsymbol{u}_1$ and then χ will be now χ_1. Because χ_1 is nothing but $\chi(\boldsymbol{u}_1)$, then f will be f_1 and f_1 will be f and same thing for f' and $f_1{}'$.

Now, so, in this integral I simply blindly because I mean the this is an integral where the integration is done on both $\boldsymbol{u}$ and $\boldsymbol{u}_1$ space then I can just say that if I interchange between $\boldsymbol{u}$ and $\boldsymbol{u}_1$ nothing will change in the final value of the integration right.

Now, I can easily say that, I will now say my $\boldsymbol{u}_1$will be $\boldsymbol{u}$ and $\boldsymbol{u}$ will be $\boldsymbol{u}_1$. So, then the one thing I have to check that what will be the change in, $d^3\boldsymbol{u}_1 d^3\boldsymbol{u}$ if they are changing no problem because integrations on both the space this one will also not change because this depends if you remember the previous discussion depends on the set of initial velocities $\boldsymbol{u}$ and $\boldsymbol{u}_1$. So, a swapping between them cannot change $\sigma(\Omega)d\Omega$. Remember when we were deriving the collision integral we talked about this.

And what about $|\boldsymbol{u} - \boldsymbol{u}_1|$? This is also cannot be changed because this is the modulus. Here I mean if you just change between the f / f_1 and f prime and f_1 prime nothing will change. So, finally, your I will be the same.

So, as both are I. So, I can be now written as half times a sum of these two integrations

$$I = \frac{1}{2}\int d^3\boldsymbol{u} \int d^3\boldsymbol{u}_1 d\Omega\sigma(\Omega)|\boldsymbol{u} - \boldsymbol{u}_1|(f'f_1' - ff_1)[\chi + \chi_1]$$

And that means, the all the other factors are common only I have a sum of χ and χ_1 .

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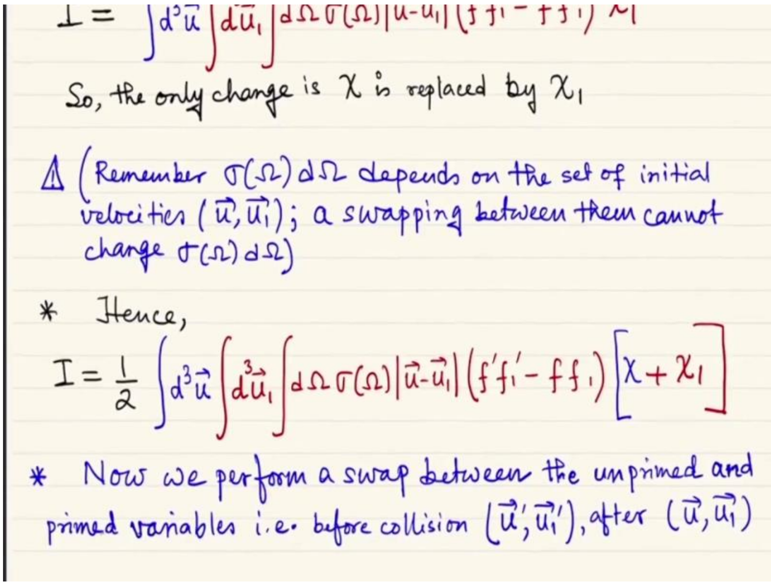

Now, in the next step we perform another swap between the unprimed and the prime variables. And that means, that before collision the variables will now be primed and after collision the variables will be unprimed.

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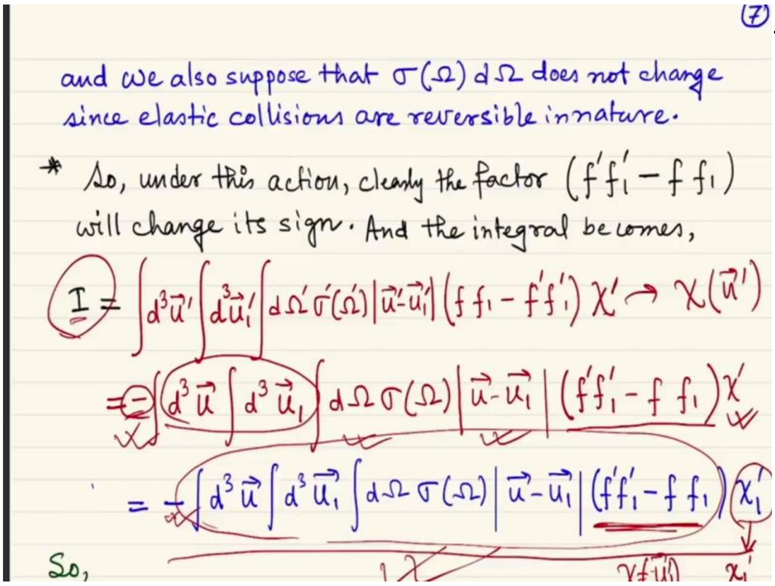

We also suppose that in our case, the elastic collisions are usually reversible in nature. So, that is why they are I mean the collisional cross section or the $\sigma(\Omega)d\Omega$ will not change. So, if I simply say that if I just swap between primed and unprimed coordinates, then I can again be blindly written as

$$I = \int d^3\boldsymbol{u}' \int d^3\boldsymbol{u}'_1 \int d\Omega' \sigma'(\Omega') |\boldsymbol{u}' - \boldsymbol{u}'_1| (f f_1 - f' f_1') \chi'$$

all these quantities inside they are just the dummy variables.

Now, if they are the dummy variables basically, I mean the quantities on which the integration is done they are dummy variables.

Now, you can simply say that I am studying the reverse collisions. So, I am simply saying that I am just trying to do the same type of integration for the reverse collision process or inverse collision process. Then all the unprimed variables will be changed with primed variables.

Now, check that according to the reversibility the collision cross section integral will be unchanged. And so will be the relative velocity, as we already discussed that in a binary elastic collision, the relative velocity before and after the collision of the two particles they are the same. That is again the same thing of saying the coefficient of restitution is 1. The term involving distribution functions will its sign.

Now, what about $\int d^3\boldsymbol{u} \int d^3\boldsymbol{u}_1$?

Remember when we were deriving the collision integral we said that if we suppose that the two particles which are colliding then during the collision or rather at the moment of the collision basically; if we assume that at that point when the collision takes place there is no other interaction other than the force of collision experienced by these two particles, then for these two particle system we can define a Hamiltonian system.

And if we can define the Hamiltonian system, then for that two particle system we can apply the Liouville's theorem. And then again just remember the corollary of the Liouville's theorem for this two particle system where you can have $d^3\boldsymbol{u}' d^3\boldsymbol{u}_1' = d^3\boldsymbol{u} d^3\boldsymbol{u}_1$, along a trajectory.

So, that we can use. If we can use that then simply you can write that I is now equal to minus everything is the same of the original I, but only χ is replaced by χ_1.

$$I = -\int d^3\boldsymbol{u} \int d^3\boldsymbol{u}_1 \int d\Omega \sigma(\Omega) |\boldsymbol{u} - \boldsymbol{u}_1| (f' f_1' - f f_1) \chi'$$

Again we are always in the being direction of the inverse collision if we now change between $\boldsymbol{u}'$ and $\boldsymbol{u}_1{}'$ then there will be actually no change. So, from the original I we again have this type of expression where we have minus sign. And this multiplicative variable will be then $\chi_1{}'$ which is nothing but $\chi(\boldsymbol{u}_1{}')$.

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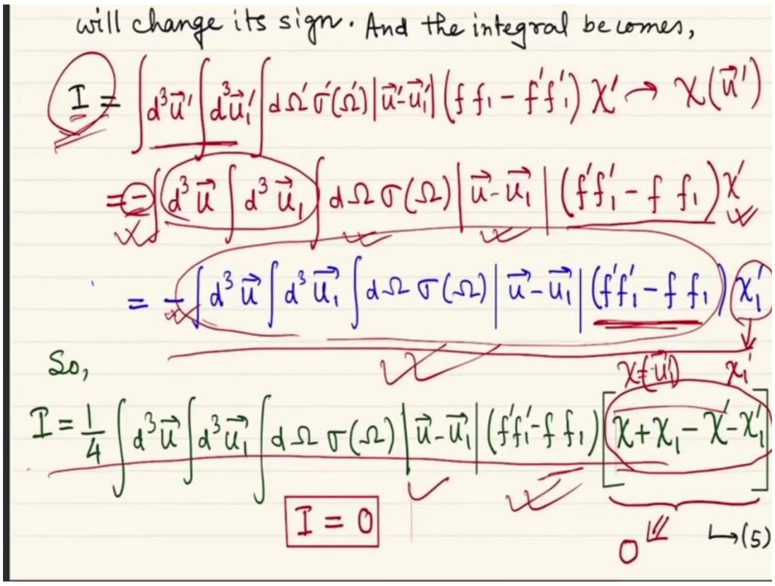

Finally, as all of them are equal and equal to I then you can write I is equal to

$$I = \frac{1}{4}\int d^3\boldsymbol{u}\int d^3\boldsymbol{u}_1 d\Omega\sigma(\Omega)|\boldsymbol{u}-\boldsymbol{u}_1|(f'f_1'-ff_1)[\chi+\chi_1-\chi'-\chi_1']$$

You see in the deep green color, the total expression.

And this is the expression for the collision integral when it is multiplied by something which is conserved in a binary elastic collision. Actually, the collision need not be binary elastic. Now, the thing is that if you have an arbitrary collision just if something which is conserved in that collision then actually you can search this for this type of thing of course, you have to be careful that, all the type of intermediate suppositions or hypothesis they are also satisfied.

For example, whether you can apply this type of intermediate Liouville’s theorem for two particle systems or not this type of thing. So, this is subtle, but now for our present case, you can simply say that, the collision integral when this is multiplied by something which is a conserved quantity of a binary elastic collision and then integrated over the velocity space gives us an expression which looks like the above equation and due to the conservation $\chi+\chi_1-\chi'-\chi_1'$ vanishes.

So, finally, I can say that the integration, *I*, will vanish. So, although in collisional Boltzmann theorem or Boltzmann equation the right hand side is not zero. For binary elastic collision at least, the thing is that when we are trying to derive the moments equations, then if it is a moment, which is a velocity moment, is conserved in a binary elastic collision then after integration the right hand side is vanishing. That is why we finally, get rid of the right hand side in the moment equation.

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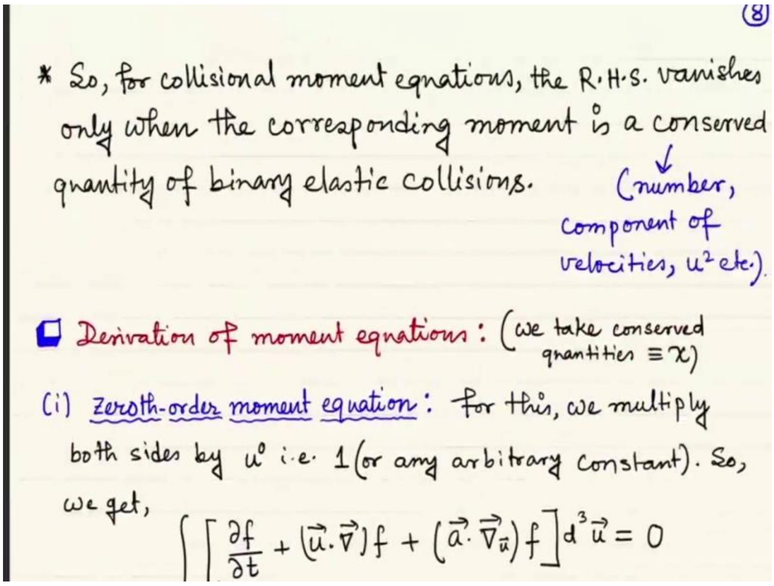

So, for collisional moment equation what I just said the right hand side vanishes only when the corresponding moment is a conserved quantity of binary elastic collision. For example, number density which is nothing but the velocity to the power 0, component of velocity which is nothing but a corollary of the conservation of momentum components.

Here this is just the component of velocity because in microscopic case all the gas molecules they have same mass, gas molecules or any system molecules. Then $\boldsymbol{u}^2$ this is also another thing which is coming from the kinetic energy conservation.

So, if we multiply the I mean for collisionless Boltzmann equation this is always 0, for collisional Boltzmann equation the right hand side is 0 only after integration if they are multiplied by this type of thing or any other conserved quantity of a binary elastic collision. So, in the next part of the lecture we will try to derive directly the moment equations.

Thank you.