Introduction to Astrophysical Fluids
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Lecture - 07
Equilibrium distribution function II

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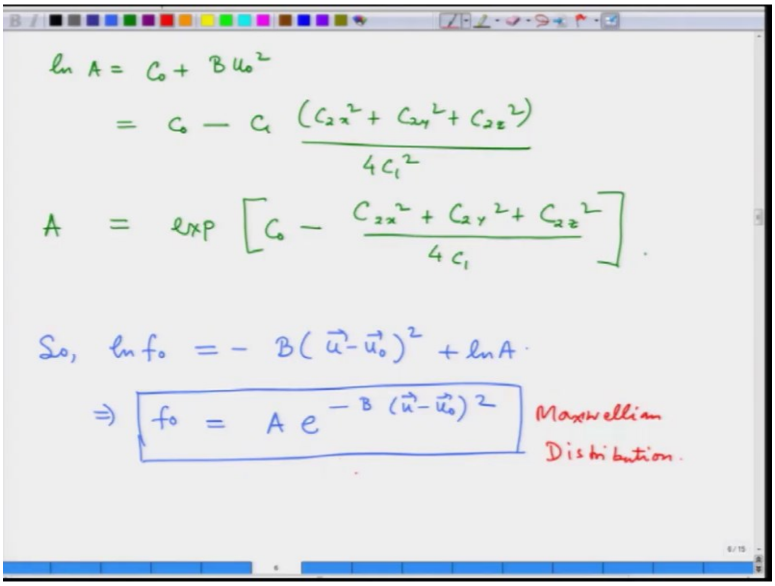

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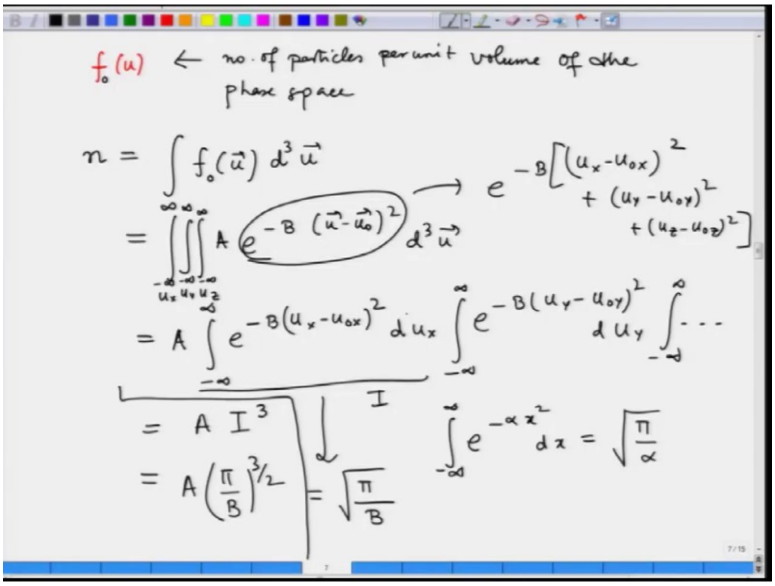

Hello. Now in the next step we try to evaluate these three constants A, B and u_0. Now, first we start by evaluating A. So, if you remember that $f(u)$ that is the distribution function that is nothing but the number of particles per unit volume of the phase space. So, if we integrate this on velocity space, then finally we will have real-space density.

That means $f(u)$ integrated over velocity space will simply give us the number of particles per unit volume in direct space right. Then you can write the number density $n = \int f_0(\boldsymbol{u}) d^3\boldsymbol{u}$**,** for every distribution function because we are talking about equilibrium now ok.

Now, we just have found that f_0 should have this form $f_0 = \ Ae^{-B(\boldsymbol{u}-\boldsymbol{u_0})^2}$. So

$n = \int_{ux}\int_{u_y}\int_{uz} Ae^{-B(\boldsymbol{u}-\boldsymbol{u_0})^2}\ d^3\boldsymbol{u}$. This is a triple integral and you have all the components of $\boldsymbol{u}$. So, I can just write explicitly u_x, u_y and u_z and they have the same limits, minus infinity to plus infinity, ok.

And finally, you can see that

$$n = \int_{ux}\int_{u_y}\int_{uz} Ae^{-B(\boldsymbol{u}-\boldsymbol{u_0})^2}\ d^3\boldsymbol{u} = \int_{ux}\int_{u_y}\int_{uz} Ae^{-B[(u_x-u_{0x})^2+(u_y-u_{0y})^2+(u_z-u_{0y})^2]}\ dxdydz \quad (1)$$

these are the summation in the index so they are nothing, but making a product of 3 terms, So, if you write this

$$n \ = A\int_{-\infty}^{+\infty} e^{-B(u_x-u_{0x})}\ du_x \int_{-\infty}^{+\infty} e^{-B(u_y-u_{0y})}\ du_y \int_{-\infty}^{+\infty} e^{-B(u_z-u_{0z})}\ du_z \qquad (2)$$

and here you can easily understand that all these integrations are evaluated from minus infinity to plus infinity must have the same value, because they are just changing the variables; the functional form is exactly the same.

So, I can write this integration $\int_{-\infty}^{+\infty} e^{-B(u_x-u_{0x})}$ as *I*. So, equation (2) would be $n = AI^3$. Now, what is this *I* and how to calculate this type of thing? So, one can show that

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \qquad (3)$$

I leave this (equation (3)) for you as an exercise to calculate; this is a very nice home task; you can do that. And if you just substitute this one over here (equation (2)), you can see that this will simply be

$$n = A\left(\frac{\pi}{\mathrm{B}}\right)^{\frac{3}{2}}$$

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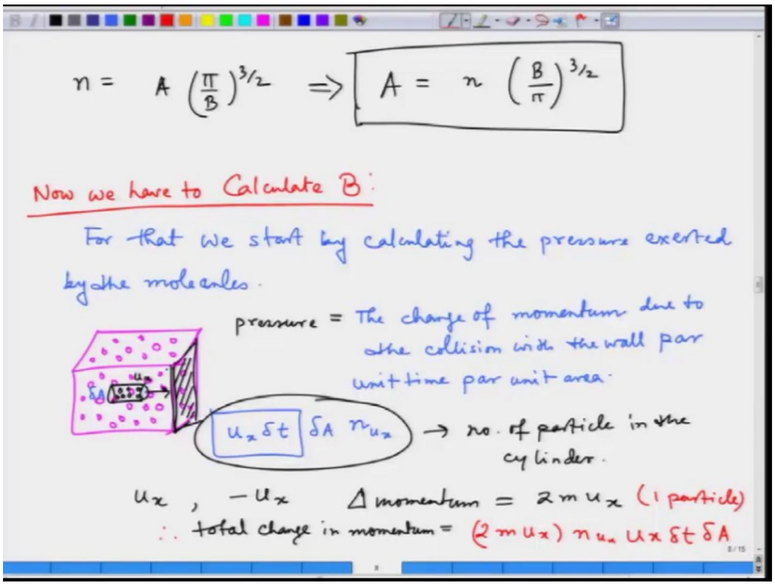

If you have this result then you can say n will be equal to A times $\left(\frac{\pi}{\mathrm{B}}\right)^{\frac{3}{2}}$. So, A will simply be n times $\left(\frac{\mathrm{B}}{\pi}\right)^{\frac{3}{2}}$. So, we find a relation between A and B, but still one characteristic variable of the system comes into play that is the number density n, but still, we cannot evaluate A and B independently of each other in terms of the other variables characterizing the system.

Now, for that we have to use our concept of kinetic theory and to know the kinetic definition of pressure. So, we have to calculate B. For that we start by calculating the pressure exerted by the molecules.

Now finally, calculating the pressure could give us the value of B, you will just see. Now, let us take a cubic box and in this box, we are considering that the particles are moving in every possible direction and sometimes they are colliding with each other sometimes, they are just moving freely.

When I am talking about the freely that does not say that they are not moving under any force, freely means without any collision. So, finally when these particles are not experiencing any collision with any other particles, they are actually colliding to the walls of the container. This is also possible for example; one particle is starting from here and it is going towards this direction and if it is not suffering any collision in this direction then it will collide to this wall ok. The same thing can be happening for the other directions.

Now, let us assume one cylinder of molecules; you see this is the typical technique; one cylinder of molecules having x component of velocity, u_x and the cylinder is moving towards the positive x direction. So, we will be calculating the pressure given by the gas molecules on this wall of the container.

So, the pressure is nothing but the change of momentum due to the collision with the wall per unit time per unit area. Because momentum change per unit time is nothing but the force and force per unit area gives you the pressure. Now, if you really understand what it means so the cylinder, we are taking such that the cross-sectional area which is perpendicular to the motion of the particles, we call that δA and we are also taking the particles inside this cylinder are all having their x component of velocity as u_x. So, along the x direction and more precisely along the positive x direction in unit time we can see a cylinder of length $u_x \delta t$.

And this is the length of the cylinder which basically will be evolved in time δt with the particles having x component of velocity is equal to u_x and moving along the positive x direction. So, when I am just multiplying $u_x \delta t$ with δA, gives me the volume of this cylinder and volume of this cylinder when gets multiplied to the number of particles inside the cylinders, which I just write as n_{ux} that gives me the total number of particles contained inside this cylinder.

Now, every such particle will collide to the wall and when it is just before the collision it had a velocity component u_x and after the collision it had a velocity component $-u_x$, that we all know. So, the change in momentum is $\Delta_{momentum} = 2mu_x$. we are only considering the momentum change along the x direction.

We are not considering any other momentum changes in the other directions. So, they can have any other momentum change in y and z directions, may be zero or maybe non-zero, but we are now considering what happens in the x direction. So, in the x direction they will have a momentum change of $2mu_x$.

The total change in momentum for molecules inside the cylinder will simply be $(2mu_x)n_{ux}u_x\delta t\delta A$. So, the pressure will now be $(2mu_x)n_{ux}u_x\delta t\delta A$ divided by $\delta t\delta A$.

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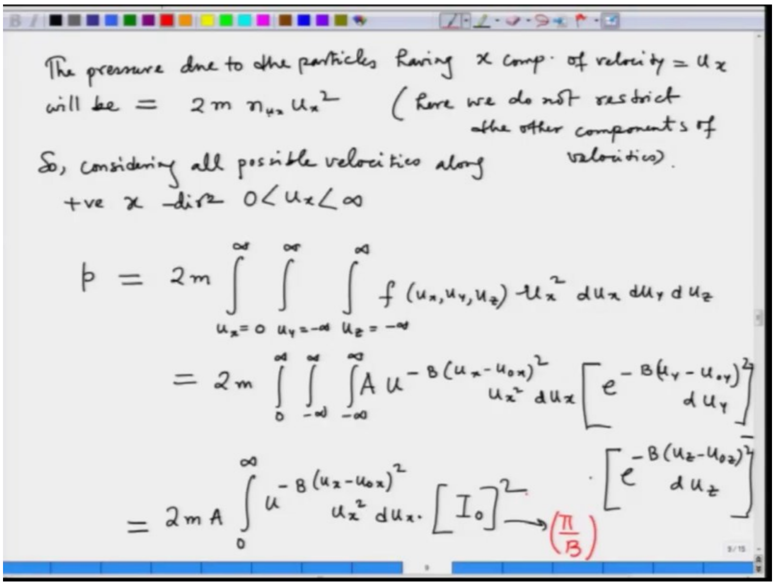


The pressure due to the particles having x component of velocity is equal to u_x will be equal to $2mu_x^2$. So, considering all possible components of velocity (here we do not restrict the other component of velocities) So, considering all possible velocities along positive x direction and other possible velocities. That means, the range should be 0 to $+\infty$

We finally, have our p to be

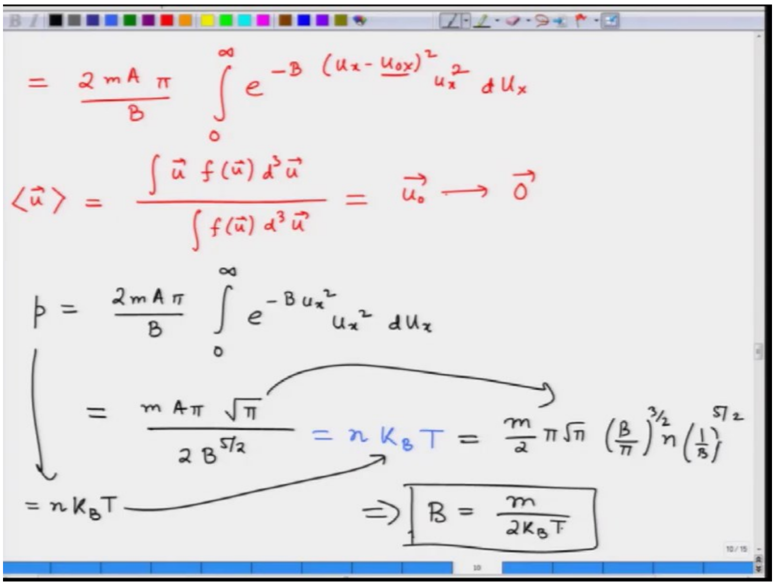
$$p = 2m \int_{u_x=0}^{+\infty} \int_{u_y=-\infty}^{+\infty} \int_{u_z=-\infty}^{+\infty} f(u_x, u_y, u_z) u_x^2 du_x du_y du_z \quad (4)$$

So, this part is giving me the pressure, but this is not an arbitrary pressure, this pressure is given on this wall. So,

$$p = 2m \int_{u_x=0}^{+\infty} \int_{u_y=-\infty}^{+\infty} \int_{u_z=-\infty}^{+\infty} Ae^{-B(u_x-u_{0x})^2} u_x^2 du_x \left[e^{-B(u_y-u_{0y})^2}\right] du_y \left[e^{-B(u_z-u_{0z})^2}\right] du_z \quad (5)$$

Now, finally, this is exactly equal to

$$p = 2m\int_{-\infty}^{\infty} Ae^{-B(u_x-u_{0x})^2}u_x^2du_x\,[I]^2 = \frac{2mA\pi}{B}\int_{-\infty}^{+\infty} Ae^{-B(u_x-u_{0x})^2}u_x^2du_x \qquad (6)$$



Now, you can actually try to evaluate this integration (equation (6)), this is not very intricate. Now, one thing that you have to do is you can just make u_{x0} to be equal to 0 and that will make our life simple and we can easily evaluate the integration and what is the physical justification for that? If you calculate the mean velocity and you can just say that my mean velocity will simply be given by this definition that is,

$$\langle \boldsymbol{u} \rangle = \frac{\int \boldsymbol{u}\, f(\boldsymbol{u})d^3\boldsymbol{u}}{\int f(\boldsymbol{u})d^3\boldsymbol{u}} = \boldsymbol{u_0} = 0 \qquad (7)$$

I have the distribution function in the velocity space which is of course, my equilibrium distribution function and the average velocity should be just $\boldsymbol{u}$ times the distribution function integrated over in the velocity space by the density; that means, it is done for normalization.

If you calculate this integration (equation (7)) you will see that this will simply be equal to $\boldsymbol{u_0}$. That simply gives you that physical significance of $\boldsymbol{u_0}$. So $\boldsymbol{u_0}$ is nothing but the average velocity of the whole system. So, let us say if we have a container of gas then the velocity with which the center of mass of the system will move will be equal to $\boldsymbol{u_0}$.

Now, let us put this constraint so that the $\boldsymbol{u_0}$ is equal to 0; we made $\boldsymbol{u_0}$ by hand is equal to 0. For this type of system, you can calculate the pressure which will be

$$p = \frac{2mA\pi}{B}\int_{-\infty}^{+\infty} Ae^{-Bu_x^2}u_x^2 du_x = \frac{mA\pi^{\frac{3}{2}}}{2B^{\frac{5}{2}}} \qquad (8)$$

Now, for this type of integration, I will give you a sheet by which you can learn how to evaluate this type of integration. Now, we know from our knowledge of kinetic theory p is equal to n times Boltzmann constant times the kinetic temperature of the system, n is the number density. So, finally, we can say $p = nk_BT$.

And finally, if we remember what A was, then

$$p = nk_BT = \frac{m\pi^{\frac{3}{2}}n}{2B^{\frac{5}{2}}}\left(\frac{\mathrm{B}}{\pi}\right)^{\frac{3}{2}} = \frac{mn}{2B} \qquad (9)$$

If you do this simplification, you can see you have a very elegant result $B = \frac{m}{2k_BT}$. You have B is expressed in terms of the property of the system exclusively. Now of course, here additionally we have introduced the temperature of the system.

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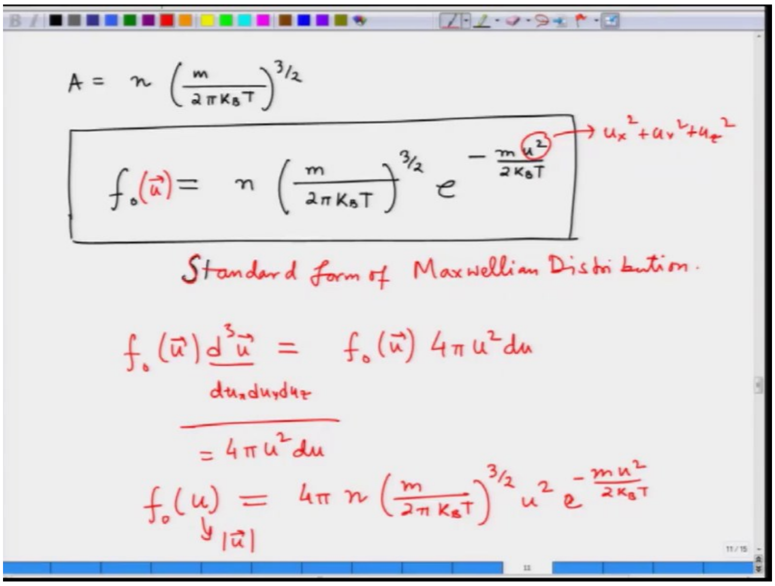

Now, A will simply $n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}$. Finally, inserting all these things we have the final form of f_0 which should look like

$$f_0(\boldsymbol{u}) = n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} e^{-\frac{mu^2}{2k_BT}} \qquad (10)$$

So, this is the standard form of standard form of Maxwellian distribution. Now, remember f_0 is a function of u vector and u^2 is $u_x^2 + u_y^2 + u_z^2$.

So, there is a priori no assumption of isotropy. Now, let us say we are trying to understand how should this distribution function should look like for a system where we have isotropy, in terms of velocities. Then you can actually say that

$$f_0(\boldsymbol{u})d^3\boldsymbol{u} = f_0(\boldsymbol{u})4\pi u^2 du \qquad (11)$$

that is exactly the isotropic form of f_0 which we then call f_0 of u, scalar u or rather $f_0(u)$, which will be nothing but

$$f(u) = 4\pi n \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} e^{-\frac{mu^2}{2k_B T}} u^2 \qquad (12)$$

So, this is the isotropic form of Maxwellian distribution where we are just depending or we are just deriving a distribution function of the speed u.

And what is u? u is nothing but modulus of the vector $\boldsymbol{u}$. So, here I end this part in the next part we will start deriving the different I mean deriving the evolution equation of different moments of velocities ok.

Thank you.