Introduction to Astrophysical Fluids
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Lecture - 06
Equilibrium Distribution Function I

Hello. Previously, we derived the collisionless Boltzmann equation or the Vlasov's equation and also collisional Boltzmann equation; which is mostly called the Boltzmann equation simply ok.

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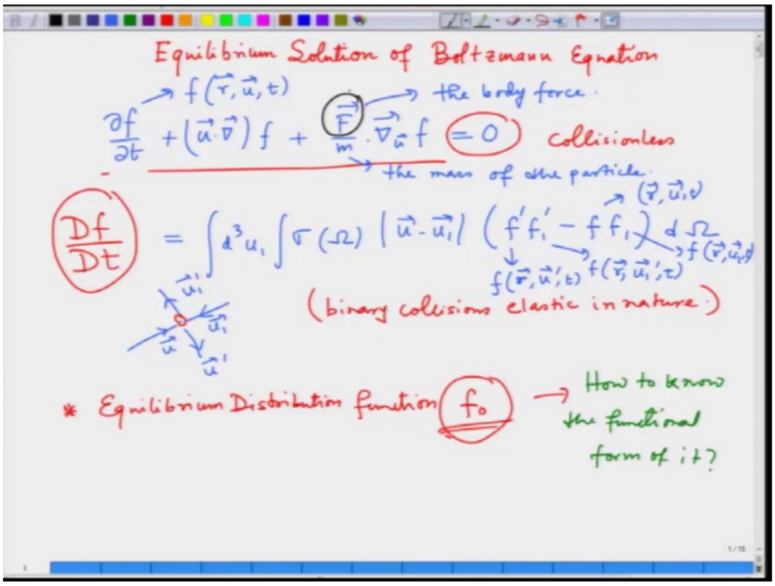

And, if you remember the form, we said that the collisionless Boltzmann equation should look like this

$$\frac{\partial f}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla})f + \frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla}_{\boldsymbol{u}} f = 0$$

where f is the kinetic distribution function right and which can be a function of the position, the velocity and time. And so, the equation should look like $\frac{\partial f}{\partial t}$ plus $(\boldsymbol{u}.\boldsymbol{\nabla})f$ plus $\frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla}_{\boldsymbol{u}} f$ the gradient in the velocity space of f and $\boldsymbol{u}$ is the kinetic velocity, $\boldsymbol{F}$ is the body force and m is the mass of the one particle.

And, then we also derived the collisional form of this where this right-hand side was not equal to zero. So, that was the collisionless form and for collisional form we had this integral

$$\int d^3u_1 \int \sigma(\Omega)|\boldsymbol{u} - \boldsymbol{u_1}|(f'f'_1 - ff_1)d\Omega$$

which we call the collision integral, where f, f', f_1, f_1' are just the distribution functions; $f^{'}$ is $f'(\boldsymbol{r}, \boldsymbol{u}', t)$ and here the other two things are kept unchanged $\boldsymbol{r}$ and t this $f^{'}_1$ will be just replaced by $f^{'}_1(\boldsymbol{r}, \boldsymbol{u}_1^{'}, t)$. f was $f(\boldsymbol{r}, \boldsymbol{u}, t)$ and this one f_1 was function of $f_1(\boldsymbol{r}, \boldsymbol{u_1}, t)$. This is true this collision integral is valid when we were considering binary collisions of two particles okay. So, they collided and before collision they had velocities $\boldsymbol{u}$ and $\boldsymbol{u_1}$ after collision they had velocities $\boldsymbol{u}'$and $\boldsymbol{u_1}'$ okay.

So, this was only model for binary collisions, elastic in nature. So, binary elastic collisions should be there and then only we can model the effect of the collision or rather this is the, I mean this is

$$\int d^3u_1 \int \sigma(\Omega)|\boldsymbol{u} - \boldsymbol{u_1}|(f'f'_1 - ff_1)d\Omega$$

the net change so, if you remember this one this is $\frac{\partial f}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla})f + \frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla_u} f$ nothing but Df/Dt. So, I mean if you just follow one trajectory of evolution in the phase space so, the time rate of change of the distribution function will be exactly equal to this one Df/Dt ok. So, this model is not a general model for any arbitrary collision, it is for a very simplistic case right.

Now, using this integral we now want to fetch some information about the nature of the of the distribution f, I mean; the nature of the distribution function for a gas or a classical system of particles which are left for long time and we will see that towards which distribution the system tends to relax itself and that will we will call the equilibrium distribution function of the system f_0.

Now, one thing I have to mention here, if you remember here, I just changed in these four terms involving the distribution function, I only changed the $\boldsymbol{u}'s$ ok; here $\boldsymbol{u}^{'}, \boldsymbol{u_1}^{'}, \boldsymbol{u}$ and $\boldsymbol{u_1}$, but I did not change $\boldsymbol{r}$ that was because the collisions were assumed to take at one point in space. So, just before and after the collision, no change in position or no considerable change in position took place ok, that was the assumption.

Now, we will try to formulate f_0. Let us consider, how to know the functional form of it ok.

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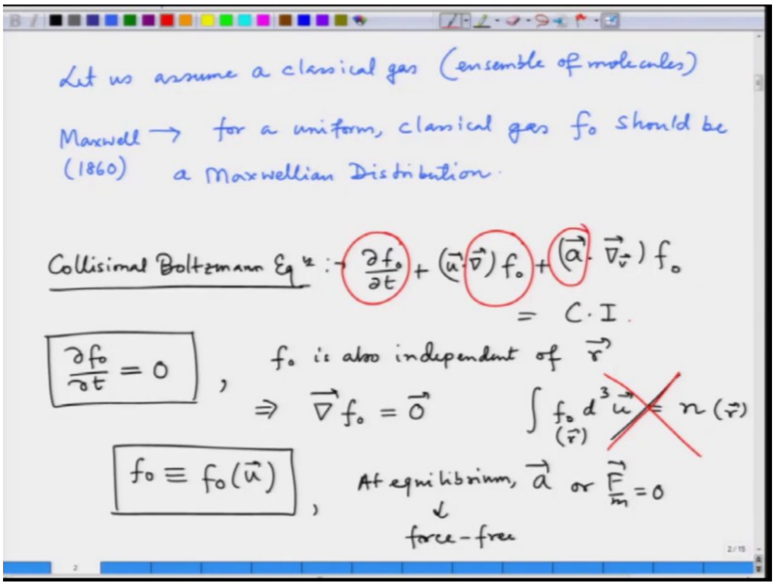

Let us assume a gas a classical gas or rather an ensemble of molecules; here I am not talking about the ensemble at phase space, but ensemble of molecules means just a collection of molecules, classical molecules which are interacting within themselves just by in terms of elastic binary collisions and then we just let it evolve without any force or anything ok. So, the system is just left to evolve freely and then we will try to understand what is the distribution function for that.

Now, Maxwell in the year 1860, he already pointed out that such a classical gas if it is uniform or homogenous; that means, the density of the gas is equal at every point in space. So, for a uniform classical gas f_0 should be Maxwellian distribution. Now, what is that and how to really show that; that exactly we will do here ok.

So, here we will try to do this directly from our collisional Boltzmann equation. So, collisional Boltzmann equation will give us $\frac{\partial f}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla})f + \frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla}_{\boldsymbol{u}}f = C.I$; $\boldsymbol{a}$, I just write the acceleration which is $\boldsymbol{F}/m$, is equal to the collision integral; $C.I$ this is the collision integral the whole integral.

Now, we have to construct by, I mean using step by step assumptions the form of f_0 ok. So, if this

$$\frac{\partial f}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla})f + \frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla}_{\boldsymbol{u}}f = C.I$$

is true for all f then of course, this is also true for f_0 ok, it is true.

$$\frac{\partial f_0}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla})f_0 + \frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla}_{\boldsymbol{u}}f_0 = C.I$$

Now, one thing we have to understand that if the gas is left for long time, then if it will tend towards an f which is equal to f_0 that f_0 should be independent of time, should not explicitly depend on time ok. That is why we can write $\frac{\partial f_0}{\partial t}$ will be equal to 0, that is somehow reasonable ok.

So, because when the system is left for long time then the system is expected to attain a steady state. So, all the properties including the very basic distribution function should not be a function of time then, right. The system will always try to tend to the equilibrium and when it attains the equilibrium point, then it will hardly change with time, this is the concept behind it.

And, again we can say that the equilibrium distribution function is also independent of $\boldsymbol{r}$; that means, $\boldsymbol{\nabla} f_0 = 0$ ok. So, it simply says that if let us say if f_0 is a function of $\boldsymbol{r}$, then you all know now that when f_0 is just integrated in the velocity space then it will give us density and if there is a $\boldsymbol{r}$ dependence then when it is integrated over velocity space, here it will have also an $\boldsymbol{r}$ dependence;

$\int f_0 d^3\boldsymbol{u} = n(\boldsymbol{r})$

that means, uniform gas will no longer be uniform right. So, for any uniform gas, so, basically here you do not need to wait for long time, you just say my gas is homogenous or uniform then the density should be a constant, right and it should be uniform everywhere in the space, right and then of course, the thing is that this type of $\boldsymbol{r}$ dependence should not be there ok so, this will be thing. So, the only dependence should come from the velocity, then f_0 will be simply a function of velocity that much we can conclude till now ok.

$$f_0 \equiv f_0(\boldsymbol{u})$$

Now, if you see that the left-hand side of the whole equation, whole collisional Boltzmann equation, this term $\frac{\partial f_0}{\partial t}$ is vanishing, this term $(\boldsymbol{u}.\boldsymbol{\nabla})f_0$ is vanishing because $\boldsymbol{\nabla} f_0$ is 0 and what about this term $\frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla}_{\boldsymbol{u}} f_0$? Is this also vanishing? Well, when we are talking about equilibrium, at equilibrium $\boldsymbol{a}$ or $\boldsymbol{F}/m$ is 0 because equilibrium is assumed from mechanical consideration: a state where you have 0 force so it is a force free state.

So, the third term is also 0 because $\boldsymbol{F}$ is 0 there, I can actually I mean even reason from here. So, this one $\frac{\partial f_0}{\partial t}$ is 0, this one $\boldsymbol{\nabla} f$ is 0, this one $\boldsymbol{a}$ is 0 so, the whole left-hand side is 0. So, the right-hand side should also be identically 0.

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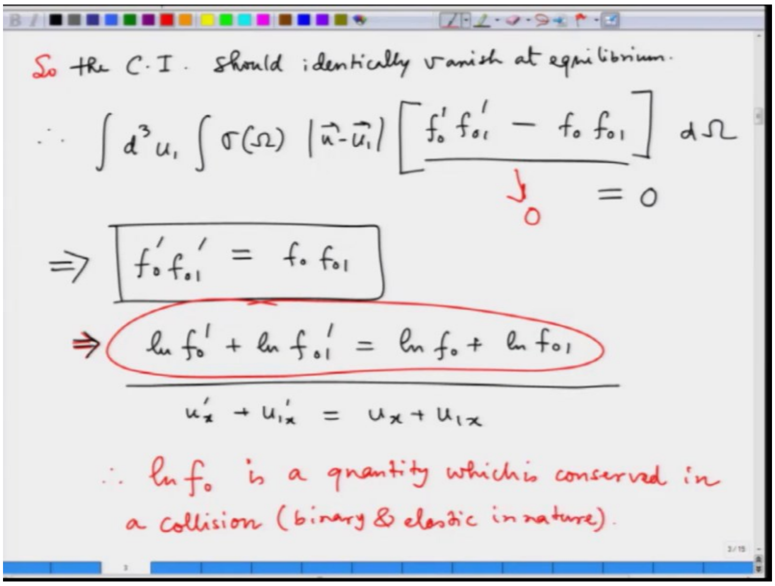

So, the collision integral should identically vanish at equilibrium ok. So, we can simply write

$$\int d^3\boldsymbol{u_1} \int \sigma(\Omega)|\boldsymbol{u}-\boldsymbol{u_1}|(f_0'f_{01}' - f_0 f_{01})d\Omega = 0$$

So, I have just replaced every f by its f_0 value because we are talking about equilibrium. So, this will be the new thing and then it is integrated over this is identically 0 and this is true for any arbitrary choice of $\boldsymbol{u_1}, \Omega$. So, the only condition by which it can be true is

$$f_0' f_{01}' = f_0 f_{01}$$

that means, that the integrand itself should vanish ok. So, you see you have the product of two equilibrium distribution functions before and after the collisions are unchanged and then if you take the natural logarithm, you can say

$$\ln\ f_0{}' + \ln\ f_{01}' = \ln\ f_0 + \ln\ f_{01}$$

So, you see that this is simply like something, I mean when you talk about the momentum conservation or energy conservation before and after the collision let us say the particles are of equal mass m then for example, for momentum conservation you say let us say for x component conservation, right. Say

$$u_x' + u_{1x}' = u_x + u_{1x}$$

This type of thing we write for momentum conservation. So, here this is the actually, it looks that this analogous thing right over here. So $\ln f_0$ is a quantity which is conserved in a collision of course, which are binary and elastic in nature; because otherwise this collision integral form will not be the same.

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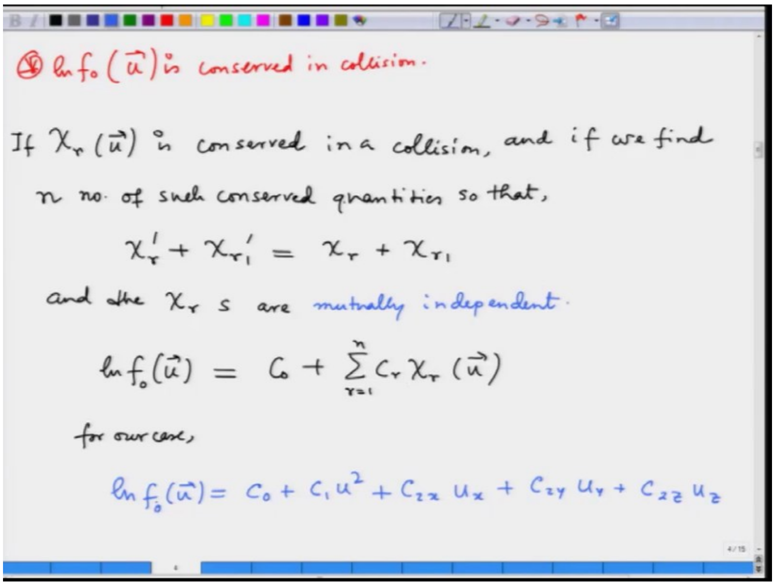

Now, you also know that $\ln f_0$ is a function of $\boldsymbol{u}$ because f_0 is a function of $\boldsymbol{u}$. Now, we will do a trick. Let us now assumed that some function $\chi_r(\boldsymbol{u})$, which is a function of $\boldsymbol{u}$, is conserved in collision and if we find n number of such quantities so that,

$$\chi'_r + \chi'_{r1} = \chi_r + \chi_{r1}$$

you have this and where all of them are function of velocity in general and the χ_r are mutually independent; now there is an important thing: mutually independent that means, they are independently conserved.

One conservation does not depend the on the conservation of the other. For example, if you are considering the conservation of momentum, you can think the three components of the momentum will be conserved in a collision separately or individually, right. So, the three components they are conserved being independent of each other and there is another independent conservation which is also a pure function of velocity that is the kinetic energy, right.

There is another conservation which we do not talk in general that is the conservation of number, right. So, the number of the particles they are also conserved before and after the collision. Then if you just put $\boldsymbol{u}^0$ over there then you will see that is 1 plus 1 will be equal to 1 plus 1. So, 2 particles will be 2 particles ok.

So, now, all this the number conservation, the conservation of the momenta: the linear momentum, component wise of course, and finally the conservation of kinetic energy, all are the mutually independent conserved quantities in a collision. We can finally, write $\ln f(\boldsymbol{u})$ which is a conserved quantity of collision as a linear combination of all these conservations ok.

$$\ln f(\boldsymbol{u}) = C_0 + \sum_{r=1}^{n} C_r \, \chi_r \, (\boldsymbol{u})$$

Of course, here we are just assuming that we have found all the conserved quantities. So, basically χ_r they will form a basis and any arbitrary conserved quantity in an elastic binary collision will then be expressible in terms of this. So, the C_0 can be absorbed inside the summation and then you have to just write r is running from 0 to n and χ_0 should be just 1, ok. So, in this way you can write the whole thing.

Now, assume that in case of simple mechanics, microscopic theory in a classical framework, we only have the conservation of number, conservation of the three components of linear momentum and conservation of the kinetic energy, that completes the basis ok. We do not have any other conserved quantities. If we assume that, then we can write; for our case

$$\ln f(\boldsymbol{u}) = C_0 + C_1 u^2 + C_{1x} u_x + C_{2y} u_y + C_{2z} u_z$$

So, $C_1 u^2$: this is the form coming representing the kinetic energy conservation; then three components are conserved individually for the linear momentum and I am just writing this as $C_{1x} u_x + C_{2y} u_y + C_{2z} u_z$, all the C_{1x}, C_{2y}, C_{2z} are constants ok.

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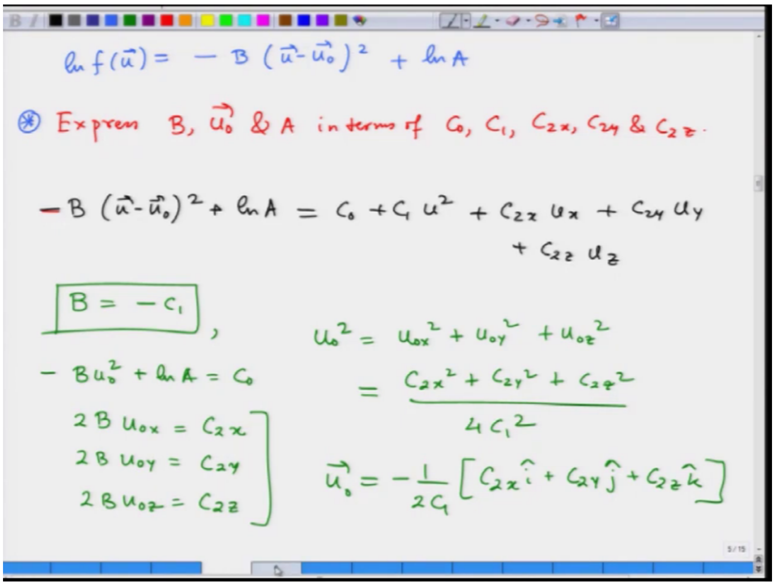

If we can write this, then by little manipulation we can also write

$$\ln f(\boldsymbol{u}) = -B(\boldsymbol{u} - \boldsymbol{u_0})^2 + ln\, A$$

So, you can easily express $B, \boldsymbol{u_0}$ and A in terms of C_0, C_1, C_{1x}, C_{2y} and C_{2z}.

I can do the first one. So, if you just write this thing like

$$-B(\boldsymbol{u} - \boldsymbol{u_0})^2 + ln\, A = C_0 + C_1 u^2 + C_{1x} u_x + C_{2y} u_y + C_{2z} u_z$$

just by matching the powers of $\boldsymbol{u}$ in both sides you can simply say

$$B = -C_1$$

that is the first thing that you can easily see from match, I mean by matching the powers of u^2.

Then you can also say

$$-Bu_0^2 + \ln A = C_0$$

but here you have one unknown B is known, it is expressed in terms of C_1. Now, you have two other unknowns I mean unknown constants u_0 and A but you have other conditions as well

$$2B\, u_{0x} = C_{2x}$$

$$2B\, u_{0y} = C_{2y}$$

$$2B\, u_{0z} = C_{2z}$$

If you combine all, you can actually see that you can simply say that my u_0^2 is nothing but

$$u_0^2 = u_{0x}^2 + u_{0y}^2 + u_{0z}^2 = \frac{C_{2x}^2 + C_{2y}^2 + C_{2z}^2}{4C_1^2}$$

$$\boldsymbol{u_0} = -\frac{C_{2x}\,\hat{\imath} + \ C_{2y}\,\hat{\jmath} + \ C_{2z}\,\hat{z}}{2C_1}$$

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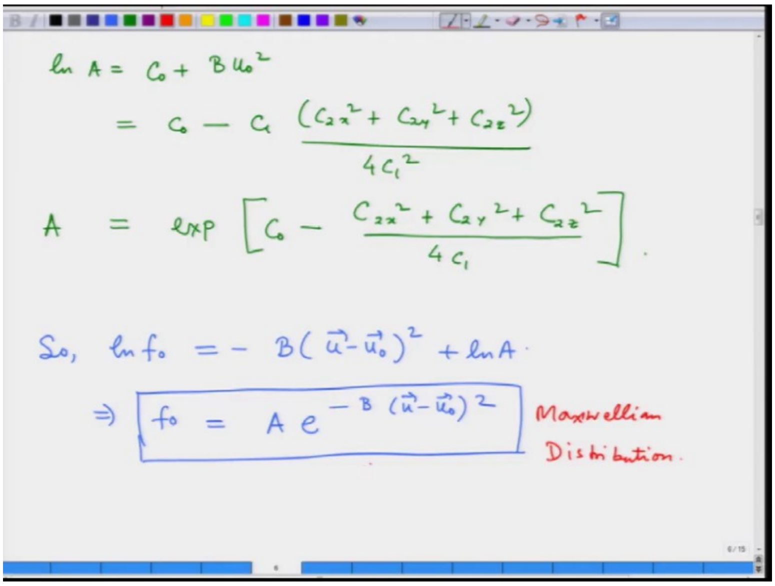

Now, finally, you can write

$$\ln A = C_0 + Bu_0^2$$

$$= C_0 - C_1 \frac{C_{2x}^2 + C_{2y}^2 + C_{2z}^2}{4C_1^2}$$

$$A = \exp(C_0 - \frac{C_{2x}^2 + C_{2y}^2 + C_{2z}^2}{4C_1})$$

So, finally, we could express all the constants B, A and $\boldsymbol{u_0}$ in terms of the $C's$.

Now, we coming back to our previous thing where we wrote that

$$\ln f(\boldsymbol{u}) = -B(\boldsymbol{u} - \boldsymbol{u_0})^2 + ln\, A$$

If we can write that let me just tell you one thing: in case you are just lost that here f_0 is actually should be written over ln $f(\boldsymbol{u})$. So, every f should be replaced by f_0 because that is only true for f_0 and f_0 will be there and finally you can say that

$$\ln f_0(\boldsymbol{u}) = -B(\boldsymbol{u} - \boldsymbol{u_0})^2 + ln\, A$$

So,

$$f_0 = A\, e^{-B(\boldsymbol{u} - \boldsymbol{u_0})^2}$$

you can easily recognize this is the famous Maxwellian distribution Maxwellian distribution.

So, finally, we could show that f_0 has this form ok, but our assumption was that only the number, the kinetic energy and the three components of linear momentum are conserved in binary elastic collision and f_0 is a function of $\boldsymbol{u}$ only ok. So, here you can see that f_0 basically obeys

$$f_0 = A\, e^{-B(\boldsymbol{u} - \boldsymbol{u_0})^2}$$

this type of distribution where A and B these two are constants and $\boldsymbol{u_0}$ this is also a constant ok.

So, this distribution looks very much like Maxwellian distribution, but the distribution which you have possibly encountered in the kinetic theory was not exactly looking like this because A and B had some specific value in terms of the given characteristic quantities of the system and also there was no $\boldsymbol{u_0}$, right. Now in the next step we try to evaluate these two constants ok.

Thank you.