Introduction to Astrophysical Fluids
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Lecture - 54
Turbulent Viscosity

Hello, and welcome to another lecture session of Introduction to Astrophysical Fluids. In this lecture, we simply continue our discussion which was inspired from the last discussion that is another mode of transport phenomena in turbulent system: the viscosity. That means, the transport of the linear momentum. So, in the previous discussion, we talked about the diffusion in a turbulent medium where mostly the density is transported.

Now, in this lecture, we will discuss the transport of linear momentum. As you know that whenever in a system you have some fast-moving layer, which is moving being adjacent to some slowly moving layer, then both the layers by the virtue of viscosity will try to reduce the relative velocity. So, faster one will try to accelerate the slower one and the slower one will try to return to the faster one, that is the story.

Now, what happens if this type of transport phenomena takes place in a turbulent medium? Is there any additional role of turbulence? For example, as we just saw that in case of a turbulent medium; in general, if you start from the first principles just because the turbulent displacements are random; you can actually show that their mean square displacements actually make some linear relation with for times which are very greater than the correlation time of the system okay. So, now in this lecture, we will try to investigate whether we have same type of effect as in case of the transport of density, the diffusion process was actually enhanced by the turbulence process right.

That was actually very easy and intuitive to see, as turbulence means that the random motions of fluid packets from here to there with random velocities then to be very honest this type of motion would surely help in mixing, there is no big surprise right.

Now, whether this is the same case for the transfer of linear momentum, we have to investigate. If instead of a laminar motion, your system has some viscous effect in a turbulent medium, does the same type of effective viscous effect would be obtained or can it modify the viscous effect to some other value? That is exactly the question we will try to answer here.

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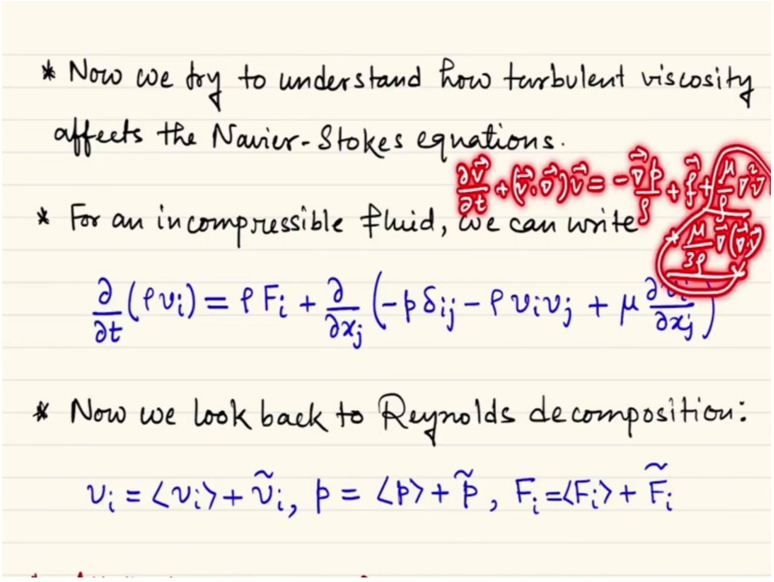

So, for that, we have to understand how turbulent viscosity affects the Navier-Stokes equation. Now, it is true that we have talked several times about turbulent viscosity, but we do not know how to introduce them. So, first of all, we will try to formally introduce turbulent viscosity.

So, of course, we know in Navier-Stokes equation; if you remember Navier-Stokes equation; so you have terms like $\frac{\partial}{\partial t}$ plus the non-linear term $\boldsymbol{v}.\boldsymbol{\nabla}\boldsymbol{v}$, and then there should be some pressure like term $\frac{\boldsymbol{\nabla}p}{\rho}$ plus there will be something some external force term plus there will be something called this viscosity like term, $\frac{\mu}{\rho}\nabla^2 v + \frac{\mu}{3\rho}\,\boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{v})$.

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\boldsymbol{v} = -\frac{\boldsymbol{\nabla}p}{\rho} + \boldsymbol{f} + \frac{\mu}{\rho}\nabla^2\boldsymbol{v} + \frac{\mu}{3\rho}\,\boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{v}) \tag{1}$$

These last two terms are originated from the off-diagonal part of the stress tensor and they constitute the viscous effects. Now, if for simplicity we just take an incompressible fluid, then you know that the resultant viscous effect is totally represented by $\nu\nabla^2\boldsymbol{v}$, where ν is nothing but $\frac{\mu}{\rho}$.

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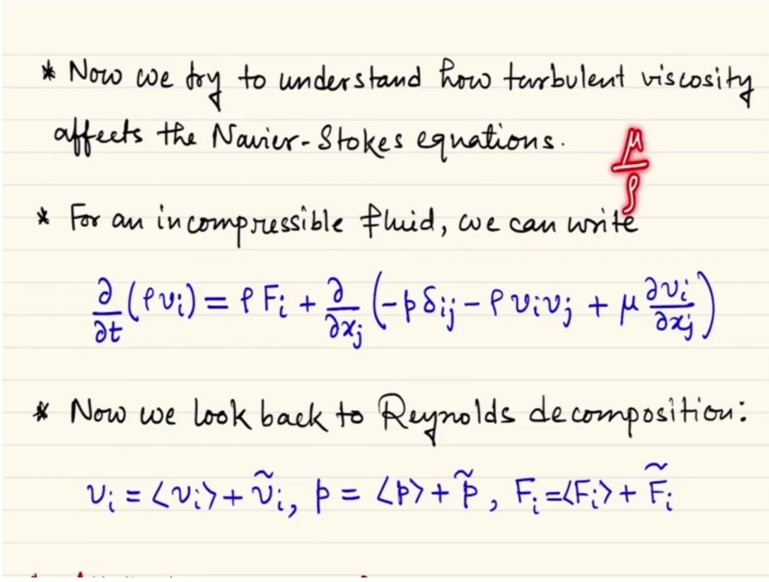

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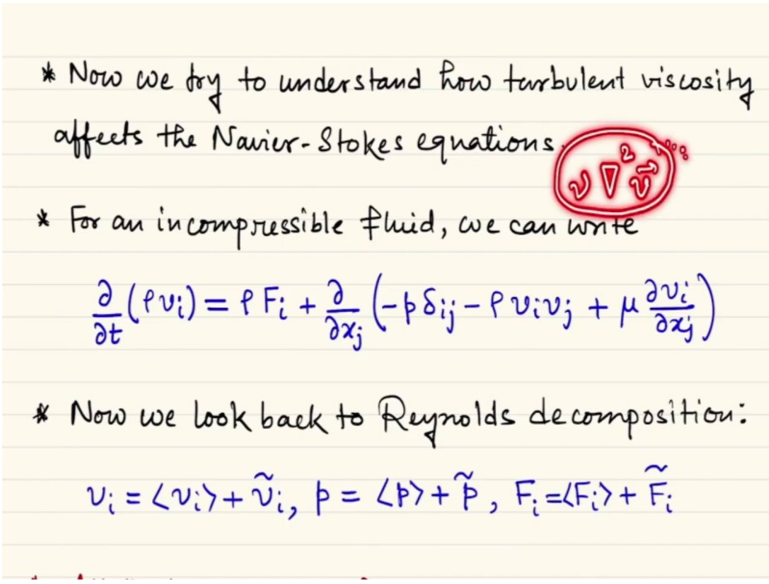

So, ν is the kinematic viscosity, $\nu\nabla^2\boldsymbol{v}$, this will be the resultant viscous term or the surviving viscous term.

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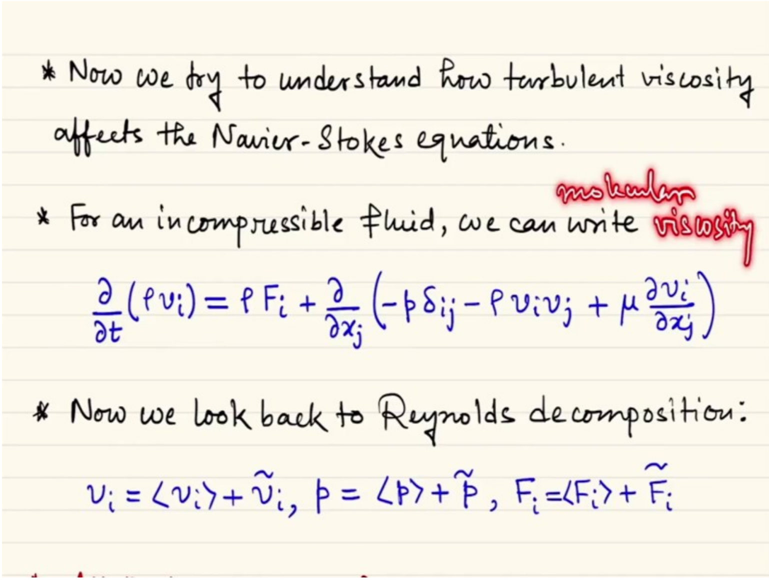

So, of course this $\nu\nabla^2\boldsymbol{v}$ term is there, we can easily understand that this is the normal molecular viscosity and the question is that in the presence of turbulence whether the viscosity or the viscous effects are only due to this $\nu\nabla^2\boldsymbol{v}$ or there should be some extra effect?

So, we just write down the incompressible Navier-Stokes equation at first. One can easily study with compressible equation, but for simplicity let us first check that what happens to the linear momentum transfer for an incompressible fluid? Because this is first of all analytically simple to handle.

So, we just take

$$\frac{\partial}{\partial t}(\rho v_i) = \rho F_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij} - \rho v_i v_j + \frac{\mu \partial v_i}{\partial x_j}\right) \tag{2}$$

F_i is the i$^{\text{th}}$ component of the body force, $\frac{\partial}{\partial x_j}(-p\delta_{ij})$, this is the pressure gradient force, $\frac{\partial}{\partial x_j}(-\rho\, v_i v_j)$ is nothing but the inertial non-linear term, $\frac{\partial}{\partial x_j}\left(\frac{\mu \partial v_i}{\partial x_j}\right)$ is the viscous term.

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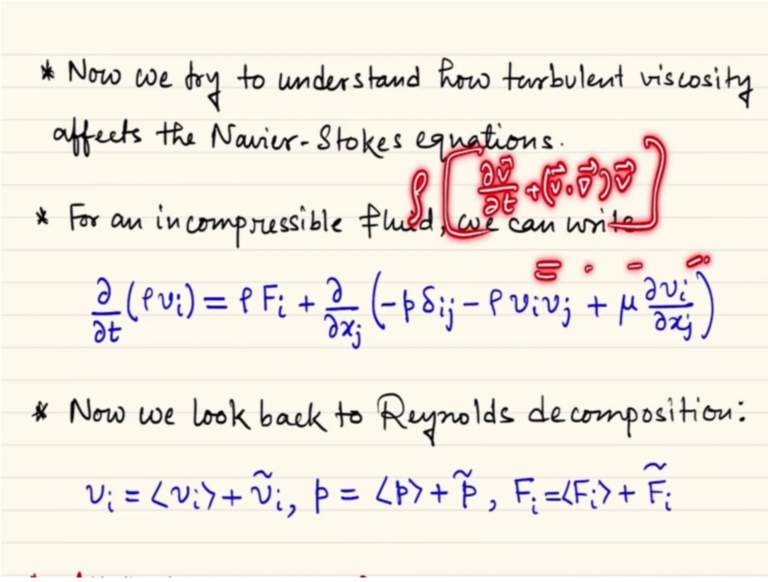

So, if you remember that one way of writing the Navier-Stokes equation is like $\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{v}\right] = \boldsymbol{\nabla} p + \cdots$ and another way of writing is $\frac{\partial}{\partial t}(\rho\boldsymbol{v}) + \boldsymbol{\nabla}.(\rho\boldsymbol{v} \otimes \boldsymbol{v}) = \boldsymbol{\nabla} p + \cdots$. So, we are using exactly the second way of writing in equation (2).

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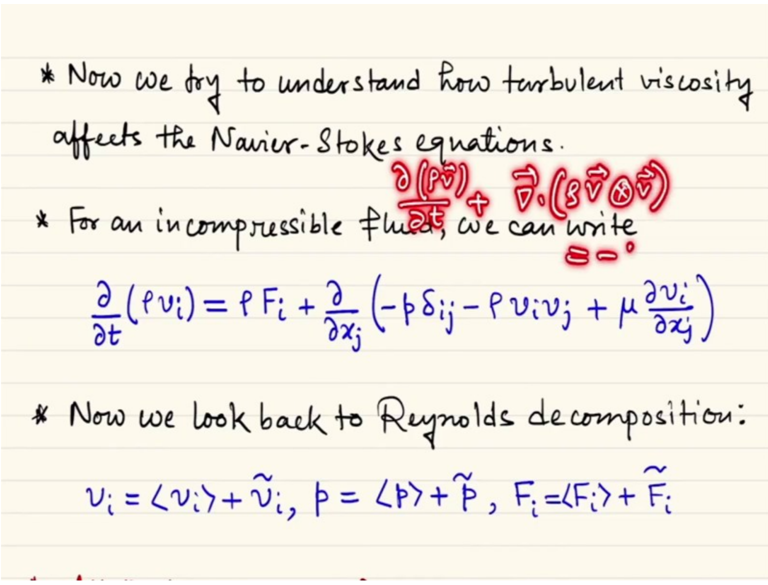

Apriori, we have not assumed that μ is a constant, we have just made μ inside this $\frac{\partial}{\partial x_j}$, actually both for ρ and μ, we have not yet assumed explicitly that they are constant, but one thing is true that here the divergence of $\boldsymbol{v}$ part is assumed already to be 0, because otherwise the viscosity would have two terms. So, this is incompressible fluid and ρ is practically a constant in both time and space.

Now, we just recapitulate or we recall the Reynolds decomposition method, which I introduced at the very outset of this section when I was trying to introduce the statistical approach to turbulence. So, Reynolds decomposition is nothing but the decomposition of the physical quantities at every point in space and time into two parts. One will be statistical average plus another will be this fluctuation with respect to that average value.

So, the i$^{\text{th}}$ component of the velocity can be written like this $v_i = \langle v_i \rangle + \tilde{v}_i$ so, this $\langle v_i \rangle$ is mean velocity plus the fluctuation velocity $\tilde{v}_i$. The pressure can be written as $p = \langle p \rangle + \tilde{p}$, $\langle p \rangle$ is mean pressure plus fluctuating pressure $\tilde{p}$ and the i$^{\text{th}}$ component of the body force would be equal to the mean force or the average force plus the fluctuation part of the force i.e., $F_i = \langle F_i \rangle + \widetilde{F}_i$.

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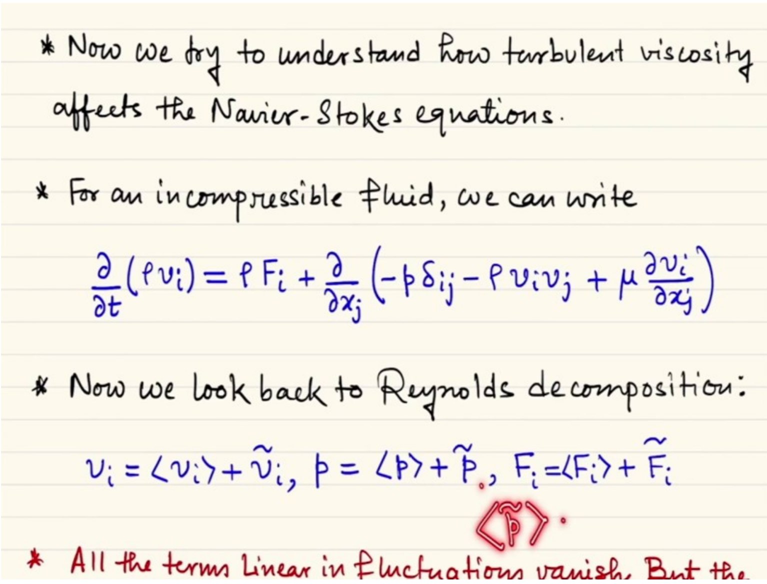

Of course, you know that by definition or by construction, any quantity which is fluctuating, for example, p fluctuation its average should be 0 i.e., $\langle \tilde{p} \rangle = 0$, right that is simple to understand.

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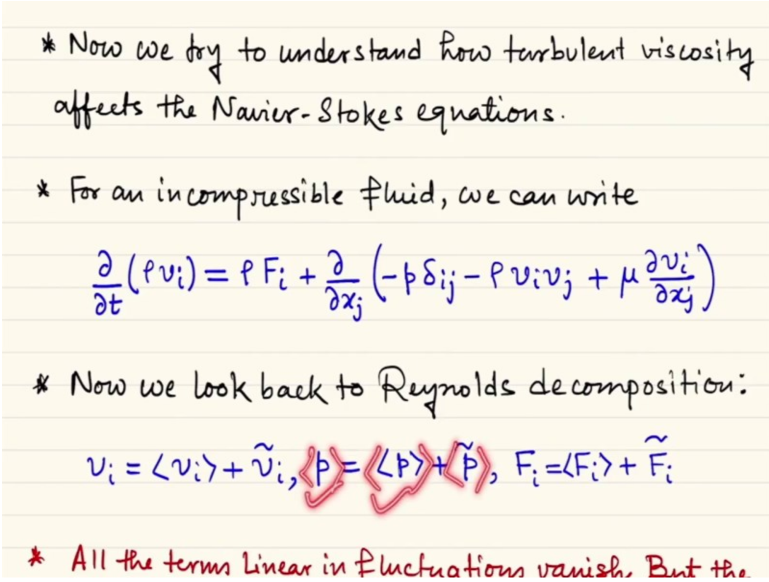

If you take the average on both sides of $p = \langle p \rangle + \tilde{p}$, so we would have $\langle p \rangle = \langle p \rangle + \langle \tilde{p} \rangle \Rightarrow \langle \tilde{p} \rangle = 0$.

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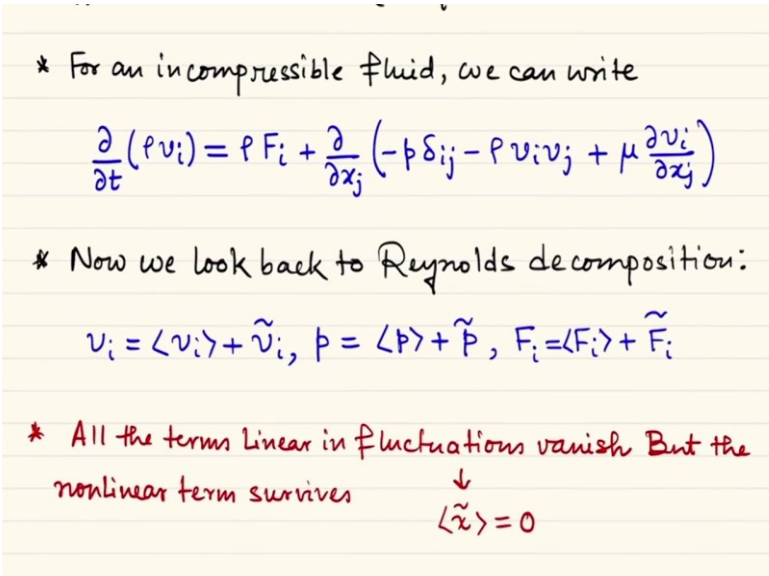

That is why we can conclude that all the terms which are linear in the fluctuations, they actually vanish but for the non-linear term we cannot say this. Why we cannot say like this? what do exactly we do; actually that is not very easy to understand; we can try to see here.

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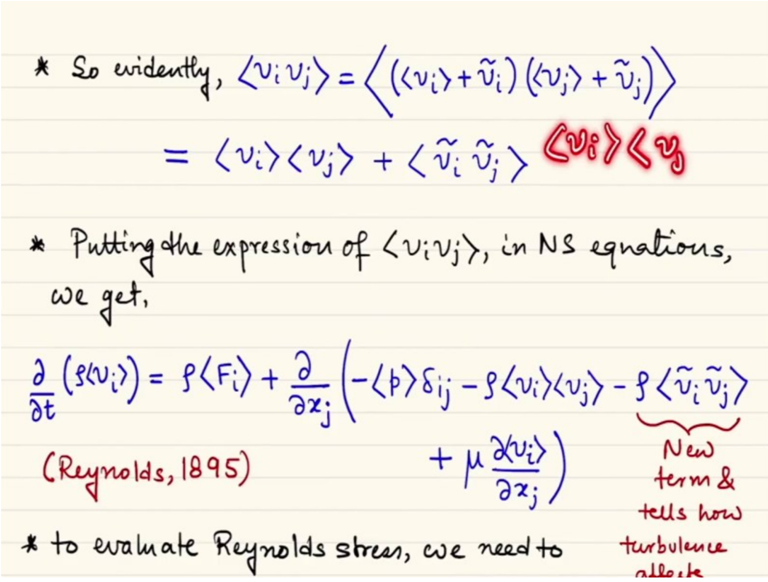

Just by incorporating the Reynolds decomposed expressions in the Navier-Stokes equation and specially to the non-linear term of the Navier-Stokes equation. So, the non-linear term practically comes out to be like this (because rho is constant); we can simply play with $\langle v_i v_j \rangle$;

$$\langle v_i v_j \rangle = \langle (\langle v_i \rangle + \tilde{v}_i)(\langle v_j \rangle + \tilde{v}_j) \rangle \Rightarrow \langle v_i \rangle \langle v_j \rangle + \langle \tilde{v}_i \tilde{v}_j \rangle \quad (3)$$

Second and third term will go away since $\langle \tilde{v}_i \rangle = \langle \tilde{v}_j \rangle = 0$.

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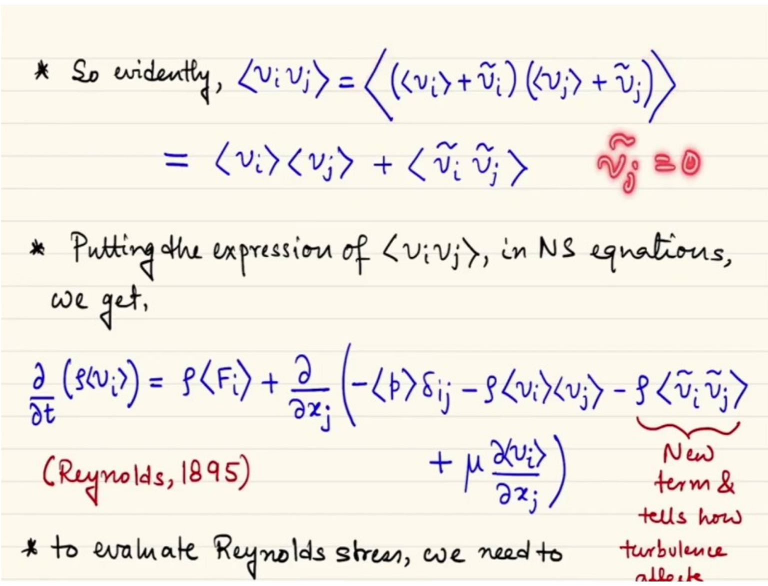

The fourth term which is $\langle \tilde{v}_i \tilde{v}_j \rangle$ this does not go away because this is non-linear in the fluctuation, so that is why its average is no longer 0.

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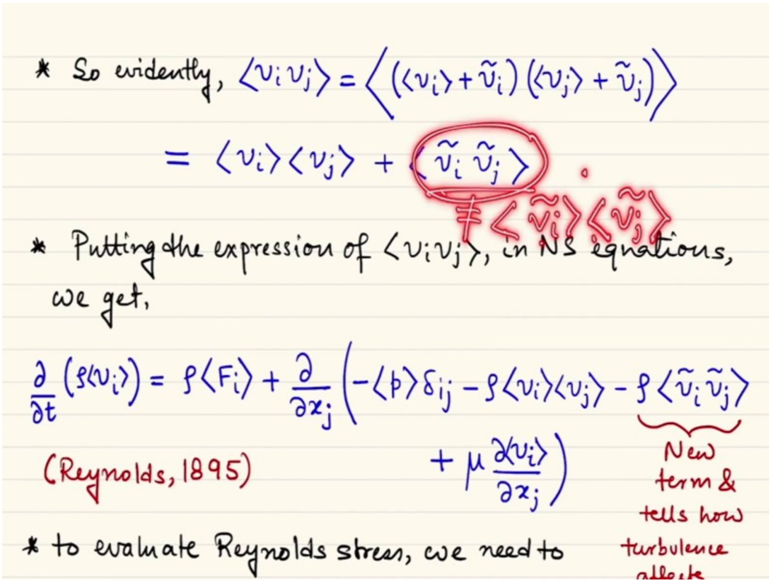

Of course, you have to understand that if this term $\langle \tilde{v}_i \tilde{v}_j \rangle$ is equal to $\langle \tilde{v}_i \rangle \langle \tilde{v}_j \rangle$, then we are done. But this is not true, this one $\langle \tilde{v}_i \tilde{v}_j \rangle$ is not equal to average of $\langle \tilde{v}_i \rangle \langle \tilde{v}_j \rangle$ i.e., $\langle \tilde{v}_i \tilde{v}_j \rangle \neq \langle \tilde{v}_i \rangle \langle \tilde{v}_j \rangle$. So, this term $\langle \tilde{v}_i \tilde{v}_j \rangle$ is non-zero in general.

And if we write this expression $\langle v_i v_j \rangle$ in the original Navier-Stokes equation, we simply obtain that

$$\frac{\partial}{\partial t}(\rho \langle v_i \rangle) = \rho \langle F \rangle_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij} - \rho \langle v_i \rangle \langle v_j \rangle - \rho \langle \tilde{v}_i \tilde{v}_j \rangle + \frac{\mu \partial \langle v_i \rangle}{\partial x_j}\right) \qquad (4)$$

Now, for the non-linear term we have two terms one will be $\rho \langle \tilde{v}_i \rangle \langle \tilde{v}_j \rangle$ and others is $\rho \langle \tilde{v}_i \tilde{v}_j \rangle$. So, this equation is nothing but the evolution of the mean angular momentum or mean linear momentum density.

Now, you see although this (equation (4)) is the evolution equation for the mean linear momentum and we have a quantity $\rho \langle \tilde{v}_i \tilde{v}_j \rangle$ from the fluctuation. This quantity is of course, non-linear in nature. This equation for the first time was derived by Reynolds in the year 1895.

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$$= \langle v_i \rangle \langle v_j \rangle + \langle v_i v_j \rangle$$

* Putting the expression of $\langle v_i v_j \rangle$, in NS equations, we get,

$$\frac{\partial}{\partial t}(\rho \langle v_i \rangle) = \rho \langle F_i \rangle + \frac{\partial}{\partial x_j}\left(-\langle p \rangle \delta_{ij} - \rho \langle v_i \rangle \langle v_j \rangle - \underbrace{\rho \langle \tilde{v}_i \tilde{v}_j \rangle}_{\text{New term \& tells how turbulence affects mean-field evolution}} + \mu \frac{\partial \langle v_i \rangle}{\partial x_j}\right)$$

(Reynolds, 1895)

* to evaluate Reynolds stress, we need to study $\frac{\partial}{\partial t}\langle \tilde{v}_i \tilde{v}_j \rangle \rightarrow$ But problem of closure due to hierarchy $\frac{\partial}{\partial t}\langle \tilde{v}_i \tilde{v}_j \tilde{v}_k \rangle$ (Reynolds Stress)

And this new term $\rho\langle \tilde{v}_i \tilde{v}_j \rangle$ which comes here, this simply says that how turbulence affects the mean field evolution of the linear momentum. This term $\rho\langle \tilde{v}_i \tilde{v}_j \rangle$ is known as Reynolds stress in similarity with the normal tensors. Now, why this $\rho\langle \tilde{v}_i \tilde{v}_j \rangle$ gives us an account for the effect of turbulence? Because, this is constituted by the fluctuations and as we have already discussed in the first lecture on turbulence that the turbulence is mainly manifested by the fluctuations right.

So, now, in order to solve this equation, we have to have a knowledge of this $\rho\langle \tilde{v}_i \tilde{v}_j \rangle$. So, for mean field evolution, we have to have a knowledge of the fluctuations, and that is exactly the point where turbulence is important, where we have to think that something will be different from the normal laminar case. Because in case of the laminar flow, this type of fluctuations is negligibly small. So, then we are just happy with saying this $\frac{\partial}{\partial t}(\rho\langle v_i \rangle)$ is equal to this $\rho\langle F \rangle_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij} - \rho\langle v_i \rangle\langle v_j \rangle + \frac{\mu \partial\langle v_i \rangle}{\partial x_j}\right)$ and this term $\rho\langle \tilde{v}_i \tilde{v}_j \rangle$ is not there because this is very small, near to zero. But the problem is that how to determine this $\langle \tilde{v}_i \tilde{v}_j \rangle$? One way is to determine that again we have to study the evolution equation of this one $\rho\langle \tilde{v}_i \tilde{v}_j \rangle$.

But if you do the evolution equation of this one $\rho\langle \tilde{v}_i \tilde{v}_j \rangle$ and constitute the equation for this, you will see that you will again be trapped in the classic problem of the closure due to hierarchy.

What is this $\frac{\partial}{\partial t}\langle \tilde{v}_i \tilde{v}_j \rangle$? This is the fluctuations of two-point functions right so this is a measure of correlation.

So, the correlation of two point, when we are trying to study its evolution is associated; you can see actually by calculating analytically; with three-point correlation functions and if you want to calculate $\frac{\partial}{\partial t}$ of this three-point correlation function let us say $\langle \tilde{v}_i \tilde{v}_j \tilde{v}_k \rangle$, then you will have a four-point correlation function. So, it will go forever. This is the same type of hierarchy problem, which we already encountered when we were talking in terms of BBGKY hierarchy. So, what to do? Of course, the solution is that we have to; as we have done also before; close the system in some artificial way.

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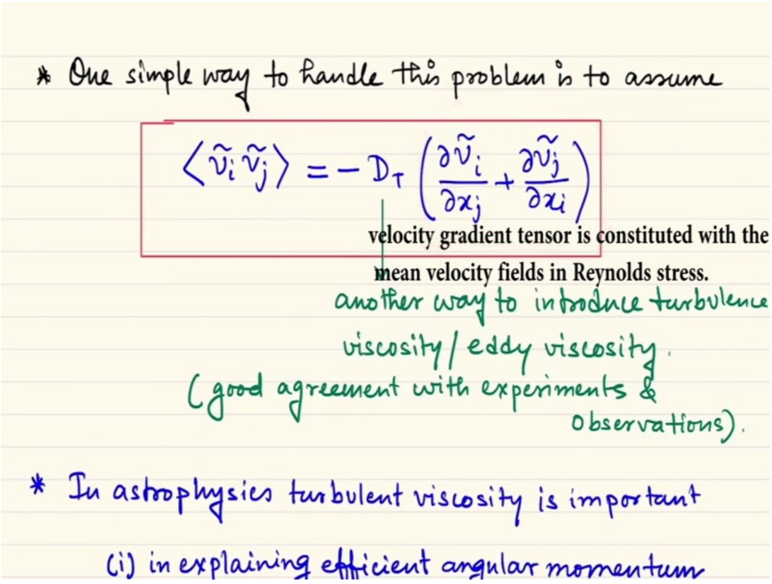

So, one simple way to close the system or to handle this problem is just to say that okay this Reynolds stress will be proportional to the velocity gradient tensor constituted with mean velocities. So, if you can do that and you say just say that the proportionality constant will be some constant, let just for instance D_T, then

$$\langle \tilde{v}_i \tilde{v}_j \rangle = -D_T \left(\frac{\partial \langle v_i \rangle}{\partial x_j} + \frac{\partial \langle v_j \rangle}{\partial x_i} \right)$$

So, this D_T is the proportionality constant, then the system is actually closed.

People tried to match this with experiment or observational results, actually surprisingly it is found to match very well. So, if we calculate the velocity gradient tensor of the mean values and you plot this one $\left(\frac{\partial \langle v_i \rangle}{\partial x_j} + \frac{\partial \langle v_j \rangle}{\partial x_i}\right)$ with this one $\langle \tilde{v}_i \tilde{v}_j \rangle$, the two-point correlation functions of the fluctuations, you will see that they will give you a linear relation and this coefficient D_T; with a negative slope of course; this quantity, now you see, so that is turbulence viscosity D_T or eddy viscosity D_T. So, of course, one way is nothing but the previous discussion of turbulent diffusion okay.

So, what I was trying to say that here in this lecture I am totally talking about the viscosity for an incompressible fluid; the transported quantity is the linear momentum. Then we have encountered this constant D_T. Now, if you remember when we wrote the constant or the diffusion coefficient for turbulent diffusion, we also called them that D_T.

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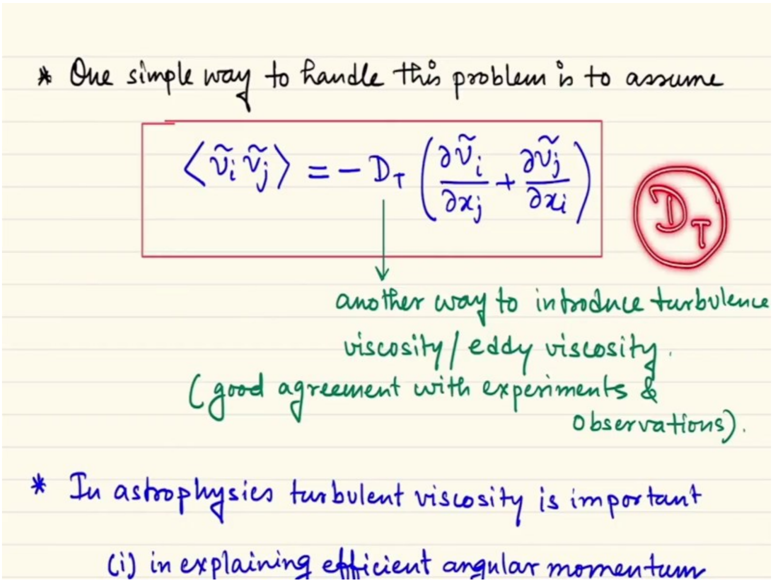

Now, was it just an accident well? surprisingly it is also seen that the turbulent diffusion coefficient and the turbulent viscosity coefficient they are also the same. So, that is why I wrote here this D_T, and this is another way to obtain this constant. So, what is the message of that? The message is that whether we are talking in terms of turbulent diffusion or in terms of turbulent viscosity, for both type of transport phenomena in turbulent medium, the same transport coefficient appeared. Why this is the same? So, that is the fundamental question. So, D_T can be obtained as a diffusion coefficient for turbulent diffusion and D_T can also be obtained as a coefficient of viscosity for turbulent flow.

Now, it is true that if you remember that when we had very briefly discussed the kinetic transport effects or transport phenomena, we showed that there are three transport coefficients. One is the viscosity, one is the thermal conductivity, another is the coefficient of diffusion, and they need not be the same. Even that you also know from your previous knowledge of kinetic theory right.

Now, for a turbulent case, what happens? This is no longer true. All the transport coefficients are the same and they are exactly equal to D_T. What is this D_T? So, if for example, for some turbulent flow, if you can in some way measure the coefficient of diffusion, just by studying the diffusion of that medium, then this coefficient will exactly be equal as the coefficient of viscosity due to turbulence.

Now, this is the discrepancy between kinetic transport phenomena and turbulent transport phenomena. It happens simply because in kinetic transport phenomena, the transport process actually modifies the molecular motion. For example, when the molecules are carrying thermal energy; more and more thermal energy; then they are actually getting faster and faster right. So, their motion is modified.

Once again when the greater number of molecules are carrying some linear momentum, this is also true that more and more linear momentum means the molecules are getting faster and faster. But, in case of turbulent viscosity or turbulent diffusion, what happens that the transport phenomena is unable to modify the turbulent motion itself. Although the turbulence causes a modification in the transport phenomena, thereby giving you some effective transport coefficient, which is the normal coefficient plus the turbulence effect turbulent part but the transport phenomena cannot give a feedback to the turbulent motion or turbulence motion by modifying the motion of the fluid as a whole.

So, the conclusion is that in microscopic kinetic transport phenomena, the microscopic motions or the motion of the transporter is modified due to the transport phenomenon that is why all the coefficients are different there. Now, here with respect to turbulence, turbulence sees every transport phenomenon as a transport of some passive scalar. So, let us say for conductivity, this is temperature; for diffusion this is density; for viscosity this is some component of the linear momentum, whatever this is, but this is a passive thing which is transported and when this passive thing is transported in turbulence, that means this transport cannot give a feedback to the flow.

So, I think this is a bit advanced topic to discuss. So, if you are interested, you can go through the reference books of this course, which was suggested. Of course, any non-equilibrium statistical mechanics book can also be helpful. So, the final message is that in case of turbulence whether this is turbulent diffusion or viscosity, the coefficient of transport phenomena remains the same. That is exactly what we are trying to say here that we have obtained this one D_T for the second time actually, one in the turbulent diffusion framework one in the turbulent viscosity or eddy viscosity framework. So, this is also known as the coefficient of turbulent viscosity or eddy viscosity.

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$$\boxed{\langle v_i v_j \rangle = -D_T\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}$$

another way to introduce turbulence viscosity / eddy viscosity.
(good agreement with experiments & observations).

* In astrophysics turbulent viscosity is important

(i) in explaining efficient angular momentum transport in galactic disc and normal accretion discs &

(ii) to prevent differential rotation in slowly rotating stars.

Now, coming at the very end, I will emphasize on two points one is that: in astrophysics specifically, this turbulent viscosity is very important. Why? Because in general for explaining efficient angular momentum transport in normal accretion disk and galactic disks just the consideration of molecular viscosity is not sufficient, and then you have to take this turbulent viscosity effect into account.

Another case is that we have already discussed, while we are discussing the effect of rotation, for slowly rotating starts, after some time even at the very beginning, it can develop some differential rotation but, at the end, due to viscous effects, the star actually ends up rotating with solid body rotation and molecular viscosity is totally inadequate to prevent this differential rotation, then also you have to take turbulent viscosity into effect. So, these are the points or astrophysical interest for which one should study turbulent viscosity more in detail.

Basically, there are research papers on the role of turbulent viscosity or eddy viscosity in accretion disks and also in the slowly rotating starts. Please go through them, so that you can have an exposure to the current status of the research of the set fields.

So, mostly this is all about the transport process in a turbulent medium. In the next lecture, I will discuss very briefly the turbulence in an MHD fluid and also a brief general overview of turbulence in the world of astrophysics okay.

Thank you very much.