Introduction to Astrophysical Fluids
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Lecture - 52
Richardson-Kolmogorov phenomenology of turbulence

Hello and welcome to another lecture session of Introduction to Astrophysical Fluids. In this lecture we simply continue the discussion which we started in the previous lecture on the turbulence in fluids. In the previous lecture we started by saying you, different elements or piece of elements by which we can try to approach the problem of turbulence.

So, because of the non-linearity and very high irregularity, as you can easily understand that a very easy analytical treatment for turbulence is not possible and then there came the suggestion by Taylor in the year 1935 who said that, we should not be interested in the one-point quantities rather we should be interested in the study of the two-point correlation functions of velocity pressure for example and their evolution.

Then following that von Karman and Howarth the year 1938, they started that and they obtained analytical relations for those correlation functions in the year 1938 for homogeneous and isotropic turbulence. Now, I did not present the exact analytical results by von Karman and Howarth because they are really a bit technically non-trivial.

So, just take it as an information that they did it and actually the next piece was done by Kolmogorov in the year 1941 and that is the so called the first exact relation in turbulence. So, that I will talk a bit about in this discussion. But in this discussion, we will start by discussing the heuristic phenomenological approach to turbulence which was initiated by Richardson in the year 1921.

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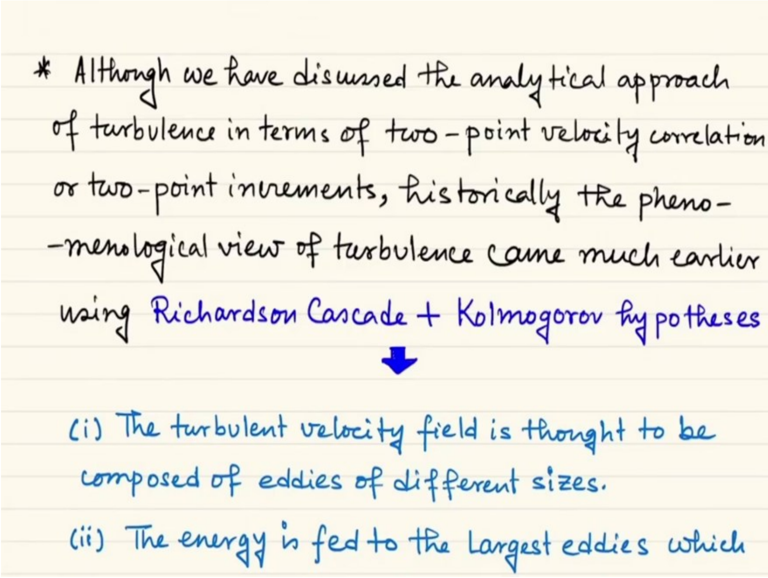

So, what I am just trying to say is that, historically the style of attacking the problem of turbulence was not very much mathematically rigorous. Richardson who was an expert in the department of weather, he was basically much more interested in modeling the atmospherical turbulence and he observed that turbulent field always mostly consists of eddies or vortex like structures of different sizes.

Then he proposed his picture of Richardson cascade and it was merged with different hypothesis of Kolmogorov. So, they are called the Kolmogorov's assumptions of universal equilibrium turbulence or Kolmogorov's universality hypothesis. I am not going into the detail of all this hypothesis step by step and the Richardson cascade separately, but here what I will try to tell you is the net picture of all these things.

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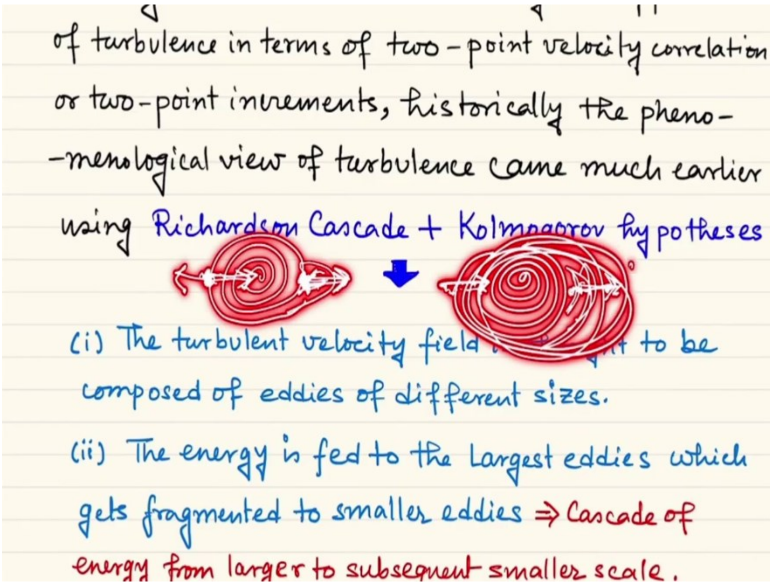

So, if you can compile all this Richardson cascade image along with the Kolmogorov hypothesis, what you will get is the following. So, the turbulent velocity field is thought to be composed of eddies of different sizes.

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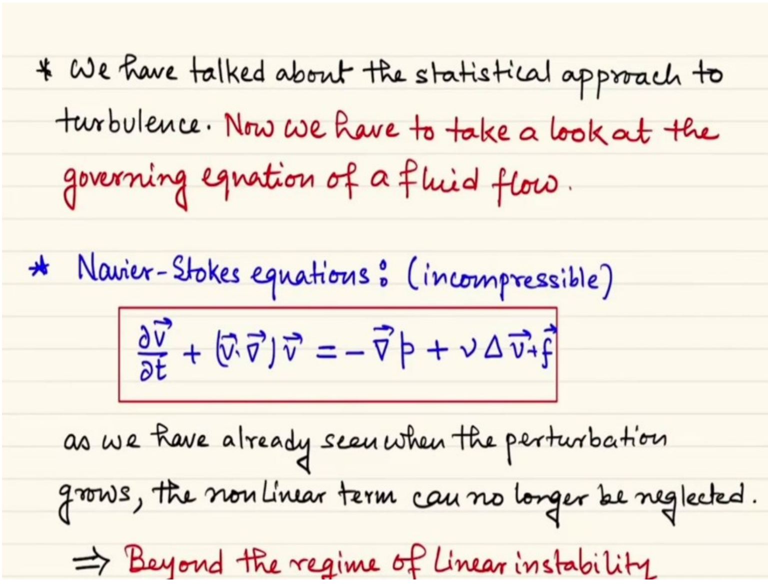

The energy by the forcing term if you remember, this is $\boldsymbol{f}$, the forcing term by which energy is injected into the system, that can be of course some body force or some random force. So, whether just by regular body force like gravity we can energize turbulence or not, that is a matter of debate, I am not going into that, but simply what when I am writing

$\boldsymbol{f}$, is simply some external energy source, I am not saying anything else okay. But this external energy source should be such that the energy is fed only to the largest eddies of the system; that means, the force function should be such that it is only working for very large scales or the scales near the macroscopic scales okay. So, I mean, the energy is only injected from exterior to the system at macroscopic scale and one of the way of saying this is to the largest eddies okay.

Now, these largest eddies get fragmented subsequently to smaller eddies and this fragmentation is mostly due to the incompressibility of the fluid. A very easy explanation of this type of fragmentation can be easily understood by simply saying that let us say, you have a vortex like this and of course as you are in a turbulent field, so, there will be some discrepancy in the velocity at this point and this point (Refer Slide Time: 04:16).

Let us say at this point the velocity is greater and once again let us make it simpler, let us say the component of velocity in this direction (let us say x direction), the component of velocity at this point in this direction is larger than that of the velocity of this point. So, what happens after sometime, this point will go farther and this point will not go farther okay.

Let us say if you have this type of velocity direction so, what happens, after sometimes let us say your vortex structure will be somehow distorted in one direction because the fluid particle which was previously at this point will be now much farther.

Then the fluid particles at this point now the fluid particles at this point let us say they come from here to here and the fluid particles from this point to that, they have come from here to here. So, thereby they have some distortion.

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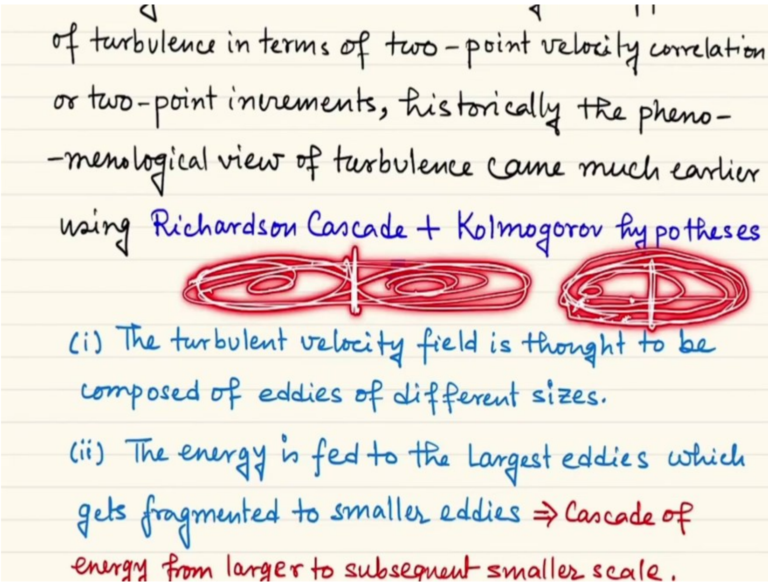

So, due to this discrepancy you will see that in time you will have in one direction an elongation type of thing of the vortex and after sometimes you can have a section of the vortex the structure that would try to tear almost in the middle and thereby making two subsequent eddies and basically here incompressibility plays the role.

So, once again I am just drawing the picture, this particle has a greater velocity. So, it goes there and it may go there; so now, the structure is more like this and then the structure with time is more like this okay, then what happens? You have to easily understand that, since this is incompressible and we are just tracing of one single vertex. So, we are just concentrating on a given mass and we are just saying, I mean we are just following that with time how does this evolve and so, for the given mass if the density is constant then of course, it is very necessary that the effective volume should also be constant otherwise this is not possible right.

So, of course, the volume is conserved almost, I mean the volume conservation is still guaranteed because there is an elongation in one direction and a compression in the perpendicular direction right. So, afterwards when the elongation is too much, then in order to conserve the volume, the lateral dimension should be so small that it can actually tear at this point and then you can see that you have two smaller eddies which are roughly the half of the size of the bigger one.

So, that is simply the story of the fragmentation of the eddies. Now, when the larger eddies get fragmented into the smaller eddies, then what happens is that the energy was first fed to the largest eddies and they are become smaller eddies subsequently so, the energy is now effectively in the smaller eddies. So, it is practically just by saying that the energy is cascading from the larger to the smaller eddies or if we just say that the size of the eddy is somehow equivalent to a length scale, then we can say that the energy is cascading from larger to subsequent smaller length scales and this is known as the cascade of energy in turbulence.

Of course, you understand that if the fluid is not incompressible then this type of fragmentation is not guaranteed and then this type of cascade is also not very easy to imagine, that is one of the main problems in astrophysical turbulence where most of the fluids are very much compressible.

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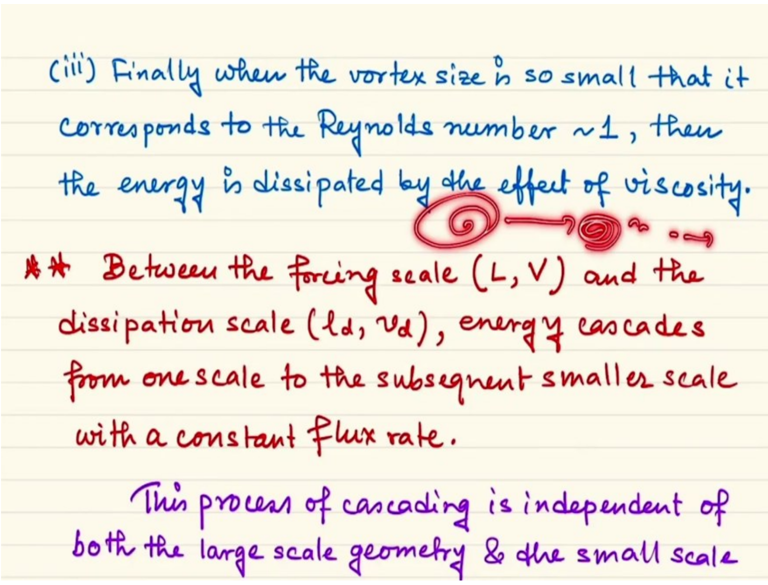

Now the next point is that, finally, when the vortex size is so small; so, there is a fragmentation process from the larger to the subsequent smaller eddies and then this fragmentation reaches to such an extent that the vortex size is very small; and it corresponds to the Reynolds number ~ 1.

So, all these things are just our phenomenological assumptions. This is the phenomenological picture and there is no hard and first proof for that, we are just saying that we believe that the turbulent system should behave like that, so energy can cascade up to the extent where the Reynolds number is of the order of 1 and we call this corresponding length scale to be Kolmogorov scale.

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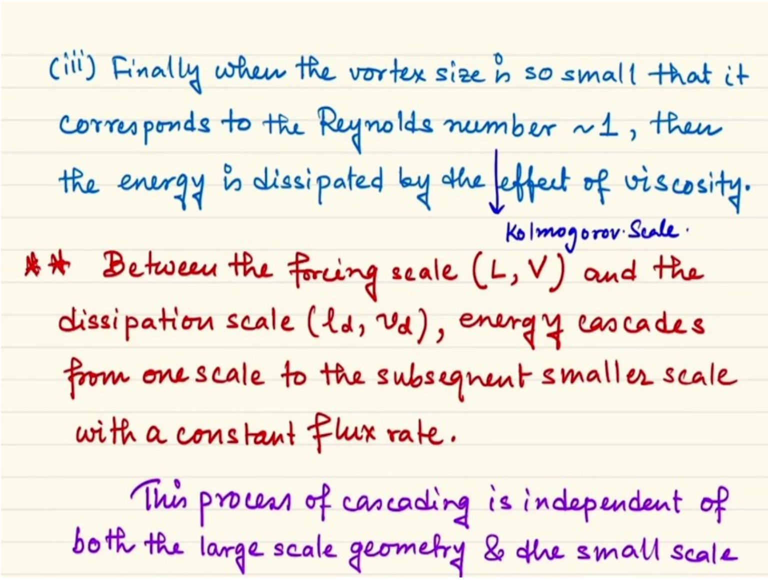

So, that corresponds to Kolmogorov scale, then we can actually say that starting from this scale, I mean this vortex of this size with Kolmogorov scale, the energy is dissipated by the effect of viscosity. So, no more fragmentation and cascade of energy takes place. Now between the forcing scale; so, forcing scale we can always say that this is the macroscopic scale, we can simply designate this by the length capital L and the velocity capital by V; and the Kolmogorov scale; which is the dissipation scale which we can designate by l_d and v_d; energy simply cascades from one scale to the subsequent smaller scale with a constant flux rate. So, this is a simply a very important assumption of the whole theory.

So, during the cascade process, the cascading is taking place in such a way that the energy flux rate is not depending on the scale chosen. So, this is basically constant throughout the scales.

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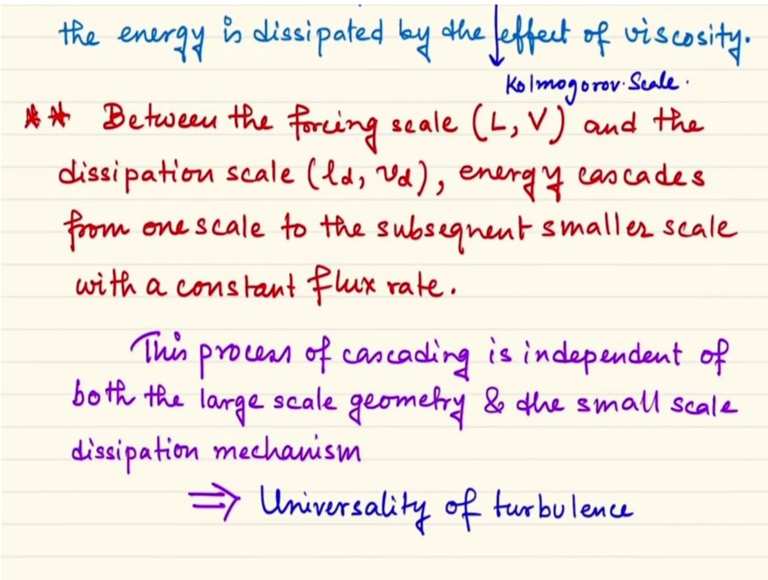

And so, this process of cascading is simply independent of both the large-scale geometry and the small-scale dissipation mechanism. Because the cascade process is only characterized by a constant flux rate and this constant flux rate does not really need any detailing of the large-scale geometry and the small scale forcing and so, that's why, this process of cascade simply gives us a concept of universality or universal behavior of turbulence.

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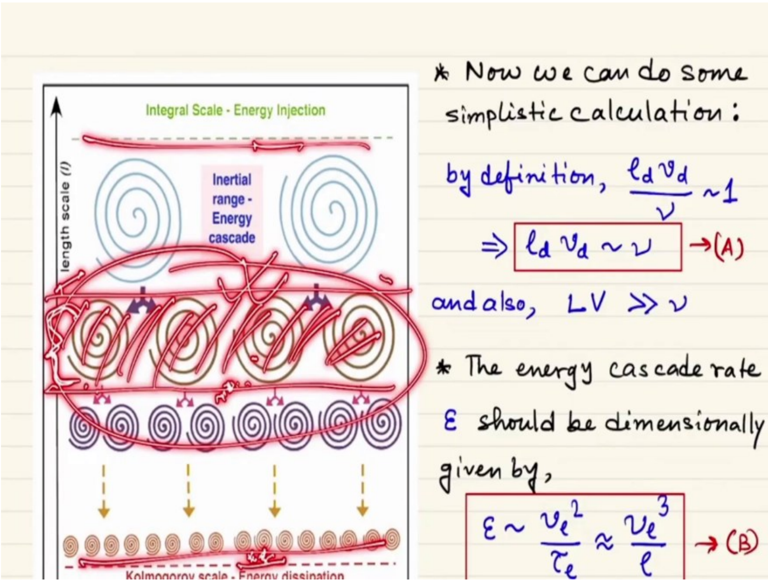

Because, roughly it is saying that the energy is injected at this scale so, this is the largest scale and or called the forcing scalar or integral scale and this is the Kolmogorov scale where energy dissipation is taking place.

This (Refer Slide Time: 15:17) picture is for the communication purpose, it is drawn like that, but between these two simply does not mean that just starting from this one and reaching to the Kolmogorov scale.

When we are talking about simply the cascading with a scale independent flux rate, then we simply talk about a range of scales which are not only in between these two scales but also very far from both the scales. So, in a sense this should be like here okay, maybe much more practical is like this. This could be the range of scales in which the proper cascading takes place with the scale independent energy rate okay.

So, these things are very subtle and this range is also far from both this scale and so this range is not really affected by any of the two and therefore this range is known as the inertial range and is universal for any turbulent system.

Because, let us say your turbulence is in a spherical jar or in a cubicle box, for both this is the same or you have a turbulence in tap water or your turbulence is in a salt water, so, it is the same. Let us say you have turbulence in glycerin or in oil, well the viscosity is different, but this cascading process is universal because this does not also care about the nature of the viscosity.

So, this is known as inertial range and you can simply see that the bigger eddies or the larger eddies, they are fragmented into subsequent smaller eddies to such an extent that the eddies reach to the scale of smallest eddies and this is the so-called Kolmogorov scale or the energy dissipation scale. Now, at this point, I mean we should remember that although we are saying that inertial range is very far from both this range, when we will do some phenomenological treatment, we will actually say that the energy is injected to the system, let us say at some rate, the energy injection rate to the system is actually equal to the energy cascade rate and that is actually equal to the viscous dissipation rate and that is the necessity of having a stationary regime in turbulence, that was one of the fundamental hypothesis of Kolmogorov.

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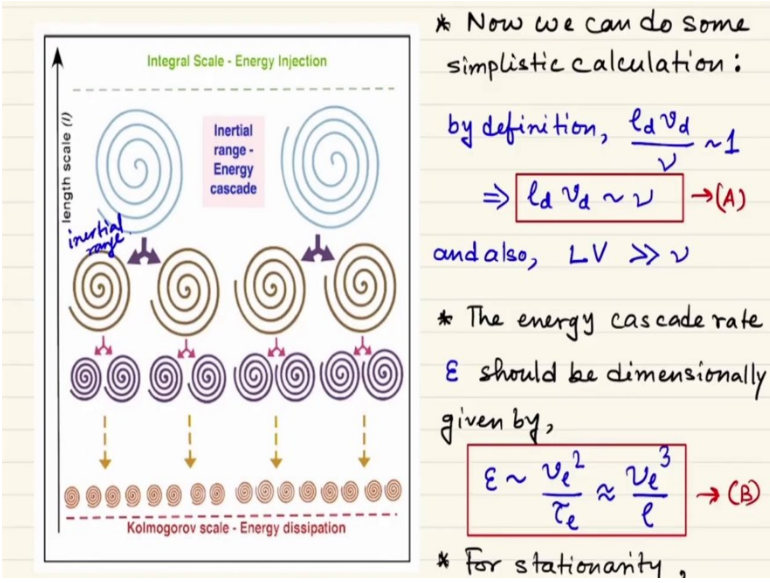

Now, we can do something here at this point we will do something which is heuristic; that means, it does not have any proper analytical rigorous mathematical background, but we will do some simplistic estimate just by using the order analysis and of course, I mean it is a very crude treatment. So, maybe you can have some complain about the mathematical regard of this part, but well this is how that historically or traditionally the turbulence field was studied at the very outset.

So, we have postulated several points and based on those points we will now trying to construct some quantitative results. So, the first one according to our definition, this is $\frac{l_d v_d}{\nu}$, the Reynolds number at the Kolmogorov scale which should be of the order of unity. So,

$$l_d v_d \sim \nu \qquad (A)$$

So, that is the Reynolds number definition in case you have forgotten. So, for our case we simply say that coefficient of viscosity is not a function of length scale, this is roughly true. So, if this is true then l_d times v_d will be of the order of ν. Once again we know that the macroscopic Reynolds number which is $\frac{LV}{\nu}$ should be very greater than 1, so $LV \gg \nu$.

Now, energy cascade rate ϵ, that is something which should be dimensionally this is the thing which is actually according to our assumption should be scale independent.

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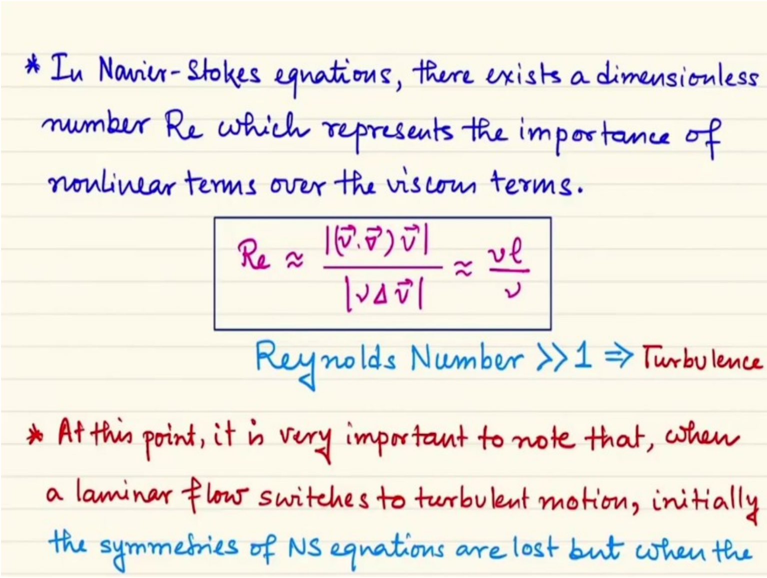

And dimensionally it is nothing, but the energy the density of energy per unit time. So, for incompressible turbulence case; traditionally we just say that there is density of course; so, ρ is 1 there and ρ is constant. So, we are just normalizing everything at which is ρ is equal to 1 ok and so, according to this normalization, ϵ is

$$\epsilon \sim \frac{v_l^2}{\tau_l} \sim \frac{v_l^3}{l} \qquad (B)$$

Now, this τ_l is the time required effectively for the fragmentation of an eddies of length scale l right and this is again roughly is nothing but $\frac{l}{v_l}$. So, if that is the only possible time scale.

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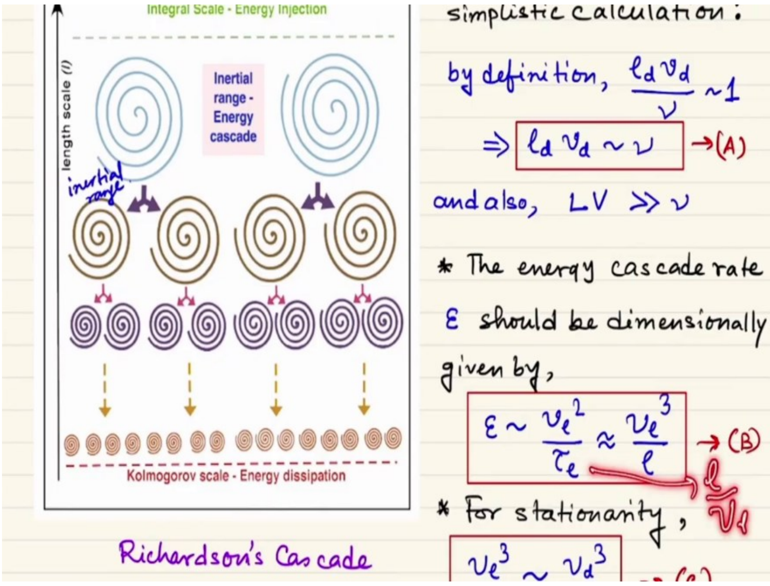

So, then we can simply replace this τ_l by $\frac{l}{v_l}$ and you have ϵ is of the order of $\frac{v_l^3}{l}$ ok. Now, we have to have stationarity; that means, that if you just write the Navier-Stokes equation,

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\nabla \boldsymbol{v} = -\nabla p + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{f}$$

and you write the energy equation; kinetic energy evolution equation from Navier-Stokes equation.

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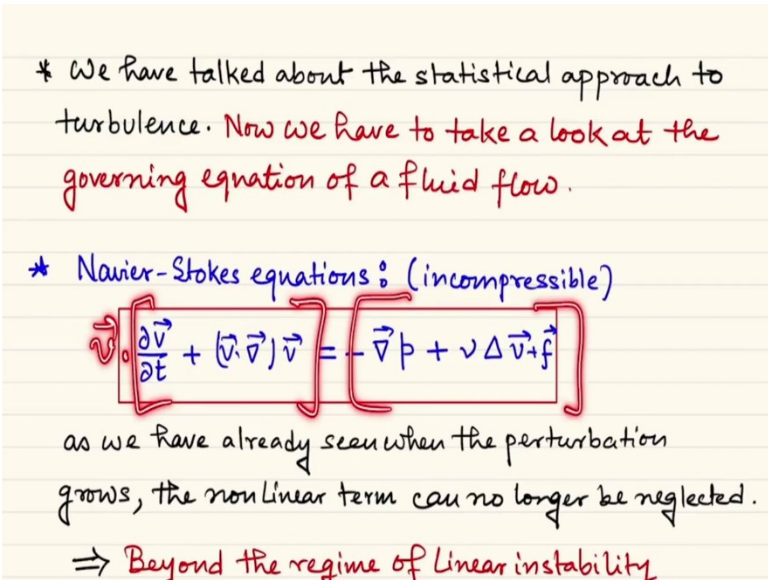

In case you are not clear at this point. Let us say we take the dot product of by velocity to both sides and then you simply do the evolution equation for kinetic energy,

$$\boldsymbol{v}.\left[\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}.\boldsymbol{\nabla}\boldsymbol{v}= \ -\boldsymbol{\nabla}p+\nu\nabla^2\boldsymbol{v}+\boldsymbol{f}\right]$$

you will see that when if you just assume a stationary state then this term $\frac{\partial\langle v^2\rangle}{\partial t}=\frac{1}{V}\frac{\partial}{\partial t}\int\frac{1}{2}v^2 d\tau$ will be zero.

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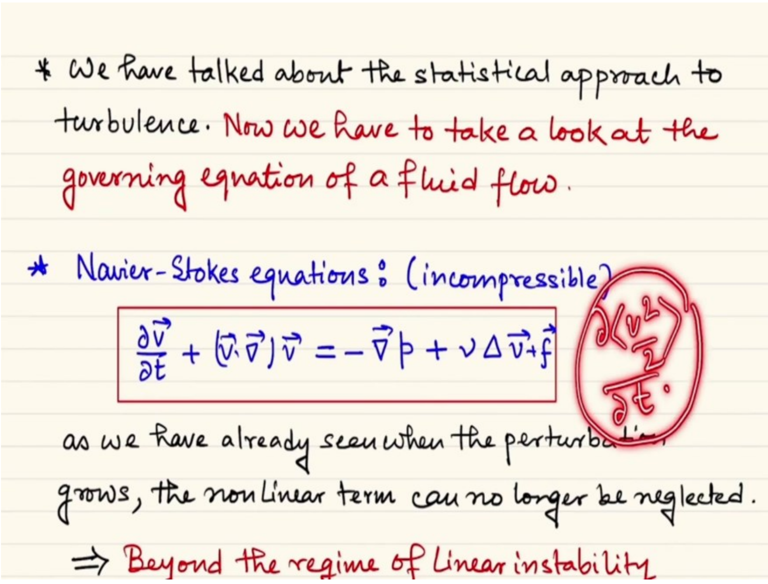

So, there will be actually a balance between non-linear part, dissipation dissipative part and the forcing part. This part $\boldsymbol{v}.\boldsymbol{\nabla}\,p$ actually does not play any role because if you just understand that $\boldsymbol{v}.\boldsymbol{\nabla}\,p=\boldsymbol{\nabla}.(\boldsymbol{v}p)$. Since for incompressible case $\boldsymbol{v}$ is divergence less. So, this will be simply $\boldsymbol{\nabla}.(\boldsymbol{v}p)$ okay. And this one. So, this $\boldsymbol{\nabla}.(\boldsymbol{v}p)$ one will be something of nature divergence.

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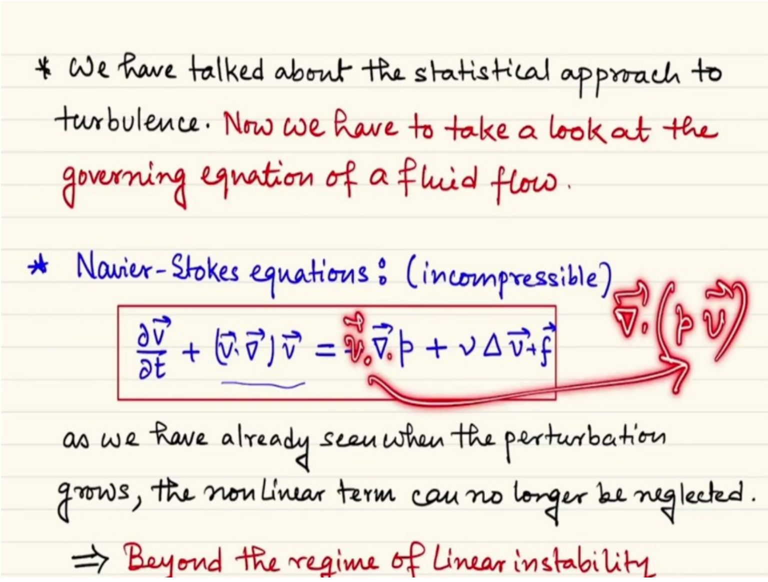

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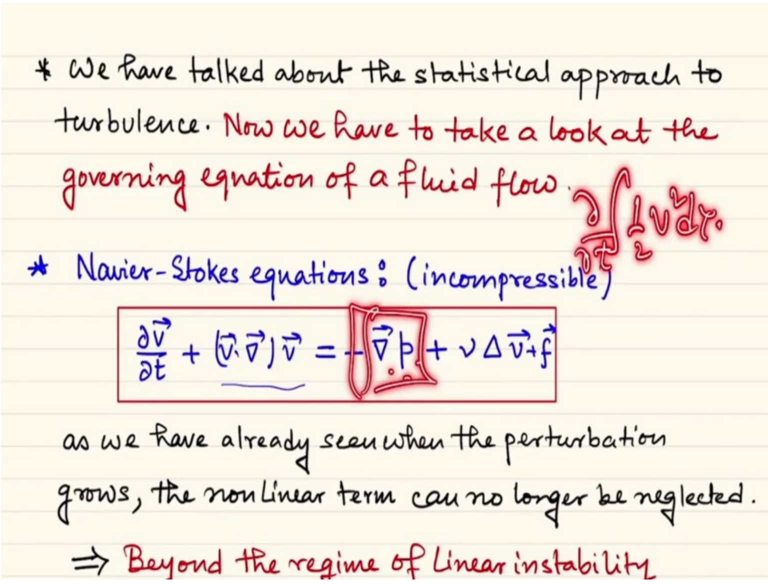

So, by Gauss's divergence theorem you can forget. So, there will be a balancing between this non-linear term and this dissipation term and this forcing term. So, at very large scale the forcing term remains i.e., the part which contributes, in the intermediate this part $\boldsymbol{v}.\nabla\,\boldsymbol{v}$ contributes and in the very small scale this one $\nu\,\nabla^2\,\boldsymbol{v}$ contributes okay.

So, now, the thing is that of course, the forcing is specifically given at the very large scale. So, once this is injected you have this. So, in the very in the intermediate scale this part $\boldsymbol{f}$

is balanced by this one $\boldsymbol{v}.\nabla\,\boldsymbol{v}$ and in the small scale simply this part $\boldsymbol{f}$ is balanced by this one $\nu\,\nabla^2\boldsymbol{v}$ okay.

So, when this part $\boldsymbol{f}$ is balanced by this part $\boldsymbol{v}.\nabla\,\boldsymbol{v}$, then we can simply say that the forcing i.e., the energy injection rate should be exactly equal to the energy flux rate by the non-linear cascade. Once again in the stationary regime of course, then you can say that the energy dissipated by this dissipation term $\nu\,\nabla^2\boldsymbol{v}$ will be exactly equal to the energy injected $\boldsymbol{f}$. So, all these three actually should have; if you think should have; the same flux rate.

So, the energy injection flux rate should be equal to the cascade energy flux rate and it should be equal to the energy dissipation rate okay or the flux or the dissipated a rate of the dissipated energy flux okay.

So, for the stationarity we first have this cascade rate; that means, the $\frac{v_l^3}{l}$ which is the ϵ, should be almost of the same to $\frac{v_d^3}{l_d}$ that is the energy dissipation rate, once again energy flux rate, I mean the density of energy per unit time. i.e.,

$$\epsilon \sim \frac{v_l^3}{l} \sim \frac{v_d^3}{l_d} \qquad (C)$$

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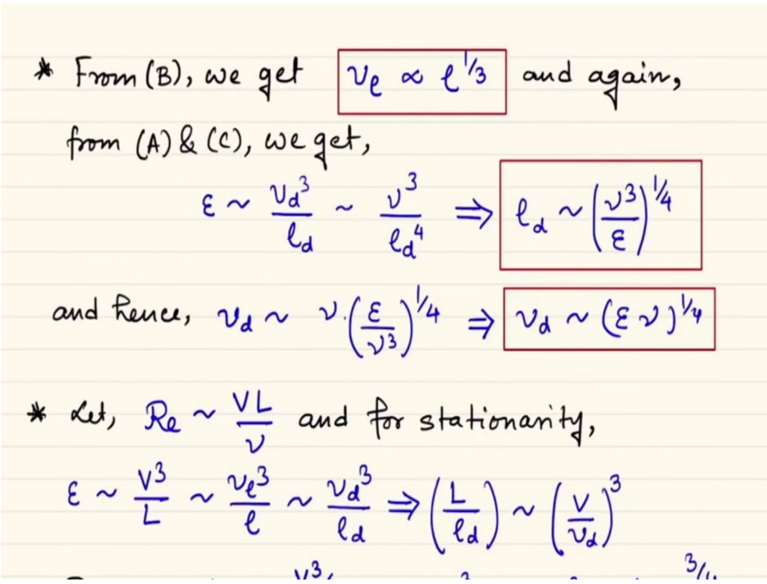

So, if you just see from (B), since ϵ is not depending on l, so, So, then you can say that v_l^3 cube will be of the same order of l. i.e., $v_l^3 \propto l$.

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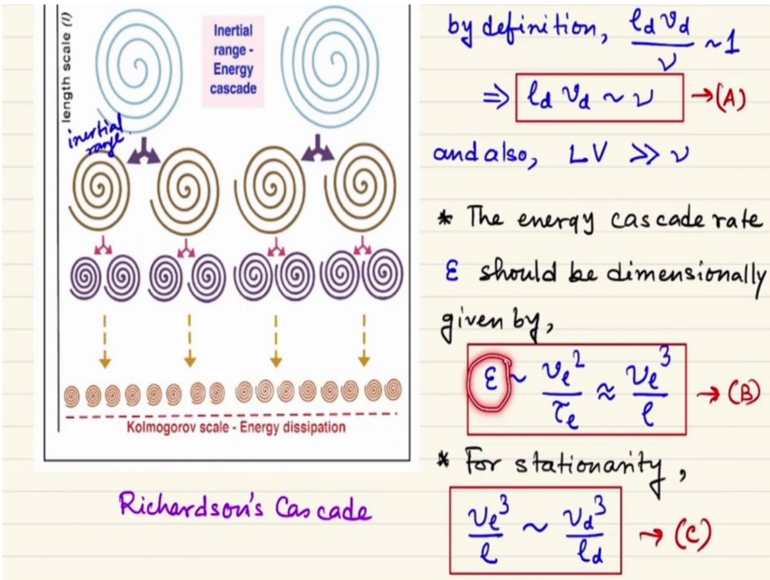

Again using (A) and (C),

$$\epsilon \sim \frac{v_d^3}{l_d} \sim \frac{\nu}{l_d^4} \Rightarrow l_d \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$

So, where you just get rid of v_d and you have ϵ will be ν^3 by l_d^4 and then you can simply write l_d will be of the order of $\left(\frac{\nu^3}{\epsilon}\right)^{1/4}$. So, that is a very important piece of result and again v_d which is of course, is also proportional to $l_d^{1/3}$. So, v_d can all actually be equal to $\nu\left(\frac{\epsilon}{\nu^3}\right)^{1/4}$.

So, above can be easily explain because from (A), you have $v_d \sim \frac{\nu}{l_d}$. So, $\frac{\nu}{l_d}$ gives you $\nu\left(\frac{\nu^3}{\epsilon}\right)^{1/4}$. So, just you can check that

$$v_d \sim (\epsilon\nu)^{\frac{1}{4}}$$

So, that is the estimation of the dissipative length scale l_d or the Kolmogorov scale and the corresponding velocity v_d in terms of the viscosity and the energy flux rate okay. Now which is also equal to the viscous rate.

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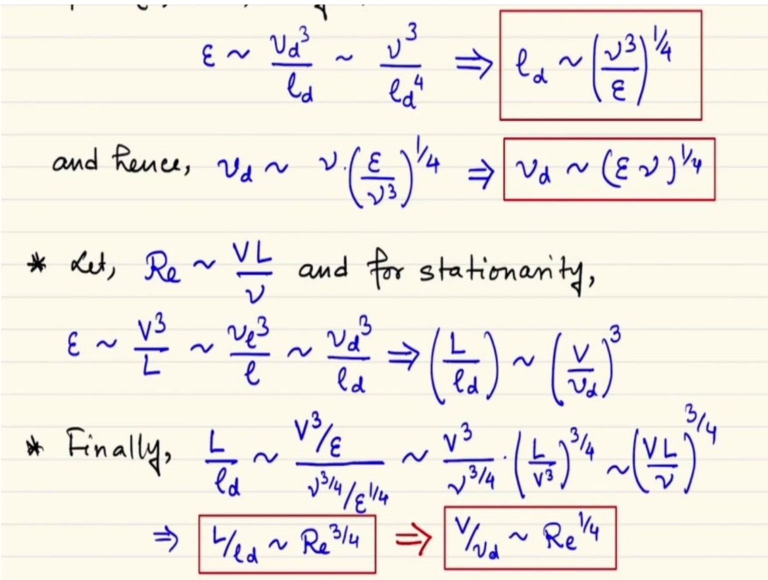

Now, if we just define the macroscopic Reynolds number to be $Re \sim \frac{VL}{\nu}$ and for stationarity then we again need the energy injection rate $\frac{V^3}{L}$ should be equal to the energy cascade rate $\frac{v_l^3}{l}$ and should be equal to the energy dissipation rate $\frac{v_d^3}{l_d}$, we simply can say that $\frac{L}{l_d}$ will be of the same order as $\left(\frac{V}{v_d}\right)^3$ that you can simply compare these two, $\frac{V^3}{L}$ *and* $\frac{v_d^3}{l_d}$, you can have this one ok. i.e.,

$$\epsilon \sim \frac{V^3}{L} \sim \frac{v_l^3}{l} \sim \frac{v_d^3}{l_d} \Rightarrow \left(\frac{L}{l_d}\right) \sim \left(\frac{V}{v_d}\right)^3$$

And finally,

$$\frac{L}{l_d} \sim \frac{V^3/\epsilon}{\nu^{\frac{3}{4}}/\epsilon^{1/4}} \sim \frac{V^3}{\nu^{\frac{3}{4}}}\left(\frac{L}{V^3}\right)^{\frac{3}{4}} \sim \left(\frac{VL}{\nu}\right)^{\frac{3}{4}}$$

$\frac{VL}{\nu}$ is nothing but the macroscopic Reynolds number Re. So, $\frac{L}{l_d}$ will be of the order of macroscopic Reynolds to the power $3/4$ and the corresponding ratio for the velocities; that means, the macroscopic to dissipative length scale velocity or rather forcing scale velocity to Kolmogorov scale velocity will be simply one third of this thing $\left(\frac{VL}{\nu}\right)^{\frac{3}{4}}$ which will be $Re^{\frac{1}{4}}$, So

$$\frac{L}{l_d} \sim Re^{\frac{3}{4}} \Rightarrow \left(\frac{V}{v_d}\right) \sim Re^{\frac{1}{4}}$$

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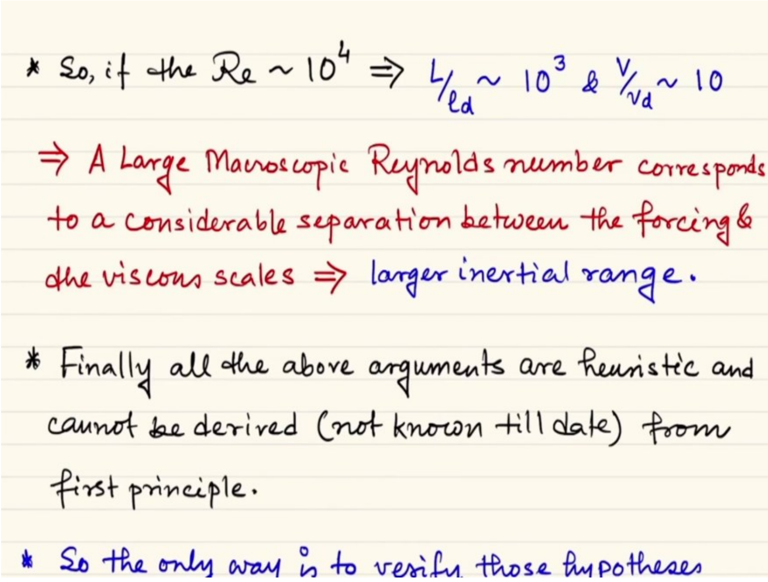

So, now just let us take one simple example, very typical example for laboratory experiment, we have seen that for water; for example, for tap water; the turbulent takes place at a macroscopic Reynolds number $(Re) \sim 10^4$; that means, that the macroscopic scale to the microscopic scale is of the order of 10^3. So,

$$\frac{L}{l_d} \sim 10^3 \ \& \ \left(\frac{V}{v_d}\right) \sim 10$$

So, we can say that the forcing length scale is 1000 times greater than the Kolmogorov length scale but the velocity of the forcing scale is just 10 times greater than the dissipative velocity scale okay.

Now, a large macroscopic Reynolds number therefore, corresponds to a considerable separation between the forcing and the viscous scales. So, if Reynolds number is larger then, $\frac{L}{l_d}$ will be larger and; that means, that the L to l_d distance will be larger. So, the separation between the forcing and the viscous scales will be larger and that creates a larger inertial range.

Finally, we have to remember that all the above arguments are heuristic and cannot be derived, not known till date at least from the first principle directly.

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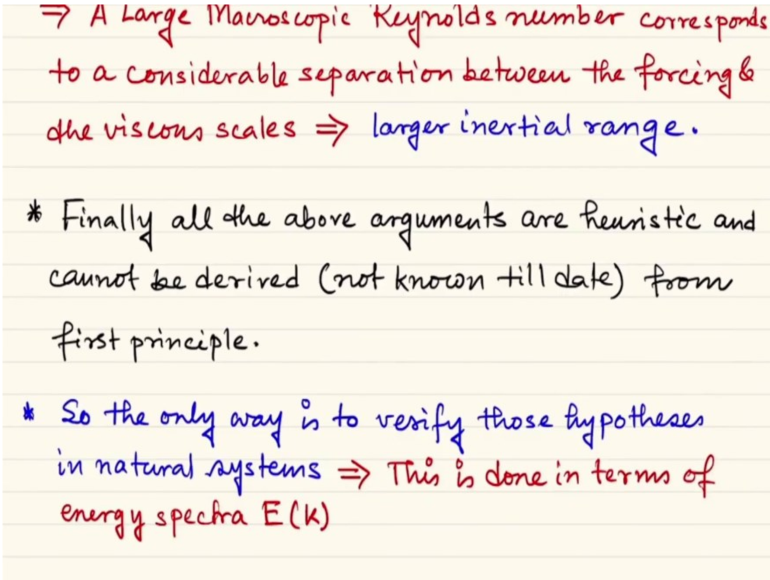

However, I mean how can we check that? So, we have made some hypothesis and depending on that hypothesis we have given at least some estimations quantitative estimations and we can simply verify those in natural systems and if that works then we are happy. That is actually systematically done in terms of the quantity which is energy spectra which is $E(k)$ as a function of k.

So, because I mean practically speaking that you can measure my macroscopic velocity microscopic length scale, but how can you measure mesoscopic, I mean the between the microscopic and macroscopic; the forcing and the dissipative scale velocities; and length scales? This is not very easy. So, therefore, what we will do in general, we calculate something which is much more traditional to calculate is the energy power spectra $E(k)$. what that is?

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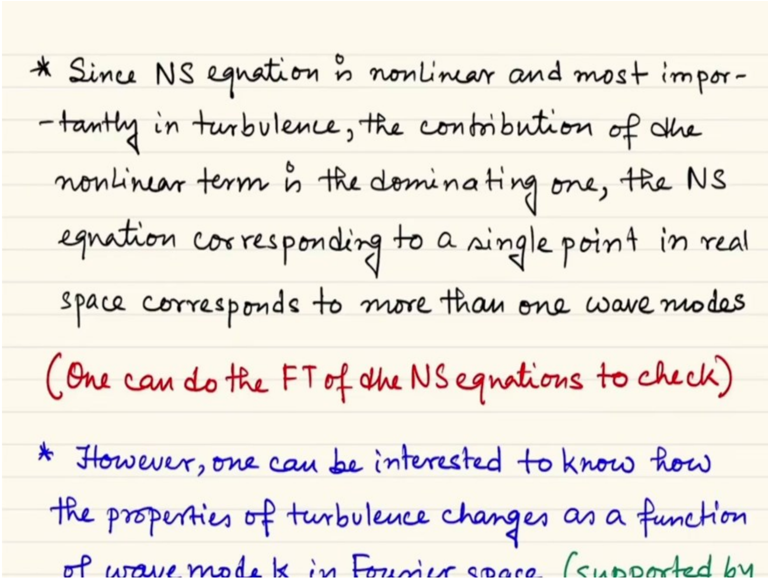

So, in order to understand that we have to go to this point that since Navier-Stokes equation is non-linear in nature and as the turbulent turbulence motion is non-linear in nature then we cannot neglect the non-linear term okay and if you do the Fourier transform in Navier-Stokes equation then simply you can understand that the non-linear terms in Navier-Stokes equation will involve more than one wave modes.

So, if the term is linear then if you just do the Fourier transform of the term then one quantity at some space point x will correspond to some other wave mode k in Fourier space, but if the quantity is non-linear; if the term is non-linear; then the any term corresponding to a single point in real space corresponds to more than one wave modes. Therefore, we talk about the triads. So, if you are interested you can see any standard text books of turbulence.

So, the non-linear term of Navier-Stokes equation involves three wave modes $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{k}$ such that $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{k}$. So, there will be a triadic relationship, I mean they should follow the rule of triangle addition. So, we see that one-point description in real space is not one point description in Fourier space.

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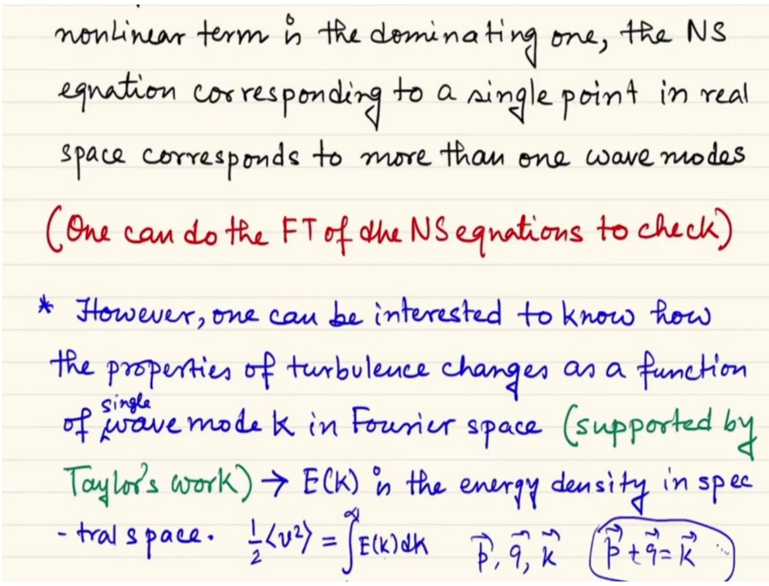

So, one can say okay, now we can actually be interested in the one-point description in the Fourier space right, one can be interested to know how the properties of turbulence changes as a function of wave mode $\boldsymbol{k}$ in a Fourier space; that means, a single wave mode okay. So, I have to write that a single of single wave mode $\boldsymbol{k}$ in Fourier space and that means, it cannot be given by a single point description in direct space. So, then we will actually have to think of two-point correlation function like this and this is exactly supported by Taylor's work of 1935 which says that we have to be interested in two-point correlation functions.

Because he also thought that in order to know the main mystery of turbulence, rather to be localized in direct space, we have to be localized in spectral space and therefore, we have to consider more than one point correlation functions because if you are now considering the Fourier transform of the correlation functions, that will give you something corresponding to only one single wave mode.

So, but what is this $E(k)$? So, $E(k)$ is the energy density in spectral space or you can say incompressible case when ρ is simply normalized to unity then,

$$\frac{1}{2}\langle v^2 \rangle = \int_0^\infty E(k)dk$$

So, I am just talking about homogenous and isotropic turbulence that is why we are simply taking k as scalar. So, this is the definition of $E(k)$ okay.

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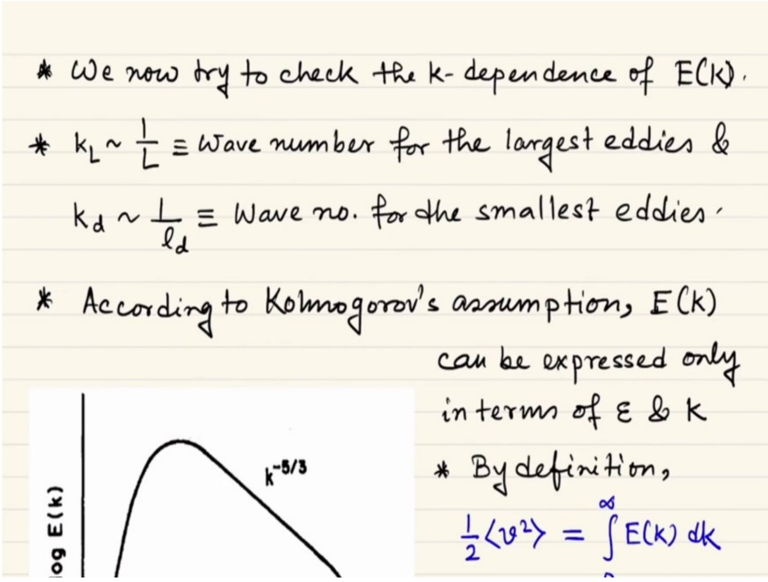

Now we try to check the k dependence of $E(k)$. So, before that, let me just designate $k_L \sim \frac{1}{L}$, which is the wave number corresponding to the largest eddies; that means, that is the smallest possible wave number in our system and $k_d \sim \frac{1}{l_d}$, that is the wave number for the smallest eddies, that is the largest possible wave number for our system. So, according to Kolmogorov's assumption $E(k)$ can be expressed only in terms of ϵ and k.

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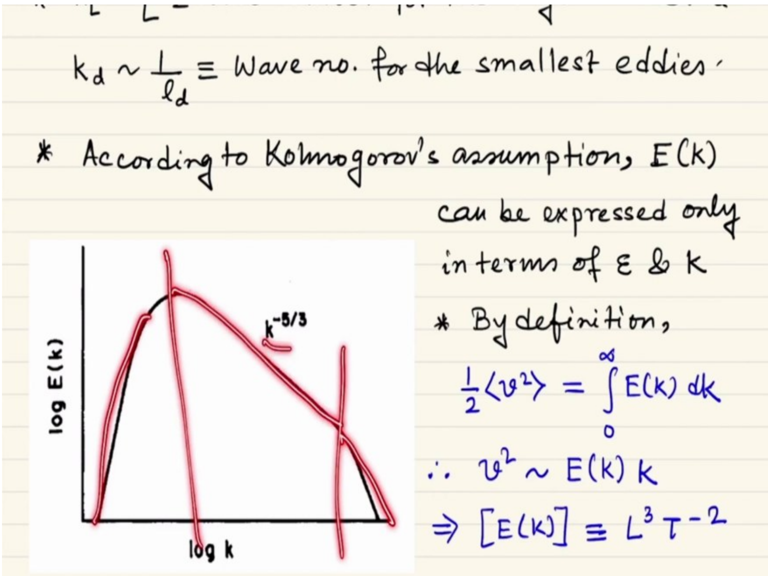

So, that actually comes also from his universality hypothesis, but here you just learn this as information. So, the definition, we have $\frac{1}{2}\langle v^2\rangle = \int_0^\infty E(k)dk$. So $v^2 \sim E(k)k$. So, $E(k)$ should have a dimension of L^3T^{-2}.i.e.,

$$\frac{1}{2}\langle v^2\rangle = \int_0^\infty E(k)dk \Rightarrow v^2 \sim E(k)k \Rightarrow [E(k)] = L^3T^{-2}$$

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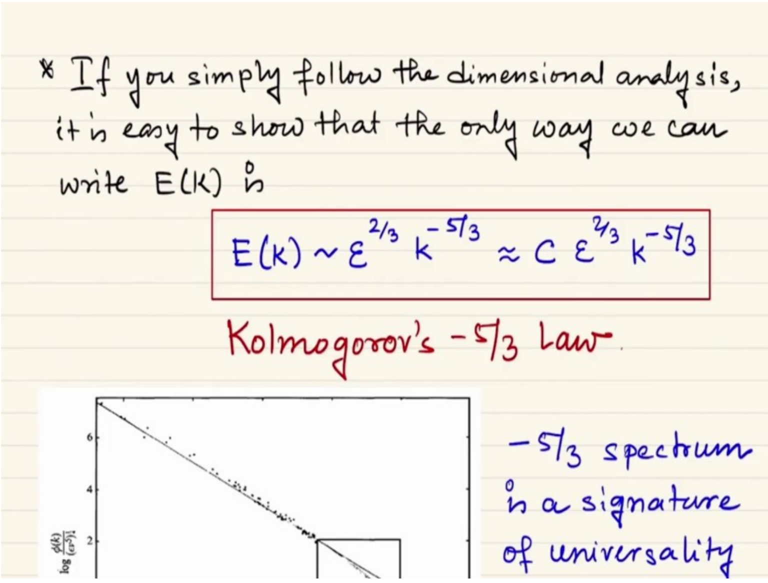

If you simply follow normal dimensional analysis and you say that $E(k)$ is only a function of ϵ and k, then just by doing dimensional analysis you will say that the only possibility is that $E(k)$ should be scale as $\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$. ϵ is something which is independent of scale and so, it is also independent of k and then finally, Kolmogorov's also said that there should be a constant of proportionality which nearly makes them equal and this is known as the constant of Kolmogorov.

$$E(k) \sim \epsilon^{\frac{2}{3}}k^{-\frac{5}{3}} = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$$

Now, this constant of Kolmogorov can be different from different turbulent system. So, here you see that the $E(k)$ basically varies as $k^{-\frac{5}{3}}$ for a given turbulent system and this is nothing, but the Kolmogorov's $-\frac{5}{3}\,rd$ law. So, what you have to do? So, if you now understand what $E(k)$ is, $E(k)$ is nothing, but the Fourier transform of the two-point correlation functions.

So, you have to calculate the two-point correlation functions for a system and you have to take the Fourier transform then we will get $E(k)$ and then you just plot $logE(k)$ versus $log\ (k)$ and you will find the slope of the graph; you will see that this is the injection, this is the dissipation between these two this is the cascade part where you have a power log which varies as $k^{-5/3}$.

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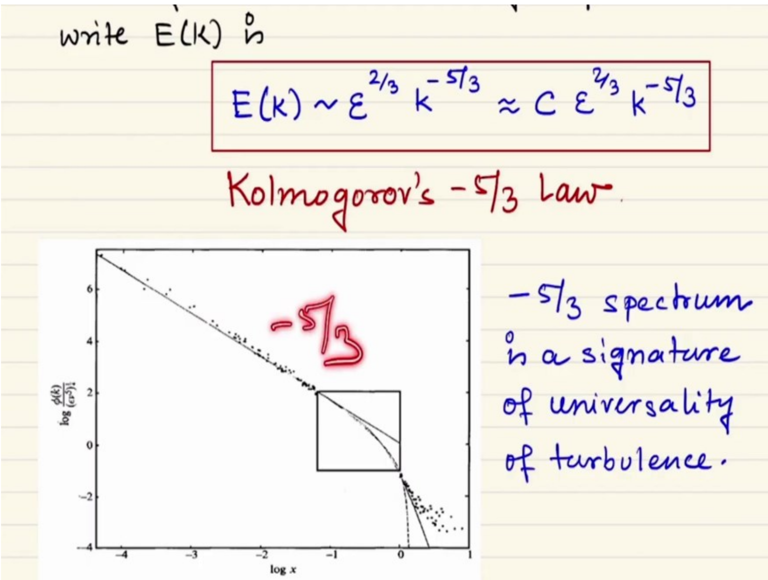

Now, you see that this relation is so universally true that it can be actually valid for most of the system. I have taken this picture from one of the books and you can see that this is consisting of different systems, one is sea water; one is like the normal tap water; one is the atmosphere; so, atmosphere can be considered to be incompressible. Actually, that is why Richardson's original image was so good that it worked for atmosphere okay.

So, here you see that although their dissipation the point where they start dissipating are different, but they all of them have some common range with this type of power law and actually the fabulous thing is that when you calculate the slope, you will get $-\frac{5}{3}$. This $-\frac{5}{3}$ spectrum is a signature of universality of turbulence.

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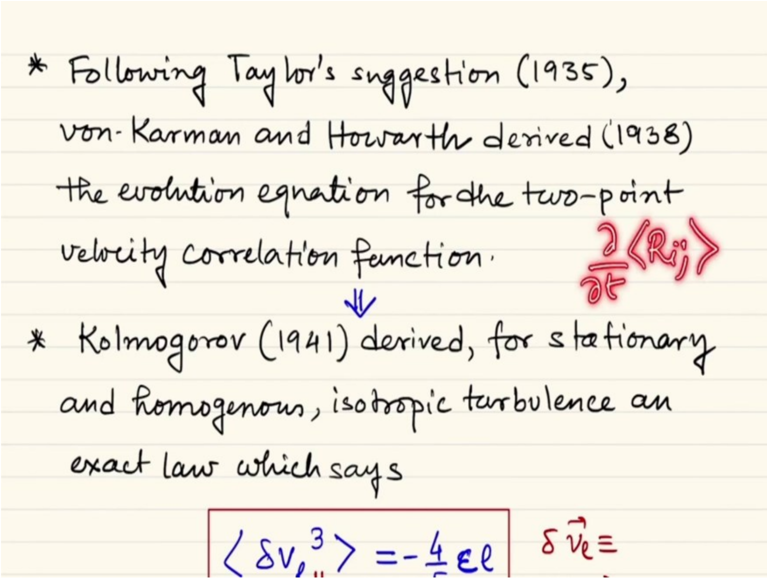

So, now in a different system for example, if the fluid is a plasma or something then this type of universality is also existing or not that is exactly what scientists always try to find out okay. So, we have seen that we are finally, using some very crude phenomenological considerations, we have given some at least a functional dependence of $E(k)$ with the wave number, but now coming back to the called the analytics, can we relate at least this?

So, up to this point I said that we cannot do anything. So, for example, when I mean up to this point, we said that all these above arguments are heuristic and it is not possible to derive them from first principle, but at least when using those things finally, we arrived at this result which is quite fantastic.

Now, the question is can we relate this to something much more analytically sound? So, for that we have to see the development of analytical approaches after that. So, as I said that after Taylor's suggestion in 19353 von Karman and Howarth, they derived in 1938 the evolution equation for the two-point velocity correlation function.

That was also an exact relation, but it was not simply, I mean a very good constraint just using two-point correlation function or something like that to the energy correlation, there was always a $\frac{\partial}{\partial t}$ of the correlators R_{ij} okay.

Now, Kolmogorov derived a law for stationary and homogeneous and isotropic turbulence; stationary means this $\frac{\partial}{\partial t}$ term was 0; the third order structure functions of the longitudinal velocities,

$$\langle \delta \boldsymbol{v}_{l||}^{3} \rangle = -\frac{4}{5}\,\epsilon l$$

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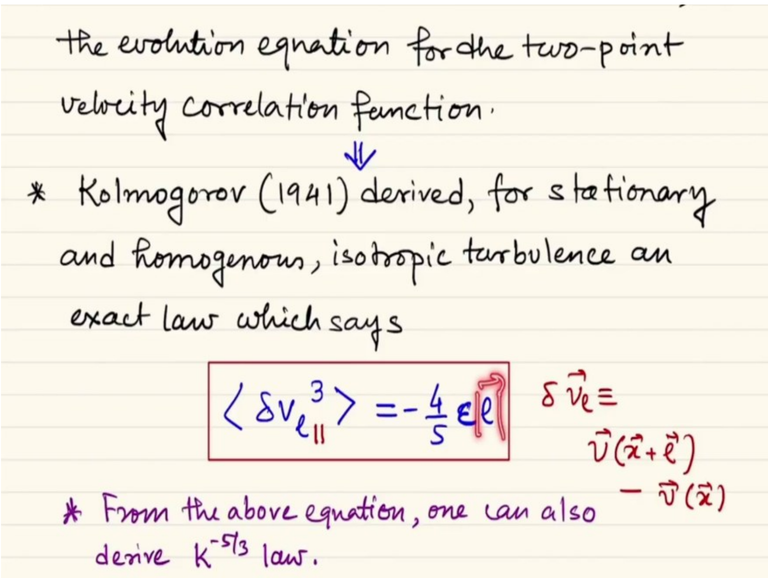

What is the third order structure function? If you take the increments of two points, that means, the velocity increments $\delta \boldsymbol{v}_l$ that is nothing but of course

$$\delta \boldsymbol{v}_l \equiv \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{l},t) - \boldsymbol{v}(\boldsymbol{x},t)$$

So, if you take this $\delta \boldsymbol{v}_l$ and then you take the its parallel component; that means, if you take some length scale or some distance then you take the projection of this vector $\delta \boldsymbol{v}_l$ along this direction then you have $\delta \boldsymbol{v}_{l||}$ parallel and then you take the cube of this.

Once you take this $\delta \boldsymbol{v}_{l||}^{3}$, you can calculate this for all the couple of points which are separated from each other by a length l and then you take the average. Here you are simply doing the space average for practical purpose although it should be ensemble average in general and then you just plot this with the modulus of $\boldsymbol{l}$ vector, which is l and you will see that the slope will be $-\frac{4}{5}\epsilon$. So, here from here you are actually also calculating ϵ and if you simply see that this one $\delta \boldsymbol{v}_{l||}^{3}$, now this one what is this?

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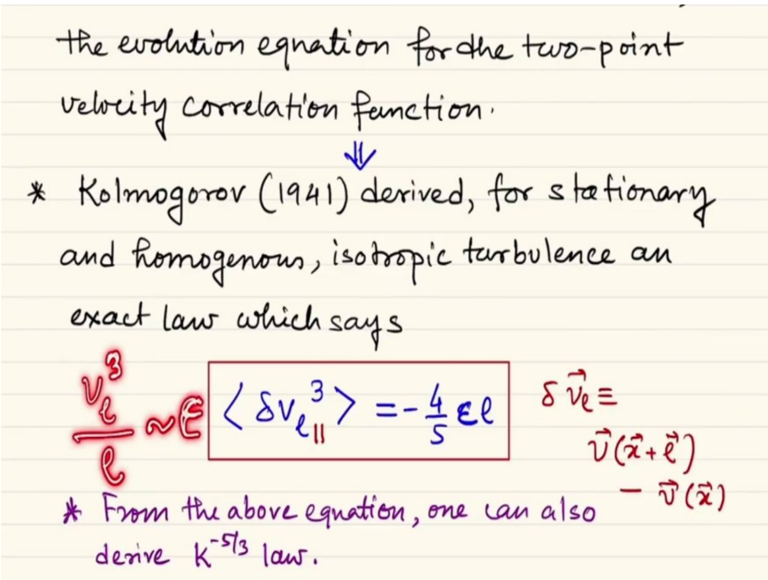

If you say that this $\delta v_{l||}^3 / l$ is nothing but roughly v_l^3, the turbulent velocity cube divided by l ok then that is proportional to the order of epsilon i.e., $\frac{v_l^3}{l} \sim \epsilon$. So, that is exactly what we found earlier.

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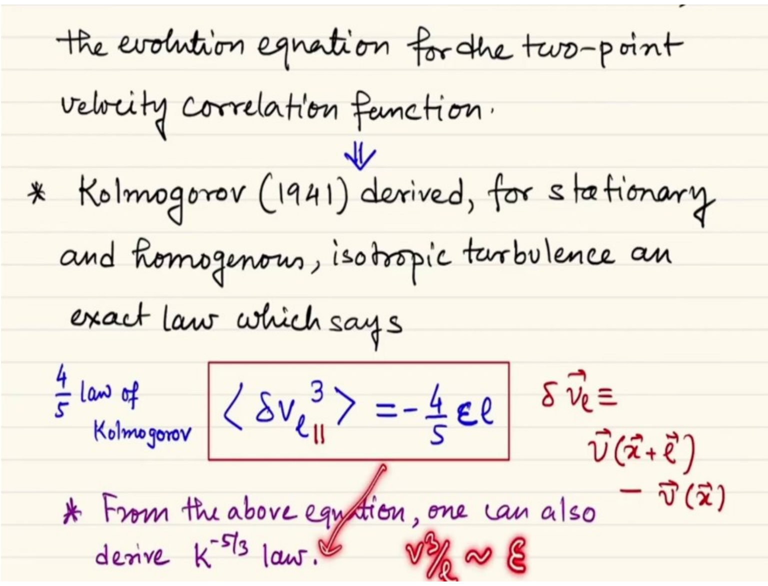

So, here we can see that this equation $\langle \delta v_{l||}^3 \rangle = -\frac{4}{5}\epsilon l$, which is known as the fourth fifth law of Kolmogorov, the previous one was the minus five-third law and this is called the four-fifth law of Kolmogorov or the exact relation of turbulence. We saw that this equation also gives the indication that we can derive $k^{-\frac{5}{3}}$ law from this because once you have this $\frac{v_l^3}{l}$ is something scale independent then actually you can simply write, I can do that.

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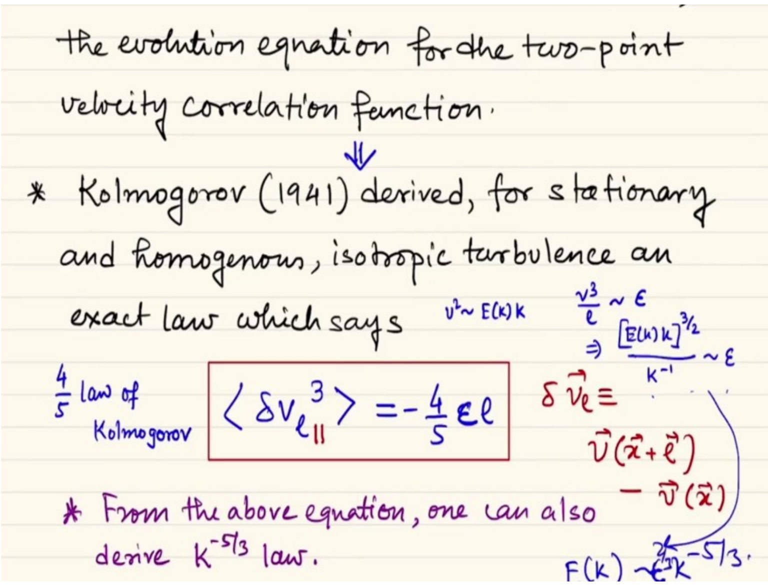

If you have $\frac{v_l^3}{l} \sim \epsilon$, which is scale independent then you that $v^2 \sim E(k)k$. So, you can simply say that $\frac{(E(k)k)^3}{k^{-1}} \sim \epsilon$. Now I leave it to you to simplify and you will see that you will retrieve $E(k) \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$.

So, you see that this result $k^{-\frac{5}{3}}$ result can actually also be obtained from exact relation as well okay. So, in the next lecture, I will discuss the Reynolds decomposition and then I will shortly introduce turbulent viscosity, because that is something very interesting for accretion disks and then I will talk very qualitatively a little bit of plasma turbulence of course, and finally, I will sum up by saying something qualitative for the turbulence in astrophysical contexts okay. But of course, not in this week that will be the program for the first lecture of next week okay.

Thank you very much.