Introduction to Astrophysical Fluids
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Lecture - 51
Introduction to turbulence in fluids

Hello and welcome to another session of Introduction to Astrophysical Fluids. In this lecture, we will discuss a new topic which is turbulence. Previously, we have discussed several properties of neutral fluids and plasma. In plasma, we have discussed different properties: the definitions of different types of plasma, how to describe a plasma from kinetic theory.

Very briefly of course we have discussed how from kinetic theory we can have a transition to fluid theory or continuum. Then, we have seen that there are multi-fluid pictures as well as the mono-fluid pictures and this depends on which length scale and which time scale, I am interested in.

Now after that, we have also discussed so the mono-fluid model as you know is the magneto hydrodynamics and then we of course, discussed the various properties of magneto hydrodynamics and then, we have discussed a little bit about their application or importance in the framework of space and astrophysics. Now, in this lecture it will discuss a different topic: the turbulence.

So, mostly, I will discuss in this session and also in the next lecture, the general properties of turbulence for normal incompressible fluids and only then, I will just discuss very briefly about the turbulence in plasmas and also their importance in astrophysical context or I mean in space physics context. For example, turbulence in solar wind is a very hot topic of research.

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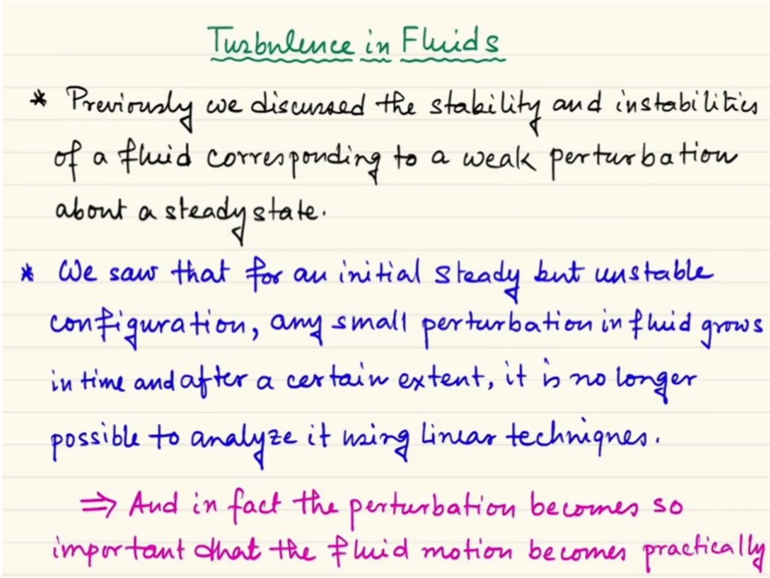

So, as you can easily understand here that previously when we discussed the stability and the instabilities of the fluid, then we said that we start from some arbitrary steady state and then, depending on whether the steady state is stable or unstable, the it responds to an external weak perturbation.

So, if the steady state is stable in nature, then it responds in terms of the linear wave mode so the system tries to get back its original configuration or if the original steady state is unstable, then basically the small perturbation which is applied actually grows in time and after a certain extent, it is no longer a small perturbation and you can we cannot any longer analyze this using linear techniques.

The we are talked about general instabilities. The onset is of course, the linear instability; that means, this system is treated under linearization but of course, when the perturbations are growing large and larger, then we have to take either non-linear techniques into account or we just have to say that okay, we cannot do analytical treatment and then, we can do some numerical treatment for those type of problems.

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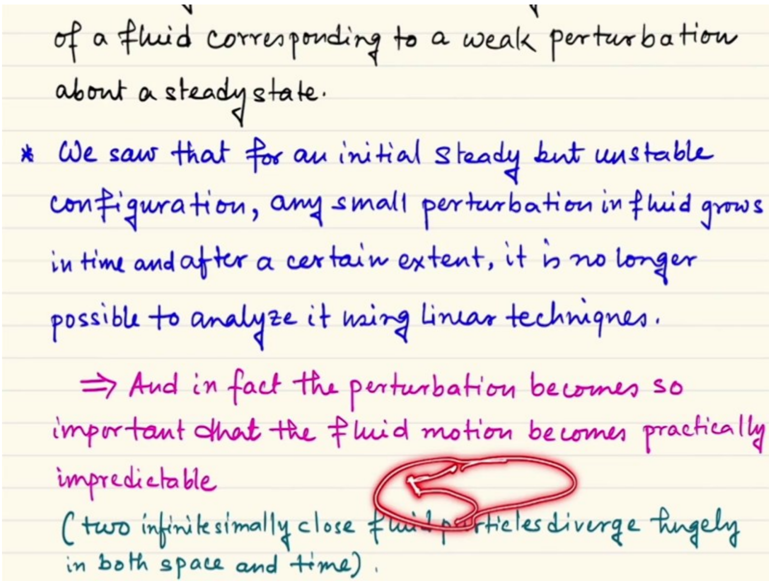

Now, it is true that in such cases since the perturbation becomes very large or important, then actually the fluid motion becomes practically unpredictable. Now, what is the meaning of that? That means, let us say you have two infinite similarly closed fluid particles and now with time, they are now diverge hugely both in space and time okay.

So, that is something very important, the irregularity is observed in both space and time.

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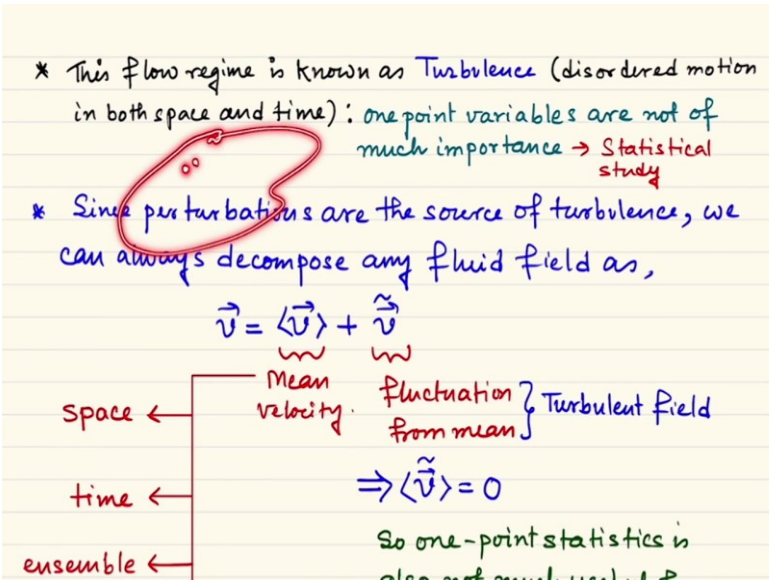

So, such a flow regime, where the system is undergoing or the flow is undergoing some very non equilibrium state of motion with irregularities in both space and time; we sometimes say disordered motion in space and time; is called turbulence ok.

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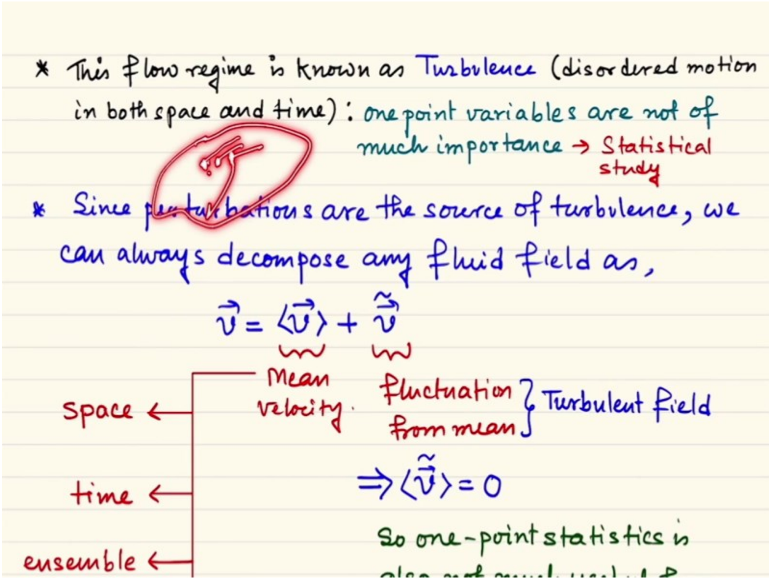

And of course, you can easily understand that if for example, you have a flow field like that and the velocity at this point is $\boldsymbol{v}$ and then, the velocity at this point, I mean tracking is not very easy. For example, the fluid is really moving totally erratically; just by knowing the velocity at this point and this point, you cannot really predict the nature of the flow in some subsequent instant because after a small-time interval this particle let us say here, this particle let us say here, this particle let us say here; so, three particles are totally in different places (Refer Slide Time: 06.21). When I am talking about particle, I am just simply talking about fluid particles in Lagrangian sense. So, in a in practical sense, you cannot picturize the fluid motion.

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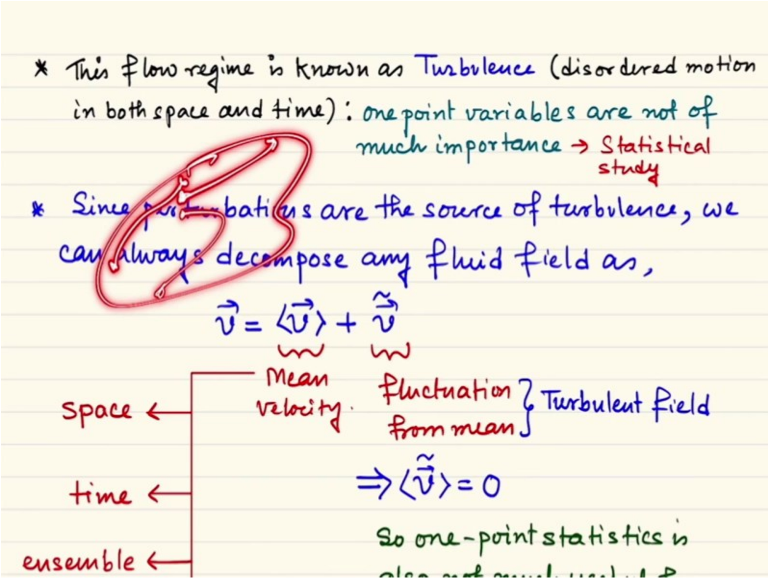

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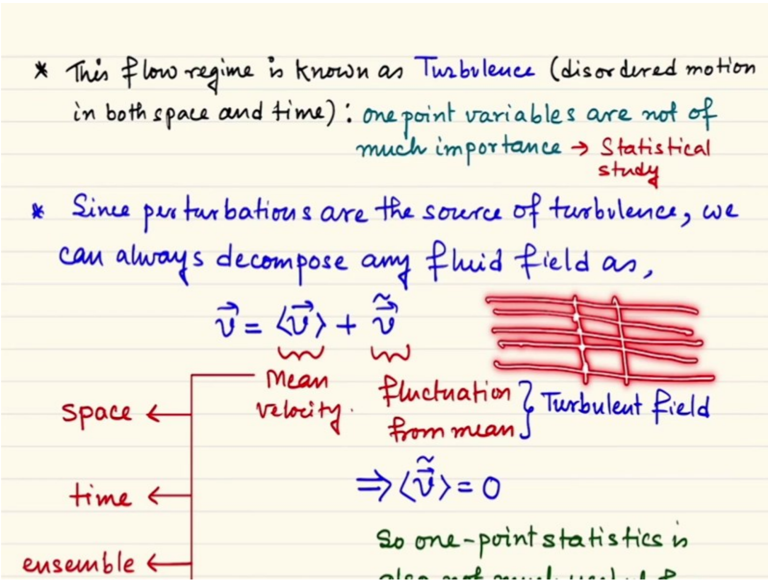

For example, if you have a laminar type of flow, then you know that layers are like this. So, just knowing this type of profile, the velocity profile here, you can have a strong idea or clear idea of the velocity profile here right. But here in turbulence, this is not possible.

So, that is why it is not very much practically important to know the one-point variables at any point of time. Of course, you can know that as an information; but to know the global picture, whether globally the system gives some vortex like structure or the system is moving in some type of two-dimensional sheet type of thing; is impossible to extract from such type of information. So, what happens that the total system is a chaotic motion; of course, the formal chaos is a different thing, but I mean just roughly speaking the turbulence motion is nothing but a chaotic motion, a totally disordered motion; so, we need the statistical study.

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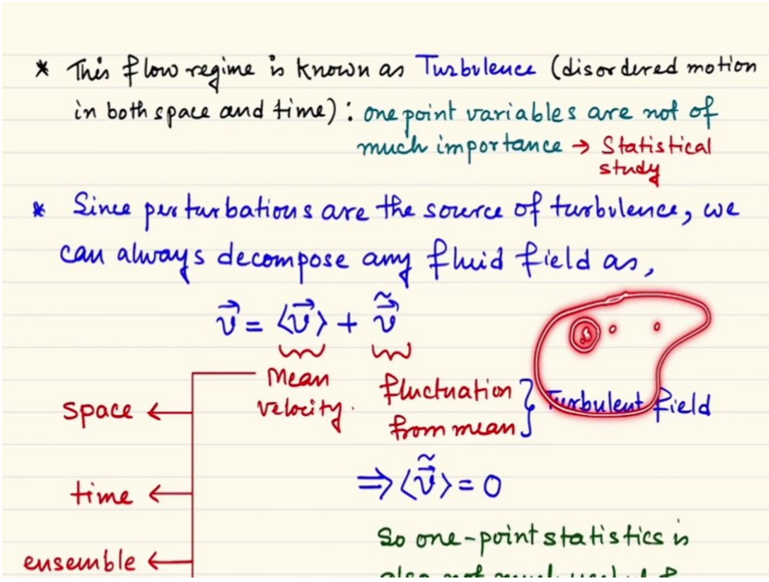

So, what we can do instead of just determining the velocity or the pressure at one-point or another point, is we can take the average of this such type of points and if you prepare an ensemble with large number of members, that means, you prepare a large number of imaginary systems ok, with identical fluid flow and then, you just specify one time point and one space point and you take that point in space and time from each member of the ensemble and you do the ensemble average. If you can do that, then this average can have some meaning ok. Why?

Because any anything which is randomly moving, we know that in statistics, this is much easier to capture the behavior because if we know its distribution actually should follow something very well-known thing either gaussian or I mean well, it is a perfectly random thing, then it should be a Gaussian type of distribution or it can be something modified Gaussian or something nearly more complicated like that; but at least we have some behavior of the distribution. That is what I am trying to say.

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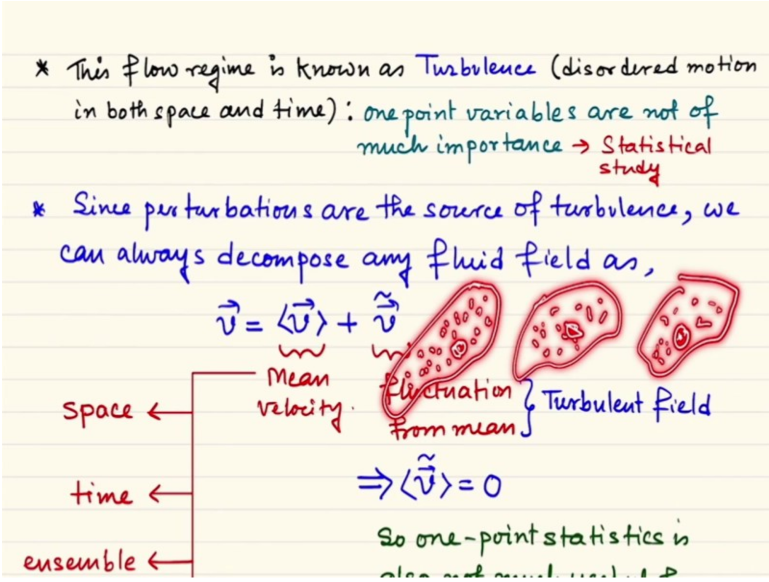

That is exactly the same type of picture what we have when we are talking about kinetic theory for example right. We have a large number of gas molecules which are moving here and there and so, we are not interested in the behavior of a single particle at one-point because this is a total mess. Then, our interest is to know that in phase space for a given $\boldsymbol{x}$ and $\boldsymbol{p}$, how many particles should be there ok. So, just analogical to that philosophy, here we are simply saying that here we are not interested in this just because once again, the movement the fluid particle is totally a mess ok.

It is totally unpredictable and that is why we are not interested in the individual values of the velocity, pressure like this at every space and time; but we are more interested in that average values and to know that, if they can give us some meaningful information ok.

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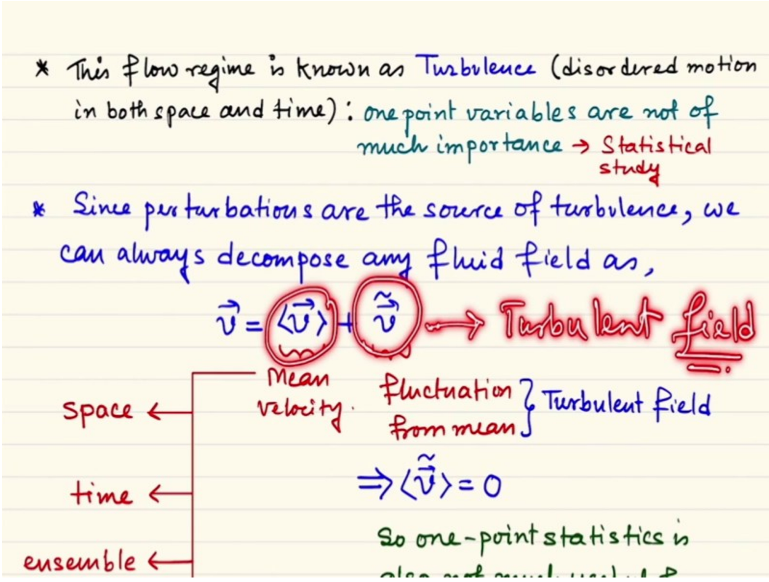

Since perturbations are the source of turbulence, then we can actually think of decomposing every fluid field ($\boldsymbol{v}$) as a mean field ($\langle\boldsymbol{v}\rangle$) plus the perturbation part or the fluctuation fluctuating part ($\tilde{\boldsymbol{v}}$) and sometimes, we call this as the turbulent velocity or turbulent field simply. I mean turbulent field is a general thing for the fluctuation part. So,

$$\boldsymbol{v} = \langle\boldsymbol{v}\rangle + \tilde{\boldsymbol{v}} \qquad \textbf{(1)}$$

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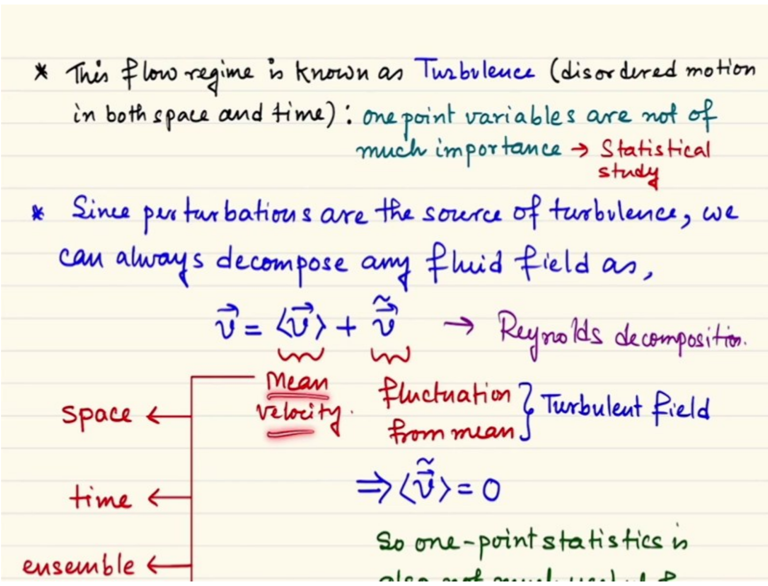

This type of decomposition was done by Reynolds and that is why this is known as Reynold's decomposition okay. So, what you have to do that you have to decompose them into two parts; one is the mean velocity that is the ensemble average $\langle \boldsymbol{v} \rangle$ once again and this is the fluctuation $\tilde{\boldsymbol{v}}$; that means the instantaneous value of velocity at every point and space and time minus the ensemble average value ok.

So, we just call this ensemble average value $\langle \boldsymbol{v} \rangle$ as the mean velocity. Now, here we are taking ensemble average that is true but I mean well theoretically, it is much more proper to talk in terms of ensemble average and it is much easier to take space average or time average.

So, ensemble can be made in numerical simulations; but in practice, you have only one system right. So, you cannot; for example, if I tell you to study the behavior of the of turbulence in a river, then, you need to create thousand rivers at the same time and this is not possible right; So, this is just something which we talk for theoretical correctness.

But in general, what we do that we assume that our system is either homogeneous enough so that the ensemble average can be replaced by spatial averages or our system is stationary so that the ensemble averages can be replaced by time averages. Now, if our system is ergodic; what ergodic is? It is a tricky definition; but at least here, just you have to simply know that this is a rough definition that if the phase space density of the ensemble points follows the Liouville's theorem or something like Liouville's theorem; let me just not going into this pinturas box; then we can say that the system is ergodic and all these three averages are practically equivalent.

So, although, we are talking in terms of ensemble average, you remember that for a practical purpose; for example, if you even if you do a new numerical simulation what would you do to take averages? Would you really start every time run the simulation 1000 times? You do not do that. If you take 1000 snapshots and take the time average or you take just for a box at a time instant, the space average, that is practically what you would do okay. So, most of the time it works. It works means it gives us something which we can expect from our intuition or from the first principle or from holistic logics okay. Now, that simply says that the turbulent systems, although they are very disordered randomly moving, ergodicity can be very well applicable for those systems ok.

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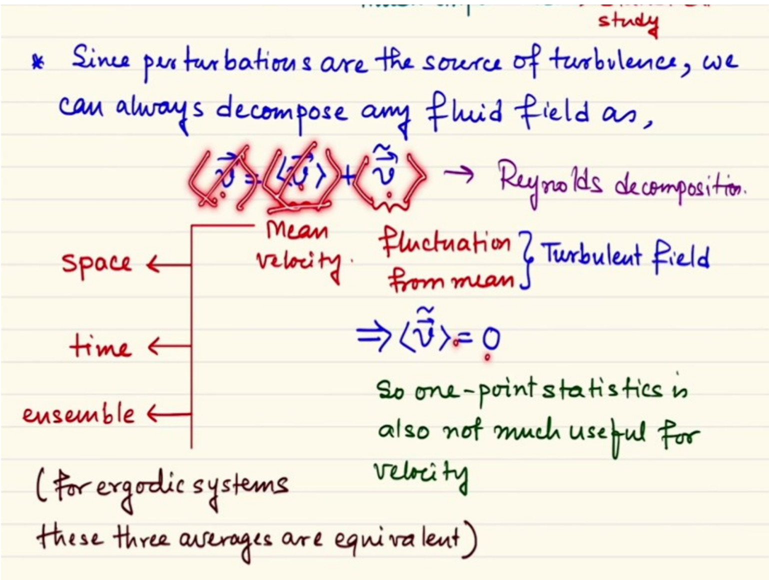

Now, here you can see that; so, these are all about these three types of averages and once again for ergodic system, these three averages are equivalent. So, for our case, what we will do is simply Reynolds decompose and we will according to our convenience from time to time, we will take space average or time average ok, which will replace the ensemble average.

So, then the fluctuation part. Now, the fluctuation part actually by definition is of mean zero and that is, the definition of fluctuation. If you take the average in both sides of equation (1).

$$\langle \boldsymbol{v} \rangle = \langle \langle \boldsymbol{v} \rangle \rangle + \langle \widetilde{\boldsymbol{v}} \rangle = \langle \boldsymbol{v} \rangle + \langle \widetilde{\boldsymbol{v}} \rangle \Rightarrow \langle \widetilde{\boldsymbol{v}} \rangle = 0$$

So, the average of the average is nothing but itself. you see that the average of $\widetilde{\boldsymbol{v}}$ is zero.

So, the one-point statistics is also not much useful for the velocity because this is simply zero. Now not in this lecture, but in the later, I mean after 1 or 2 lectures, we will properly address this Reynolds decomposition method and thereby, introducing some important things of turbulence okay.

But here just for your information, I would like to mention that just by doing Reynold's decomposition, the take-home message is that one-point statistics for this tilde is not just interesting okay.

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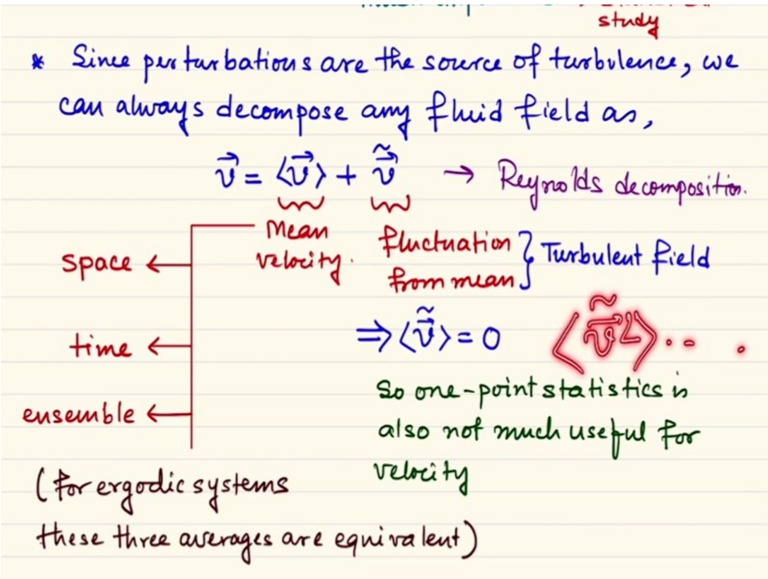

Now, you can of course, do $\langle\tilde{v}^2\rangle$ and that can give you something okay. Now, at this point, of course, this is possible. So, one-point statistics of first order is not possible. So, there is another possibility.

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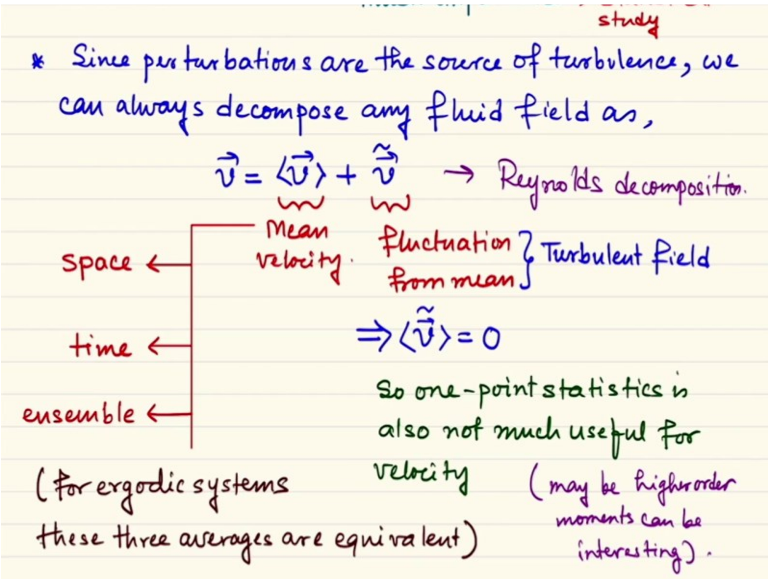

So, maybe higher order moments can be interesting ok.

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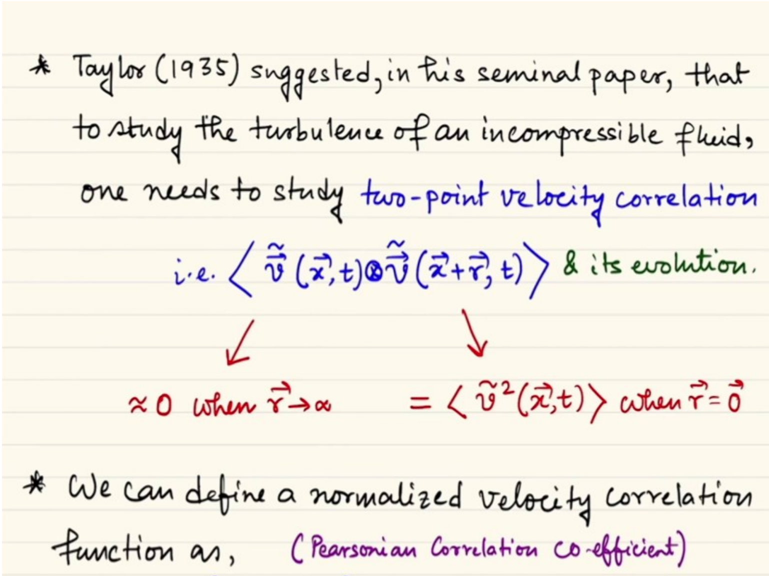

The total thoughts about this turbulence were changed after Taylor, who suggested in a seminal paper in 1935 that actually for studying the turbulence in incompressible fluid instead of one-point velocity functions one needs to study two-point velocity correlation function $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) \otimes \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle$ and its evolution.

So, that means, the fluctuating velocity at one-point for a given time instant at one-point in space contracted with another point in space and its evolution. So, that is the correct thing to study. How actually Taylor thought that, I will come later into that ok.

So, this type of quantity $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) \otimes \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle$ has to satisfy two properties; one is that it should go to 0, when $\boldsymbol{r}$ is very very large; that means, two very distance or distant points is really impossible to be correlated and another thing is when $\boldsymbol{r}$ is 0, then this is simply the exactly what I just said that the higher order moment for $\tilde{\boldsymbol{v}}$ i.e., $\langle \tilde{\boldsymbol{v}}^2(\boldsymbol{x}, t)\rangle$ when $\boldsymbol{r} \rightarrow 0$. .

We just follow Taylor and we will be interested into the study of two-point correlation functions and the one-point quantity will be simply coming as a special case of this ok, when your $\boldsymbol{r}$ is zero.

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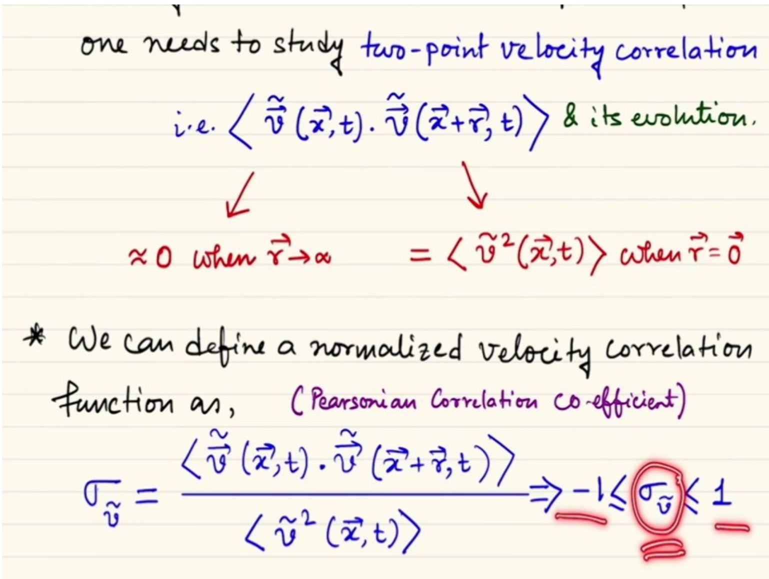

And actually, then we use this quantity $\langle \tilde{\boldsymbol{v}}^2(\boldsymbol{x},t)\rangle$ in order to normalize the velocity correlation function

$$\sigma_{\tilde{v}} = \frac{\langle \tilde{\boldsymbol{v}}(\boldsymbol{x},t).\tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r},t)\rangle}{\langle \tilde{\boldsymbol{v}}^2(\boldsymbol{x},t)\rangle} \Rightarrow -1 \leq \sigma_{\tilde{v}} \leq 1$$

which is the traditional Pearsonian correlation coefficient. People who are used to statistical methods, you should know this Pearsonian correlation coefficient. So, this is simply that this is denoted by this sign $\sigma_{\tilde{v}}$ which is equal to the ensemble average of $\tilde{\boldsymbol{v}}$ at $(\boldsymbol{x}.t)$ contracted with the $\tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r},t)$ divided by the $\tilde{\boldsymbol{v}}^2(\boldsymbol{x},t)$.

And if you do that correctly, you will see that $\sigma_{\tilde{v}}$ actually, one can show that always should be within -1 to 1 okay.

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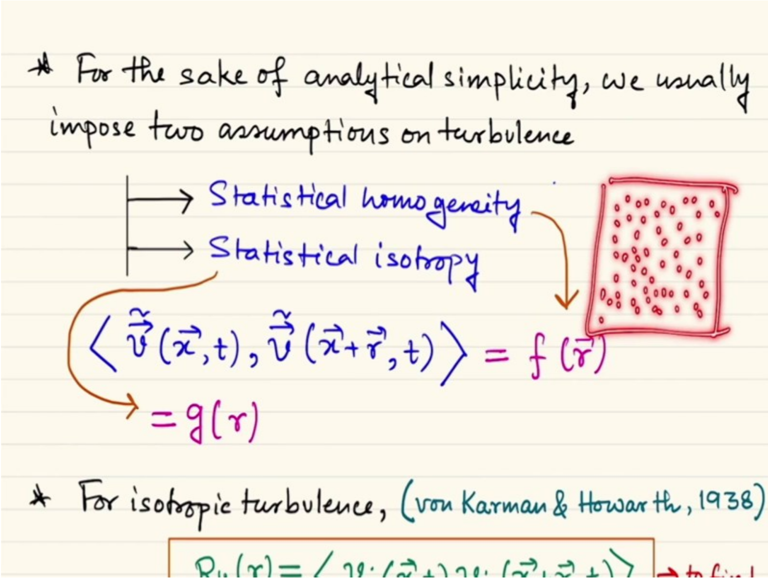

Now, up to this point, we have not said any assumption on the nature of the turbulence; that means; statistical homogeneity or isotropy anything, stationarity nothing. But at this point, for the sake of analytical simplicity, we have to introduce two assumptions of turbulence; one is statistical homogeneity, another is statistical isotropy and what is the advantage of that?

If the system is statistically homogenous, so statistically homogeneous does not say the system is absolutely homogeneous. When a system is absolute homogeneous, some property is totally uniform in space that is the homogeneous. For example, if you are taking an incompressible fluid ok, it has a density homogeneity in space; that means, at every point the density is constant ok.

Now, when we are talking about statistical homogeneity, it does not say that it has I mean at every point some quantity, for example the velocity is uniform. What it simply says that the two-point correlation functions should only be a function of the distance between these two points and not really depends on where the origin is okay i.e.,

$$\langle \tilde{\boldsymbol{v}}(\boldsymbol{x},t).\tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r},t)\rangle = f(\boldsymbol{r},t)$$

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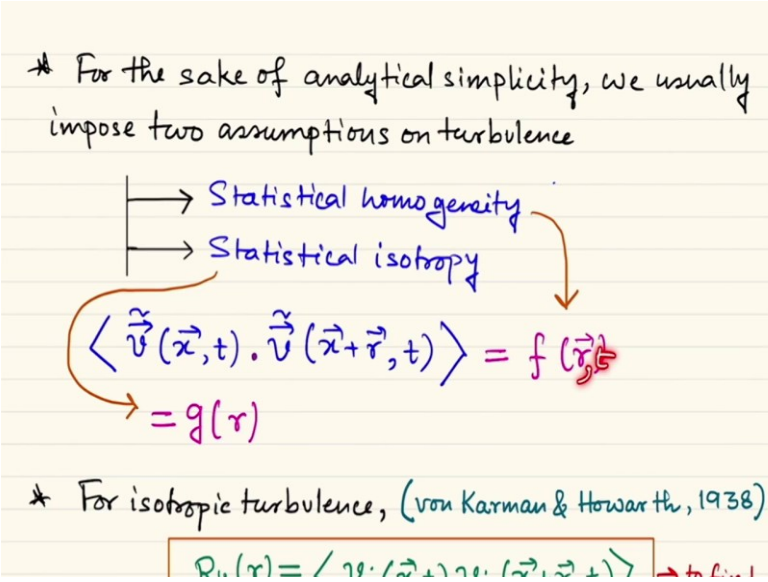

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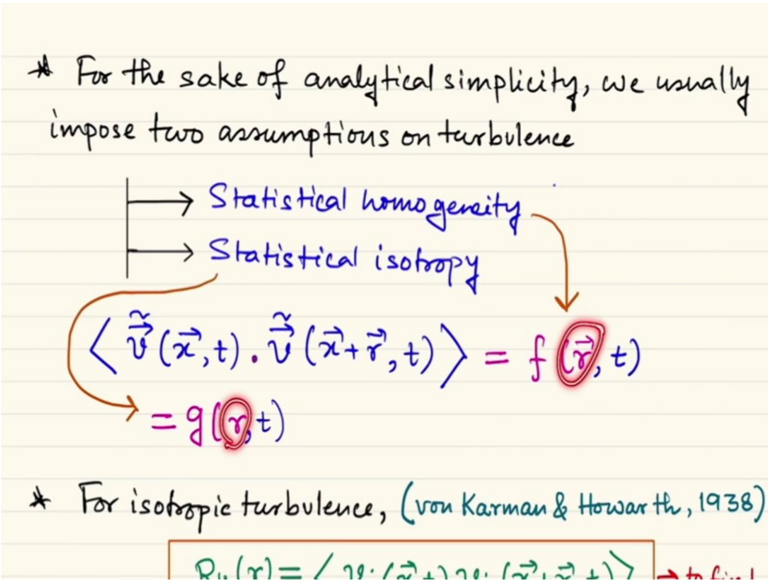

So yeah, there is a t dependence of course; but just for the space part, this is not depending on the origin, I mean this is not depending on the $\boldsymbol{x}$. So, the absolute position of the points does not matter. So, only the mutual distance between two points that matters and so, now, what happens that you can actually find in your system all the points, which are let us say r distance apart and for this type of couple of points, you can calculate this quantity $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x},t).\tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r},t)\rangle = f(\boldsymbol{r},t)$ and then, you can just take the average and that will give you a reasonable estimate for this correlation coefficient and you will see that this will only be a function of r and t okay.

If this is statistically isotropic in addition to this, this one $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) . \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle$ will simply be a function of not a vector $\boldsymbol{r}$; but only the modulus of $\boldsymbol{r}$. It is even much more simplified situation. $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) . \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle = g(r, t)$

So, it has so called spherical symmetry and of course, the t dependence.

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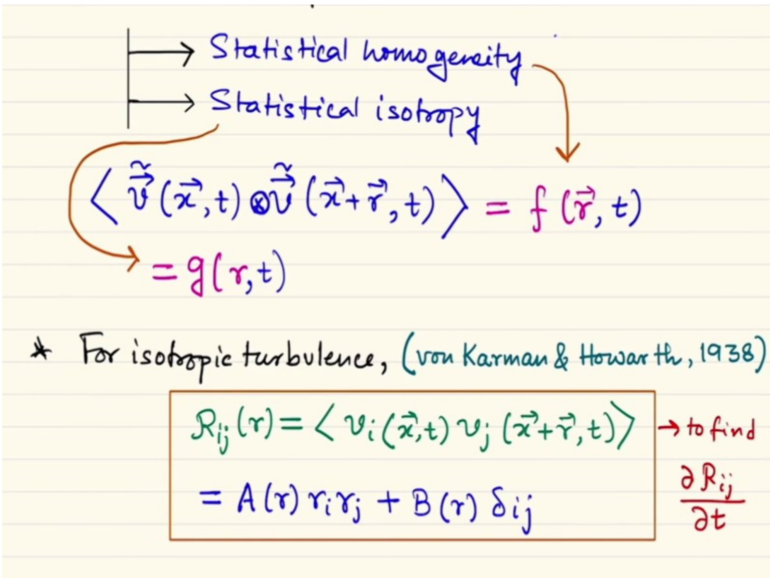

Now, in the year 1938, Von Karman and Howarth, these two persons, they after doing some maths, they actually showed by some tensorial arguments that the most general expression for two-point correlation functions which they now call it as $R_{ij}(r)$, can be written like this

$$R_{ij}(r) = \langle v_i(x,t)\, v_j(x+r,t)\rangle = A(r) r_i r_j + B(r)\delta_{ij}$$

This because this is a tensor right.

So, in general this $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) . \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle$ is a tensor. So, here although, I have written here this dot product, Von Karman and Howarth actually took that as a tensor product; that means, in the normal form, you have this $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) \otimes \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle$ type of thing. Of course, in several literature sometimes you can have this $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) . \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle$.

So, this $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) . \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle$ is nothing but the trace of that tensor $\langle \tilde{\boldsymbol{v}}(\boldsymbol{x}, t) \otimes \tilde{\boldsymbol{v}}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle$. In year 1938, they said that the most general form of this tensor is $A(r)r_i r_j + B(r)\delta_{ij}$, which is some function of some function of r only; that means, modulus of $\boldsymbol{r}$ vector; so, once again,

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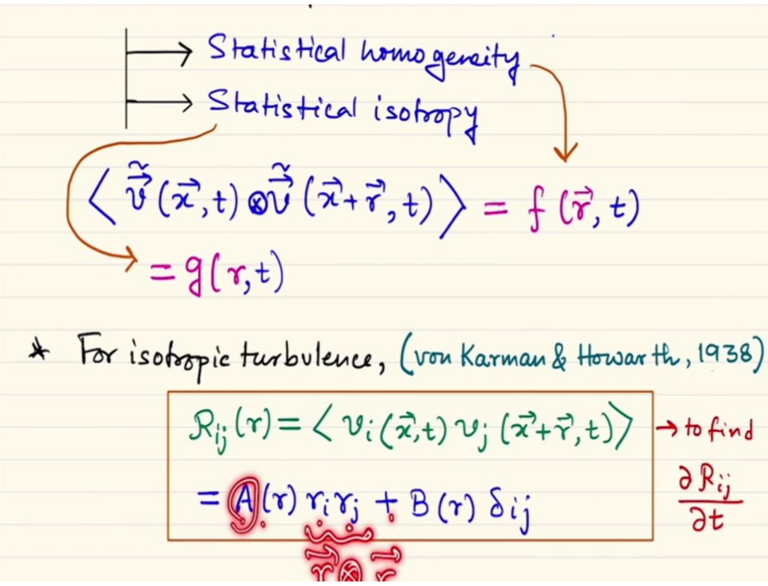

what is this $r_i r_j$? you can easily understand this is nothing but $\boldsymbol{r} \otimes \boldsymbol{r}$; plus $B(r)$ which is another function of modulus of $\boldsymbol{r}$ times Kronecker delta δ_{ij}. So, using this formula, it was much more-simpler for them to find according to the Taylors suggestion $\frac{\partial R_{ij}}{\partial t}$; that means, the evolution of the correlation function.

But first let me just tell you that although I mean from the very beginning of this lecture, I just presented you the analytical pathway for terminals; that means, how starting from one-point quantities people switch to two-point correlation functions and finally, they tried to study their evolution for statistical homogeneous and isotropic turbulence. But historically, we actually did not do like that; historically people actually attacked the very problem of turbulence in a slightly different way. So, analytical treatment was there of course, but even before that in the year 1921 Richardson, who was an expert of weather forecasting. So, he basically gave a very schematic picture of the phenomenological energy transfers in different scales of turbulence. So, at this point, I have to mention that the notion that a turbulent system should behave like a multiscale system was already conceived by that time. Although by mathematics, it was not properly shown or it was not properly exploited, this multiscale property; but people have in their mind that in a turbulent system, the total flow should have a multiscale nature. That means, the systems have actually different forms in different scales. So, and according to the scale, the system behaves actually differently.

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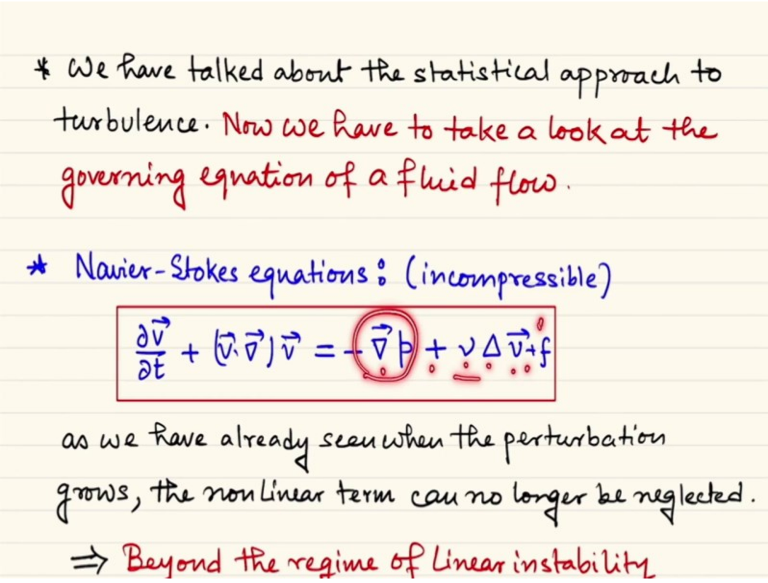

If irrespective of the skill the system behaves, I mean identically, then there is no problem okay. But I mean, what I am saying that when a system for example, say if it is a laminar flow, then you cut the layers transversely to its flow into smaller and smaller parts, well, the laminar nature will remain laminar right.

Now, in a turbulent flow that is not the correct thing. So, if you just want to go to smaller and smaller scales, the picture can be different and I am coming to that and very crude and phenomenological view was conceived by Richardson, already by the in the year 1921 which was much earlier than von Karman and Howarth and even the work of Taylor. So, you see and to understand these things, we have to just have a look at the governing equation of the fluid flow. So, we have told many things about velocity, two-point velocity correlation; but till now, we have not talked about the equation which is governing the velocity and everything. So, that is Navier Stokes Equation; just for a normal incompressible fluid,

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}\, p + \nu\, \Delta\boldsymbol{v} + \boldsymbol{f}$$

So, $\boldsymbol{f}$ is the external forcing term.

So, we have to first understand that whether starting from Navier Stokes equation, we can even roughly try to understand the turbulent regime? Because you know that the same equation actually can guide can lead to laminar flow as well as turbulent flow right. So, actually what happens that as we have already seen in previous discussions that when the perturbation grows, now if you remember that the non-linear term can no longer be neglected.

So, if you remember that when the perturbation is weak, we can linearize the system just by neglecting the second order smallness, I mean the terms with second order smallness okay for example, the term like this $\boldsymbol{v}.\nabla\boldsymbol{v}$, when your perturbation is not negligibly small, but when I mean it is not of first order basically, it is larger than that, then the non-linear term cannot be neglected anymore okay.

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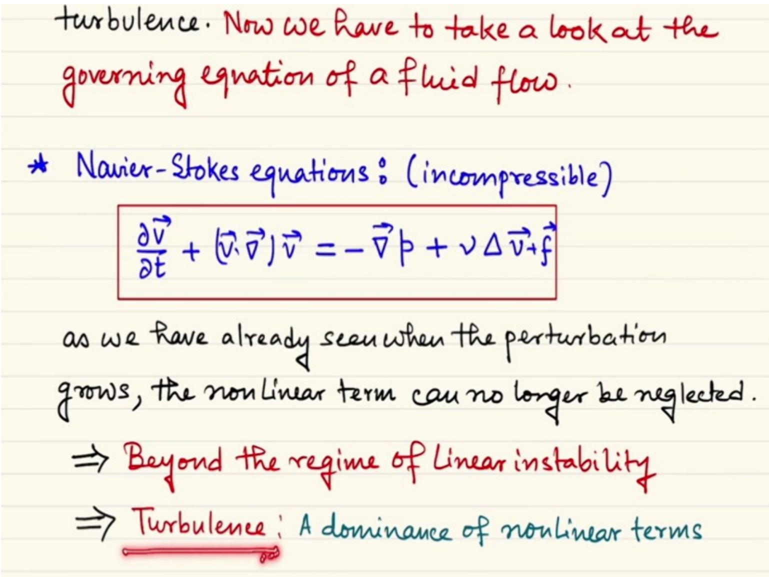

And then, simply we are beyond the regime of linear instability. So, basically how to reach to turbulence? this is another fundamental and very deep question. So, to be very honest, turbulence is a very deep and complicated subject and actually more than one course can be given on turbulence.

So, just in 1 or 2 lectures, it is almost impossible to present turbulence. So, then, I mean in this way, when we are beyond the regime of linear instability, we reach or rather attain the regime of turbulence and which is simply described by a clear dominance of the non-linear terms okay.

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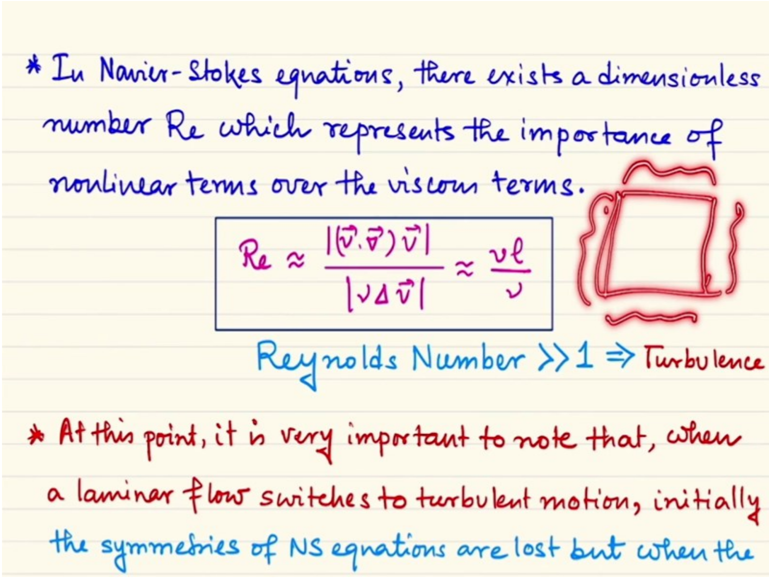

So, when the perturbation grows larger and larger, then the non-linear term basically they are being the non-linear in perturbation, they will grow much faster than the other linear terms ok. That is a very common sense. So, in Navier Stokes equation, there actually exists a dimensionless number Re which represents the importance of the non-linear terms over the viscous terms and that Re is nothing but

$$Re = \frac{|\boldsymbol{v}.\nabla\boldsymbol{v}|}{|\nu\Delta\mathbf{v}|} = \frac{vl}{\nu}$$

This is known as the scale specific Reynolds number. If the systems Reynolds number corresponding to the largest scale and the largest velocity is very greater than 1, then we say that the system has achieved turbulence macroscopically okay.

Macroscopically means so, what I am saying that in a system, the largest scale is the scale of this box for example, where there is a turbulent fluid. So, the box size is the largest scale or rather roughly larger scale is the scale, roughly the order which we can see in naked eye. So, this is the macroscopic scale.

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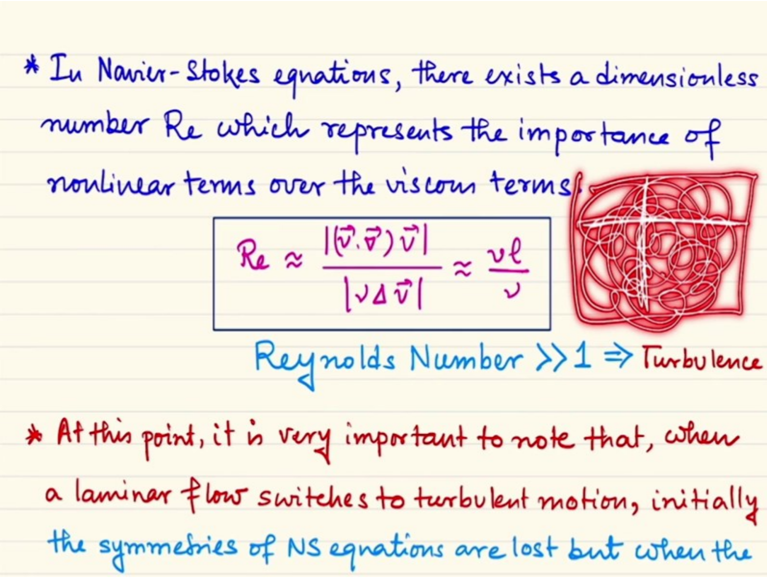

And we see that in the macroscopic scale, if there are this chaotic type of arbitrary motion, then the large scale is this one roughly or this one. Of course, they are of the same order roughly. So, if that Reynolds number is very greater than 1, then the system is turbulent; the flow is turbulent that is by definition.

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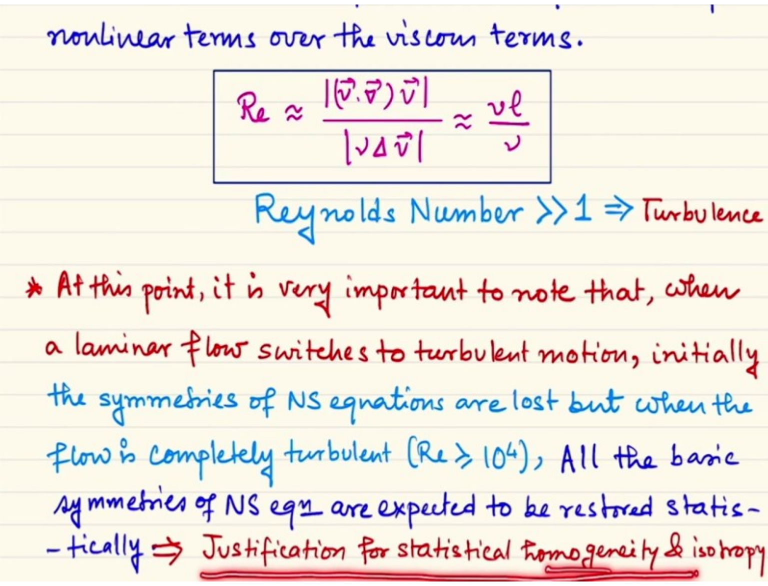

Now, at this point when we have already defined Reynolds number and the regime of turbulence depending on that. It is very important to remember that when a laminar flow switches to turbulent motion, initially, the symmetries of the Navier Stokes equation are lost but when the flow is completely turbulent, all basic symmetries are expected to restored; that means, the homogeneity and isotropy.

So, in laminar, if you are thinking of the equation itself so, it is actually invariant under the origin I mean in both space and time. So, whether you let the fluid flow here or in London, it does not matter or today or tomorrow, does not matter. I mean if the other conditions for example the temperature and pressure, they are kept constant and then, then there is no explicit depending on the space and time origin and also, Navier Stokes equation is Galilean invariant.

Navier Stokes equation is invariant under symmetric under time reversal. So, all these things are there ok. There is a there is a small catch ok. So, Navier Stokes equation is invariant under time reversal only when the viscous term is not there ok; otherwise, there can be time, I mean yeah, the viscous term can actually destroy the time reversal symmetry.

So, to be very formal, the Euler equation is symmetric under time reversal; but viscous term which dissipates energy actually kills the time reversal symmetry. But all the symmetry which it has for example, homogeneity in space and time, isotropy for example, so, this type of symmetries are gradually lost when the flow is getting more and more turbulent. That means, its Reynolds number increases from 10 to 100 like this. But when the flow is completely turbulent; that means, the Reynolds number is greater or equal to 10^4 or large even I mean even greater actually, then the good news is all the basic symmetries of the Navier Stokes equations are expected to be restored statistically. Why this is so? This is not really very much clear, but this is actually the crude approximation and somehow, people have applied this hypothesis and they have seen that this works.

So, to be very honest, this is something which already happens in nature, that is why we expect for any general system in turbulence that when the turbulence is completely developed or fully developed ok, then all the Navier Stoke symmetries are restored, but in statistical sense and that is why it somehow justifies of our previous hypothesis of previous assumptions of statistical homogeneity and isotropy. So, actually homogeneity in both space and time, to be very honest.

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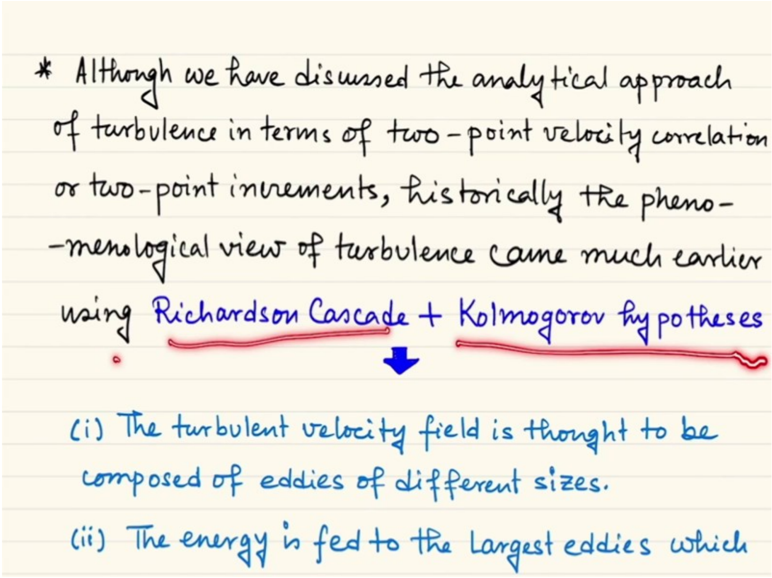

Now, we have discussed the analytical approach of turbulence in terms of two-point velocity correlation or two-point increments; but as I just said that, historically the phenomenological view of turbulence came much earlier using Richardson's cascade and Kolmogorov's equilibrium hypothesis and that was done actually in not in direct space, but in Fourier space, this part I will be discussing in the next lecture.

So, from this lecture, the message is that turbulences are disordered out I mean out of equilibrium motion in both space and time and the non-linearity is dominating and also one-point quantities are of no importance, although statistical description is important.

Then, Taylor said that the two-point correlation functions are the quantities which we have to look at and then, Von Karman and Howarth, they analyzed that and studied that their revolution. And finally, we said that okay, so all these things are there, but of course, how to really, I mean without knowing anything how to characterize a turbulent regime? For that we started looking at the governing equations; of-course von Karman and Howarth when they derived their $\frac{\partial R_{ij}}{\partial t}$, they also use Navier Stokes equation; what else they could have used okay.

So, now, just by introducing Navier Stokes equation, we said that there is a dimensionless quantity called Reynolds number, a large value of whose value can correspond to the domain of flow regime which is completely dominated by non-linearity and which is the turbulent regime. In the next lecture, I will come into the Richardson cascade picture and the Kolmogorov, I mean phenomenology okay.

Thank you very much.