Introduction to Astrophysical Fluids
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Lecture - 05
Boltzmann equation for collisional systems II

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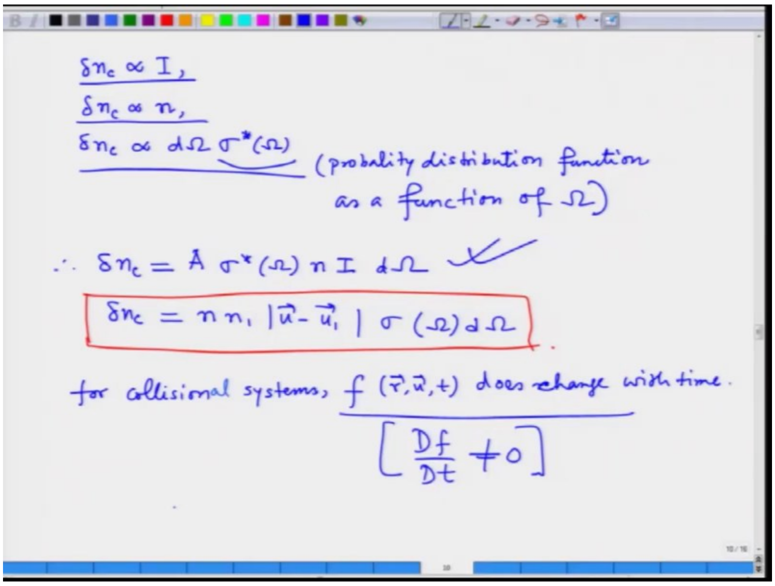

Hello and welcome to the course of Introduction to Astrophysical fluids. Now, here $\frac{Df}{Dt} \neq 0$, what is the meaning of that? And of course, this is when traced along a particle trajectory. What is the meaning of that?

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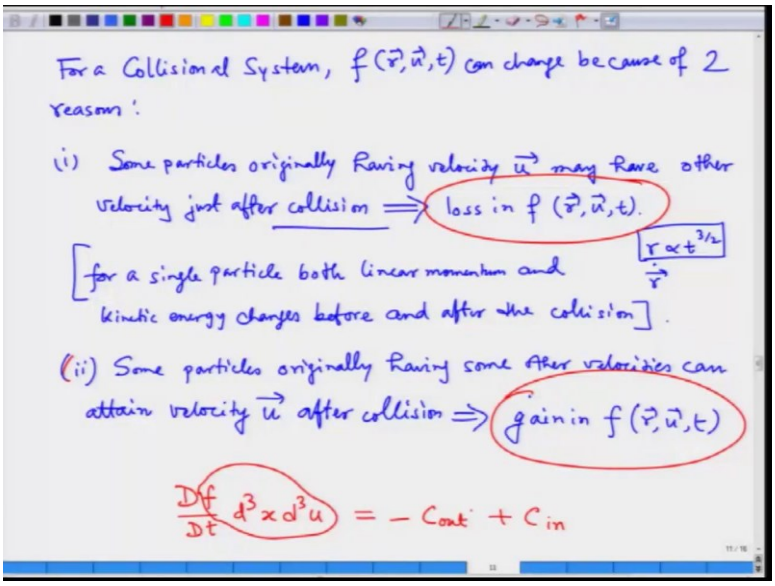

If the system allows collisions, then basically that $f(\boldsymbol{r}, \boldsymbol{u}, t)$ changes for a collisional system because of two reasons.

1. some particles originally having velocity $\boldsymbol{u}$ may have other velocity just after collision right. Thereby, leading to a loss in $f(\boldsymbol{r}, \boldsymbol{u}, t)$.

So, $f(\boldsymbol{r}, \boldsymbol{u}, t)$ is the number of single particle states. In the phase space volume, $f(\boldsymbol{r}, \boldsymbol{u}, t)$ is the density of single particle states in the μ space, that was the definition. So, you can see that if there were no collision, then $f(\boldsymbol{r}, \boldsymbol{u}, t)$ would have just evolved dynamically according to some law.

Then, you know like if the particle is now over here, then after some time where it will be and then, you can also calculate $\boldsymbol{u}$ right and you can also calculate so $\dot{\boldsymbol{r}}$. If you know this type of dependences and then you can also say that what will be its velocity at some subsequent interval okay.

Now, the problem is that if collision comes, so this is something which is not expected. This inhibits a normal evolution. So, that means, if the particle was evolving according to some evolution, then the problem is that although we are saying that kinetic energy, momentum all are conserved before and after the collision. But if you are concentrating on a single particle, basically before and after collision its individual momentum and kinetic energy, they are basically changing so, remember for a single particle, both linear momentum and energy and kinetic energy changes before and after the collision.

It simply says that if the particle was belonging to the particles having velocity $\boldsymbol{u}$ for example, then what happens just after the collision? So, just after the collision in general, we assume that the particle is momentarily localized, that means, its velocity changes, but it does not still leave the place. So, its position will be still $\boldsymbol{r}$ but it will have some different velocity okay and if this is true, then basically you can easily understand that this will lead to a loss in $f(\boldsymbol{r}, \boldsymbol{u}, t)$. Because the now the particle is moving to another club of particles having some velocity, let us say $\boldsymbol{u_1}$ or rather $\boldsymbol{u}^{'}$; sorry, I think $\boldsymbol{u}^{'}$ is okay. After the collision, I used the prime coordinates and then you can say now the particle is belonging to $f'(\boldsymbol{r}, \boldsymbol{u}', t)$.

So, in the same way another source of change is simply that

2. some particles originally having; this is actually a gain; some other velocities can attain velocity $\boldsymbol{u}$ after collision. So, this is a gain in $f(\boldsymbol{r}, \boldsymbol{u}, t)$.

These two effects are now non-zero. So, schematically, this can be written as

$$\frac{Df}{Dt} d^3u \, d^3x = -C_{out} + C_{in} \qquad (1)$$

$f d^3u \, d^3x$ is nothing but the number of the particles in this $d^3u \, d^3x$ elementary volume of phase space or rather μ space and $\frac{D}{Dt}$ is the change of the number with time, of course along a particle evolution trajectory and that will simply now be written as $-C_{out} + C_{in}$.

So, if you remember R.H.S of (1) is nothing but the rate of the particles gained or lost through collision and that we can classify into two classes; one is the loss another one is the gain.

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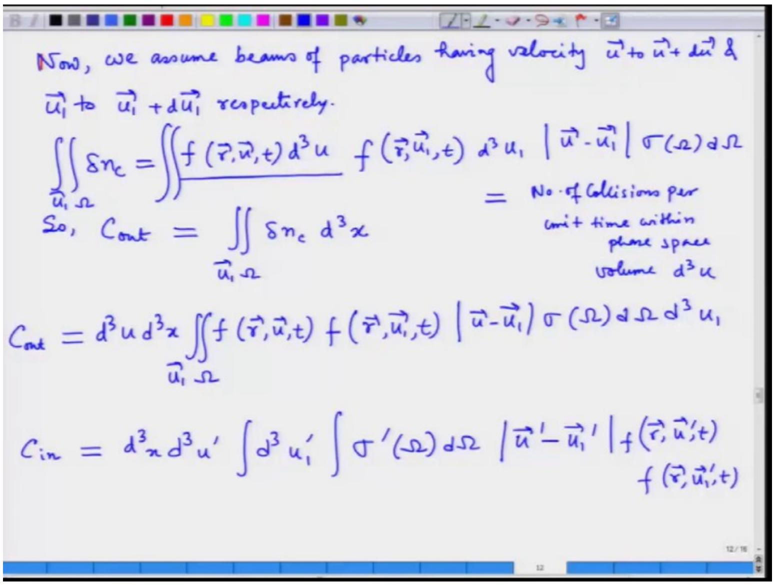

Now, we assume beams of particles. So, before we assume two beams of particles $\boldsymbol{u}$ and $\boldsymbol{u_1}$. Now, we will be interested in beams of particles having velocities in the infinitesimal range; one is that $\boldsymbol{u}$ to $\boldsymbol{u}+\ \boldsymbol{d\,u}$ and another is $\boldsymbol{u_1}$ to $\boldsymbol{u_1}+\ \boldsymbol{d\,u_1}$ respectively. If you do that, then if you remember the definition of δn_c that was this

$$\delta n_c = Ann_1|\boldsymbol{u}-\boldsymbol{u_1}|\sigma(\Omega)d\Omega$$

Then, you can easily write now δn_c will be equal to

$$\delta n_c = \ f(\boldsymbol{r},\boldsymbol{u},t)d^3u\, f(\boldsymbol{r},\boldsymbol{u_1},t)d^3u_1|\boldsymbol{u}-\boldsymbol{u_1}|\sigma(\Omega)d\Omega$$

This definition is very interesting. You can see. So, f is the number of particles per unit phase space volume. Now, if it is just multiplied with the number d^3u, it simply says that this is the number of particles having position $\boldsymbol{r}$ but having velocity ranging from $\boldsymbol{u}$ to $\boldsymbol{u}+\ \boldsymbol{d\,u}$ and that is exactly what we are seeking for. Another is for the second beam $f(\boldsymbol{r},\boldsymbol{u_1},t)d^3u_1$. Again, the rest was just $|\boldsymbol{u}-\boldsymbol{u_1}|\sigma(\Omega)d\Omega$.

Now, you remember that we are having two beams with the velocity in range $\boldsymbol{u}$ to $\boldsymbol{u}+\ \boldsymbol{d\,u}$ and $\boldsymbol{u_1}$ to $\boldsymbol{u_1}+\ \boldsymbol{d\,u_1}$. But that will simply change the number densities, but the relative velocity will not be changed by that ok. So, this basically can be related to C_{out} by this definition, I do an integration over u_1 and Ω ,

$$C_{out} = \int_{u_1}\int_{\Omega} \delta n_c \, d^3x$$

Now, what is the meaning of that? Why I did that? So, finally, δn_c multiplied with d^3x and integrated on u_1 space and Ω space simply gives us the number of loss of the particle or the number of changes per unit time from the volume element $d^3x d^3u$.

So, finally, you will have something, where you will just have the number of the particles; δn_c is nothing but the number of collisions per unit time within phase space volume d^3u and if you now multiplied with this whole thing with d^3x, then finally, you will have number of collisions per unit time within phase space volume $d^3x d^3u$ and we are saying that is exactly equal to C_{out} right.

If this is true, then finally, we can write that C_{out} is nothing but

$$C_{out} = d^3x d^3u \int_{u_1}\int_{\Omega} f(\boldsymbol{r}, \boldsymbol{u}, t) f(\boldsymbol{r}, \boldsymbol{u_1}, t) |\boldsymbol{u} - \boldsymbol{u_1}| \sigma(\Omega) d\Omega \mathrm{d}^3 u_1$$

So, that is the definition for C_{out}. In the same way basically, you can also calculate C_{in} just by tracking the collisions in back process.

Collisions are elastic in nature. So, there is no dissipation. So, whether the collisions are taking place in a forward manner or in the backward manner, they will behave identically. So, basically then you can see that if we consider the collisions of two particles which are coming and collide with velocities $\boldsymbol{u'}$ and $\boldsymbol{u'_1}$ and after collision they have the velocities $\boldsymbol{u}$ and $\boldsymbol{u_1}$.

So, if you consider this type of collisions in a backward direction, then basically you can write or you can obtain an expression for C_{in} which will simply be

$$C_{in} = d^3x d^3u' \int_{u_1}\int_{\Omega} f(\boldsymbol{r}, \boldsymbol{u}', t) f(\boldsymbol{r}, \boldsymbol{u'_1}, t) |\boldsymbol{u}' - \boldsymbol{u'_1}| \sigma'(\Omega) d\Omega \mathrm{d}^3 u'_1$$

So, $\sigma'(\Omega)$is just the angular distribution for the backward collisional.

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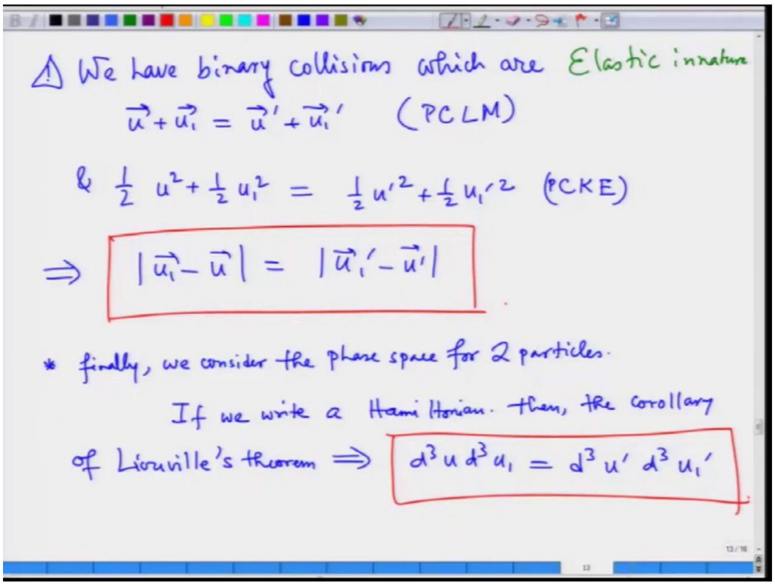

Now, remember that we have a binary collision which are elastic in nature right. Then, what is the good news? First, we have

$$\boldsymbol{u} + \boldsymbol{u_1} = \boldsymbol{u}' + \boldsymbol{u}_1'$$

before and after the collision, this is nothing but the principle of conservation of linear momentum because of all the particles are having identical mass. So, mass is not there. And we also have

$$\frac{1}{2}\, u^2 + \frac{1}{2}\, u_1^2 = \frac{1}{2}\, u'^2 + \frac{1}{2}\, u_1'^2$$

that is principle of conservation of kinetic energy. Again, mass is not there because this is cancelled. If you do this, this can be simply left to you as an exercise that you can show that it leads to a very simple result, this is

$$|\boldsymbol{u} - \boldsymbol{u_1}| = |\boldsymbol{u}' - \boldsymbol{u}_1'|$$

So, and if you remember, I do not know that if you are habituated with this mechanics of collisions, this is nothing but of saying that the coefficient of restitution is equal to 1, right. So, this is also another property of elastic collision.

Finally, we consider the phase space of two colliding particles only. So, if their interaction can be described by a Hamiltonian then that is something not very evident. So, finally, we consider the phase space for two particles, which are colliding and if we say that under the condition that these two particles system can be described by a Hamiltonian. So, usually this is only possible that if their collision is taking place under some central type of force, where the potential is basically depending on the relative distance between them or some other system so that you can write the you can write a Hamiltonian for these two-particle system.

Now, remember that these two particle systems are actually colliding with each other. If we write a Hamiltonian, then the corollary of Liouville's theorem simply says that

$$d^3u d^3u_1 = d^3u' d^3u_1'$$

Again, a very important result.

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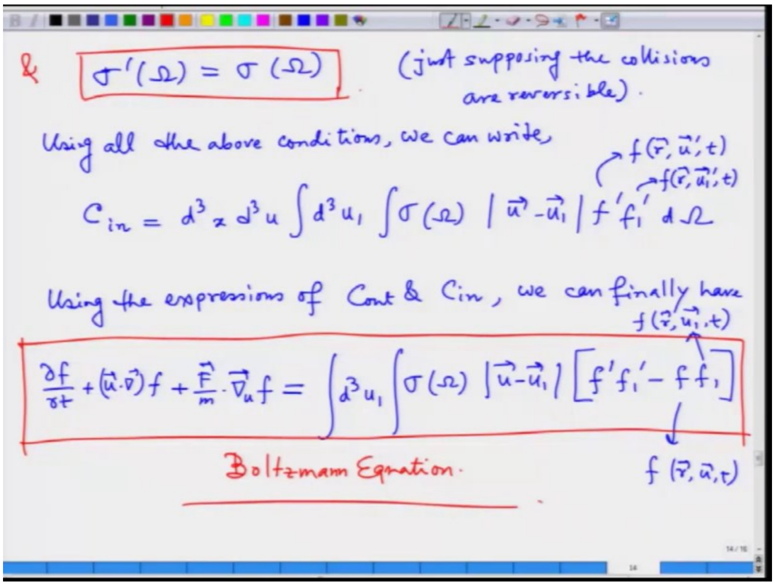

Finally, you can also say just supposing the collisions are reversible, that

$$\sigma(\Omega) = \sigma'(\Omega)$$

Above is true because we do not have any source of irreversibility like any dissipation or anything okay. Finally, you can write using all the above conditions

$$C_{in} = d^3x d^3u \int_{u_1} \mathrm{d}^3u_1 \int_{\Omega} d\Omega f(\boldsymbol{r}, \boldsymbol{u}', t) f(\boldsymbol{r}, \boldsymbol{u}_1', t) |\boldsymbol{u} - \boldsymbol{u}_1| \sigma(\Omega)$$

So, using the expressions of C_{out} and C_{in}, we can finally write

$$\frac{\partial f}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} f + \frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla}_{\boldsymbol{u}} f = \int d^3 u_1 \int \sigma(\Omega) |\boldsymbol{u} - \boldsymbol{u_1}| [f^{'} f^{'}_{1} - f f_1]$$

Where $f' \equiv f'(\boldsymbol{r}, \boldsymbol{u}', t)$ and $f_1' \equiv f_1'(\boldsymbol{r}, \boldsymbol{u}_1', t)$ are distribution function before collision and $f \equiv f(\boldsymbol{r}, \boldsymbol{u}, t)$ and $f_1 \equiv f_1(\boldsymbol{r}, \boldsymbol{u_1}, t)$ are distribution functions before collisions. So, finally, you have above expression which is known as the traditionally collisional Boltzmann equation or simply Boltzmann equation. Whenever we say Boltzmann equation, it simply defines the Boltzmann equation with the collision term.

So, here basically in this lecture after deriving the collisionless Boltzmann equation, we derived a version of Boltzmann equation, where we extended the Boltzmann equation for collisional system and the collisions are of very specific nature, they should be binary and they should be elastic in nature okay.

Thank you.