Introduction to Astrophysical Fluids
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Lecture - 49
Linear wave modes in MHD

Hello and welcome to another lecture session of Introduction to Astrophysical Fluids. Previously, after deriving magnetohydrodynamic equations that is the mono-fluid model of plasma from kinetic theory and multi-fluid models of plasma, we discussed various interesting features of magnetohydrodynamic fluids.

So, first we discuss the importance of Lorentz force, how can we decompose the Lorentz force into two types of contributions, one is like a magnetic pressure another is a tension term. Then, also we defined plasma β and it was followed by a discussion on the frozen-in theorem for a magnetohydrodynamic fluid. So, and then we also saw different types of inviscid invariants.

So, we started by mass conservation, then we talked about the very subtle case of linear momentum conservation. Then we talked about some scalar invariants of which a real scalar invariant was the total energy which had three parts, kinetic energy, magnetic energy, and compressible thermodynamic energy or compressible potential energy.

And then, we also introduced several pseudo scalars, which we called the helicities. So, one was the kinetic helicity which is a purely hydrodynamic concept. You may be accustomed with this concept already. Then, another was like the cross helicity. So, kinetic helicity density was nothing but the scalar product of velocity and the vorticity vector. Then cross helicity was introduced, it was the scalar product of velocity and the magnetic field vector. And finally, we said that there is another helicity which is of very importance in MHD turbulence and in other type of phenomena, that was the magnetic helicity which is the scalar product of magnetic field and magnetic vector potential.

So, in this part we also said a few words, if you can remember that in under which circumstances in practical case we can really assume that our system can follow ideal MHD equations. And finally, we talked about the very interesting Elsässer variables and how can the equations of incompressible MHD can equivalently be projected in terms of the Elsässer variables.

One thing was very easy to understand, but I did not really mention, I just thought that it is maybe good to tell you directly that is in incompressible MHD as $\boldsymbol{v}$ and $\boldsymbol{b}$ both are divergence less vectors, $\boldsymbol{v}+\boldsymbol{b}$ and $\boldsymbol{v}-\boldsymbol{b}$, that means, $\boldsymbol{z}^{+}$ and $\boldsymbol{z}^{-}$, they are also divergence less. So, the two Elsässer variables are actually two solenoidal vectors.

So, the whole set of MHD equations in terms of Elsässer variables are constituted by divergence of $\boldsymbol{z}^{+}$ is equal to 0, divergence of $\boldsymbol{z}^{-}$ is equal to 0 and two evolution equations, one for $\boldsymbol{z}^{+}$ and one for $\boldsymbol{z}^{-}$. That constitutes a complete dynamical theory.

In today's discussion, we will start discussing the response of an MHD fluid towards an external very weak or first order perturbation, linear perturbation. And actually we will see that there will be interesting wave modes.

And of course, that says that our chosen initial system, which is of course is steady system, is such a steady system that it corresponds actually a stable equilibrium, something analogous to stable equilibrium system. And that is why the system when it is perturbed, it responds in terms of the linear modes.

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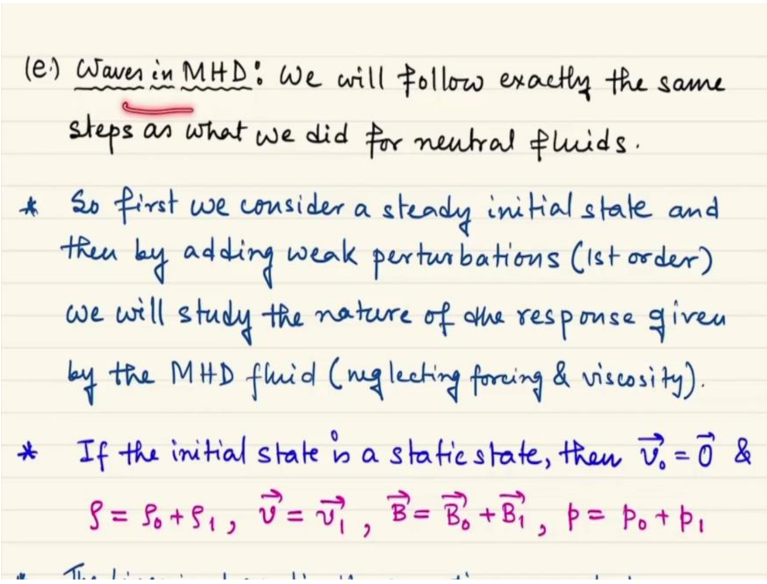

Whenever we are discussing in this part the waves in MHD fluid, we will follow exactly the same steps as what we did for the neutral fluids. So, if you remember that first of all we start with the steady initial state. And then we perturb every quantity, by a very small amount, and what we call weak perturbations are linear perturbations and sometimes we call them first order perturbations.

And then, we study the nature of the response given by the MHD fluid, after just expressing every quantity as its initial value plus the first order perturbation we try to replace all the values in the original equations.

And finally, we just drop out all the 0-th order terms which are basically nullifying each other due to the 0-th order equations, and then we also neglect the terms which are of second order smallness. And then we have a set of linear equations.

So, with that linear equation finally, we assume plane wave type of solution, and then we derive some relation between ω, the frequency and the wave vector k, which we call in general the dispersion relation. Now, let us do that formally.

So, here first of all when we study the response of an MHD fluid towards a weak perturbation, we will for the sake of simplicity at the first step neglect forcing and viscosity. It is always a very good question that what happens to the wave modes or to the nature of response, if we take the forcing or the viscosity into account! that is for you to think. It is a very good question actually.

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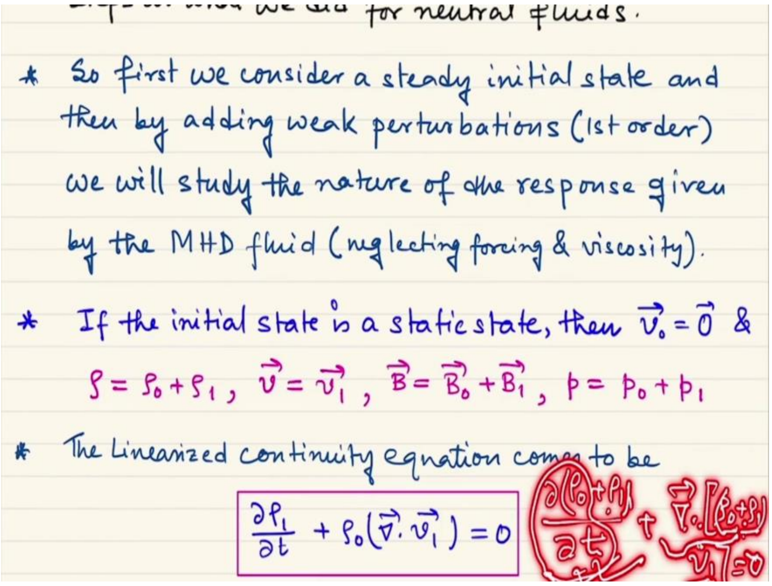

Now, we start from an initial state which is a static steady state. So, not only that it is a steady, that means, every quantity is independent of time, but the velocity in the initial state is 0. So, we start with the static case.

And then, we perturb the system by a little bit. So, ρ becomes $\rho_0 + \rho_1$, where the initial hydrostatic density was ρ_0, $\boldsymbol{v}$ becomes $\boldsymbol{v}_1$ only because $\boldsymbol{v}_0$ is 0 by definition. So, $\boldsymbol{B}$ is the magnetic field and it is just the initial $\boldsymbol{B}_0$ plus the fluctuation or plus the perturbation. And same thing for the pressure, so initial pressure p_0, plus p_1, first order perturbation.

Now, when we linearize the continuity equation then the equation becomes

$$\frac{\partial}{\partial t}(\rho_0 + \rho_1) + \boldsymbol{\nabla} \cdot [(\rho_0 + \rho_1)(\boldsymbol{v}_0 + \boldsymbol{v}_1)] = 0.$$

Now, $\frac{\partial \rho_0}{\partial t} = 0$, as $\boldsymbol{v}_0 = \boldsymbol{0}$, so that is why the 0-th order term is 0. We also neglect $\rho_1 \boldsymbol{v}_1$ because of the second order smallness.

So, $\frac{\partial \rho_1}{\partial t}$ is of course, with first order smallness plus $\rho_0 \boldsymbol{v}_1$, that is also first order smallness. So, the linearized equation of continuity is $\frac{\partial \rho_1}{\partial t} + \rho_0(\boldsymbol{\nabla} \cdot \boldsymbol{v}_1) = 0$.

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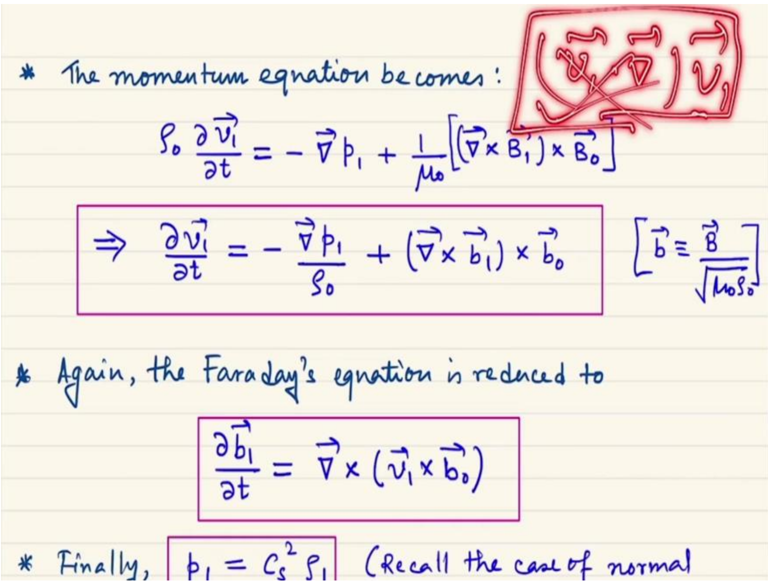

Then, we do the same thing for the momentum evolution equation, where you can easily understand that this is again, so there will be $\rho \frac{\partial \boldsymbol{v}}{\partial t}$, now only ρ_0 can be taken because there is no $\boldsymbol{v}_0$. So, in order that the total thing to be of first order, the part of ρ should contribute to 0-th order because the only nonzero contribution from $\frac{\partial \boldsymbol{v}}{\partial t}$ would come in the first order, there is no 0-th order for this part.

So, that is why we have $\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t}$ and you know that there should be no contribution from the part $(\boldsymbol{v}_1 \cdot \boldsymbol{\nabla})\boldsymbol{v}_1$ because of its second order smallness. So, we forget that.

So, ∇p_0 will be balancing with the 0-th order term of the Lorentz force of course and you see that here we can actually, if you remember, neglect the body force or for example, some force like gravity only because you have here another term other than the p, which is the Lorentz force.

If this Lorentz force term is not present here, then we have to take into account the body force and that is our very usual hydrostatic equilibrium and the hydrostatic first order perturbation under gravity or under some conservative force field. But here we really do not need that. That is simply because that we have $\boldsymbol{\nabla} p_1$ and $\frac{1}{\mu_0}[(\boldsymbol{\nabla} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0]$.

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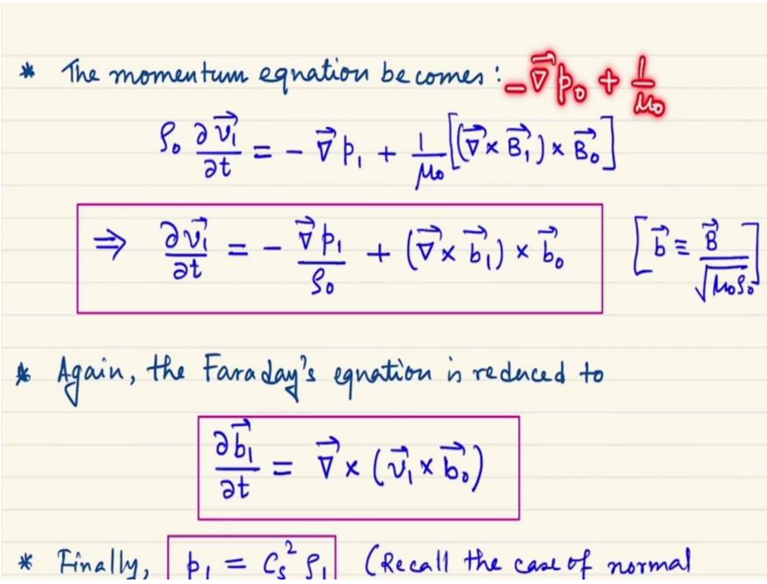

So, in the 0-th order for example, what happens? you just have $-\boldsymbol{\nabla} p_0 + \frac{1}{\mu_0}[(\boldsymbol{\nabla} \times \boldsymbol{B}_0) \times \boldsymbol{B}_0)]$.

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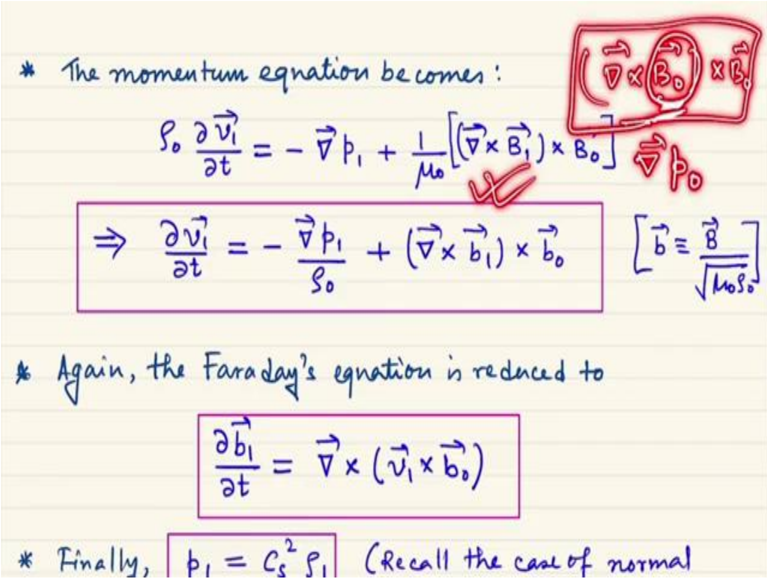

And if you just choose $\boldsymbol{B}_0$ to be a constant then $\frac{1}{\mu_0}[(\boldsymbol{\nabla} \times \boldsymbol{B}_0) \times \boldsymbol{B}_0)]$ is 0, but if you do not choose $\boldsymbol{B}_0$ to be constant in space, but it will then simply be cancelled by $\boldsymbol{\nabla} p_0$. Why? Because the 0-th order term will be taken care by the 0-th order contribution of pressure.

So, now only surviving term is $\frac{1}{\mu_0}[(\boldsymbol{\nabla} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0)]$. You also have $\frac{1}{\mu_0}[(\boldsymbol{\nabla} \times \boldsymbol{B}_0) \times \boldsymbol{B}_1)]$, so that is also a bit typical because most of the cases we have seen that the initial magnetic fields are in general uniform in space. Maybe sometimes they can have some time dependence, but $\boldsymbol{B}_0$ almost uniform in space. But there can be actually instances where you can actually keep $\frac{1}{\mu_0}[(\boldsymbol{\nabla} \times \boldsymbol{B}_0) \times \boldsymbol{B}_1)]$.

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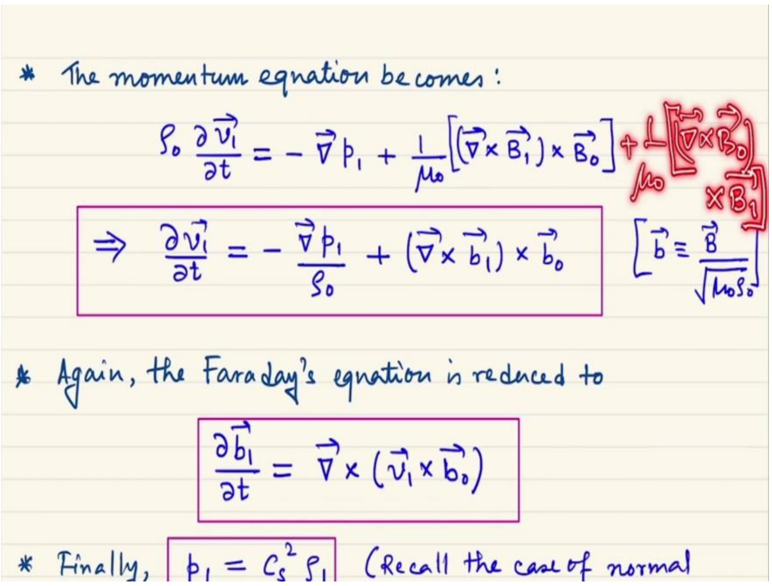

Actually there is no inhibition in writing that one, but here just for the sake of simplicity we have not written it. So, this is somehow a simplified form and we cannot write both as $\boldsymbol{B}_1$ because that would get a second order smallness. So, we forget that.

So, finally, we can write $\frac{\partial \boldsymbol{v}_1}{\partial t} = -\frac{\nabla p_1}{\rho_0} + [(\nabla \times \boldsymbol{b}_1) \times \boldsymbol{b}_0)]$.

How did we get that? We have simply got that just by replacing $\boldsymbol{B}$ by $\boldsymbol{b}$. And that was quite easy because if you simply divide everything by ρ_0 and you have one μ_0 on the RHS.

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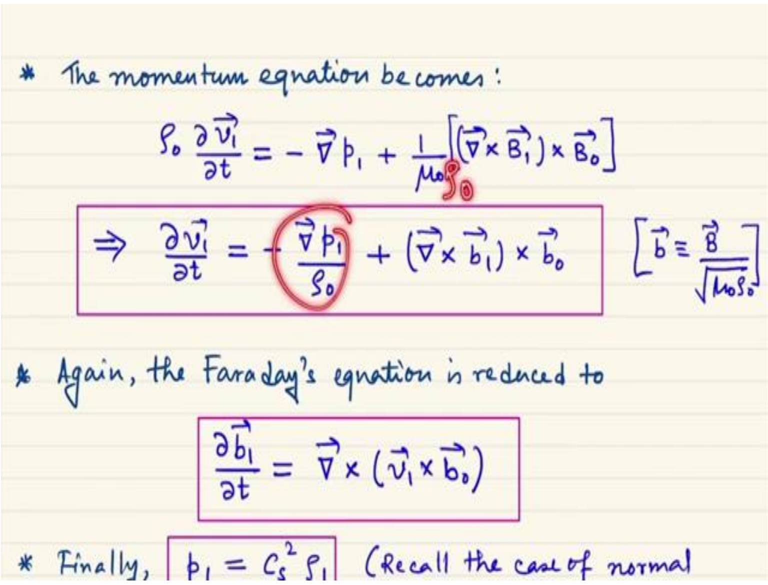

So, then you have $\boldsymbol{b}_0 = \frac{\boldsymbol{B}_0}{\sqrt{\mu_0 \rho_0}}$ and $\boldsymbol{b}_1 = \frac{\boldsymbol{B}_1}{\sqrt{\mu_0 \rho_0}}$ and you are done.

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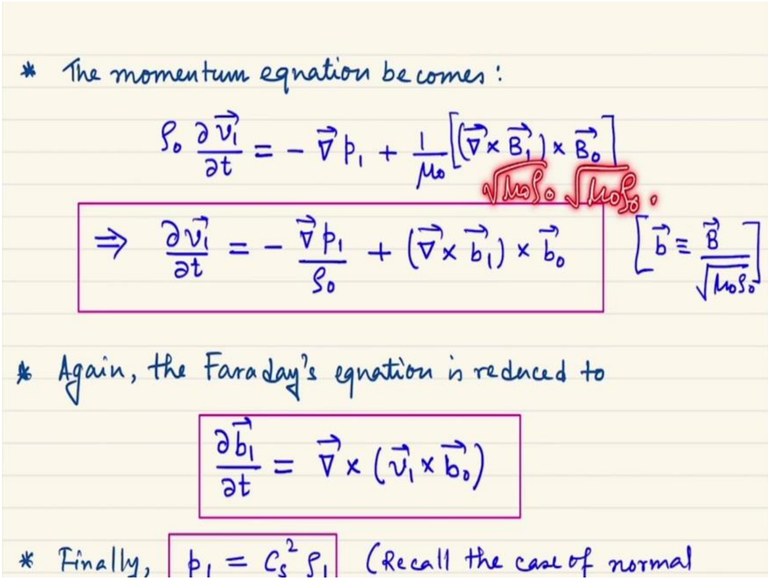

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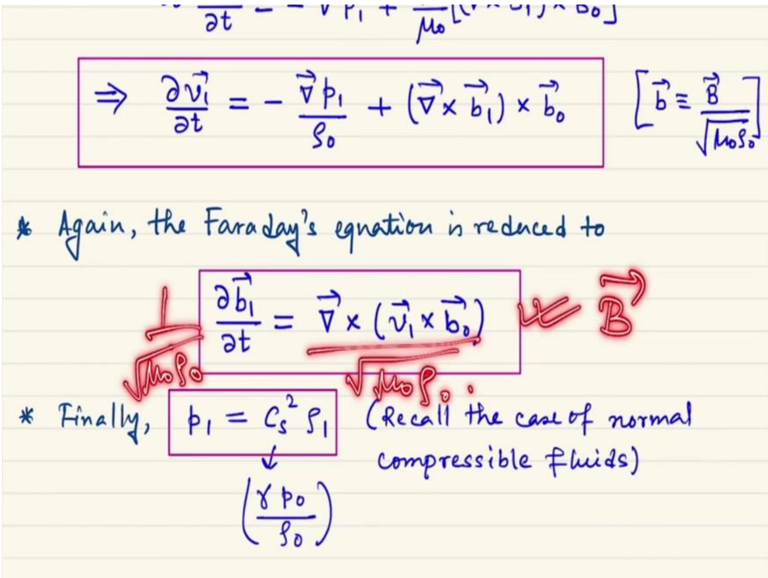

Now, the third equation of magnetohydrodynamics is the equation of Faraday. So, equation of induction, which simply says that $\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{b})$.

Now, once again $\boldsymbol{v}$ cannot have any $\boldsymbol{v}_0$ amount, so in order that the total thing should be exactly a first order, 0-th order cannot be there because 0-th order will be cancelled by the

0-th order part of the on the left side. So, there cannot be any term like $\nabla \times (\boldsymbol{v}_0 \times \boldsymbol{b}_1)$, as $\boldsymbol{v}_0 = \boldsymbol{0}$.

Here I can write the original equation in terms of $\boldsymbol{B}$, but just by dividing every side by $\sqrt{\mu_0 \rho_0}$, you can define your $\boldsymbol{b}$. So, then your Faraday equation is $\frac{\partial \boldsymbol{b}_1}{\partial t} = \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{b}_0)$, and finally we consider a polytropic MHD fluid.

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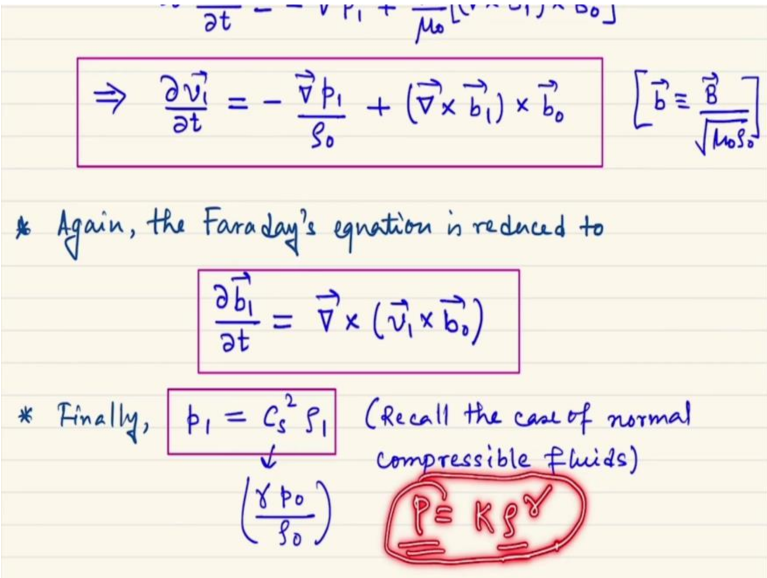

So, polytropic fluid means $p = K\rho^\gamma$. But when we are talking in terms of the first order perturbations, if you recall what we did for ordinary hydrodynamic fluid, we said that even if the equation is polytropic there will be always a proportionality relation between the first order perturbation in pressure and first order perturbation in density, $p_1 = c_s^2 \rho_1$. And the proportionality constant is nothing but the equilibrium sound speed square. So $c_s^2 = \frac{\gamma p_0}{\rho_0}$.

So, we are done. We have four equations which we need.

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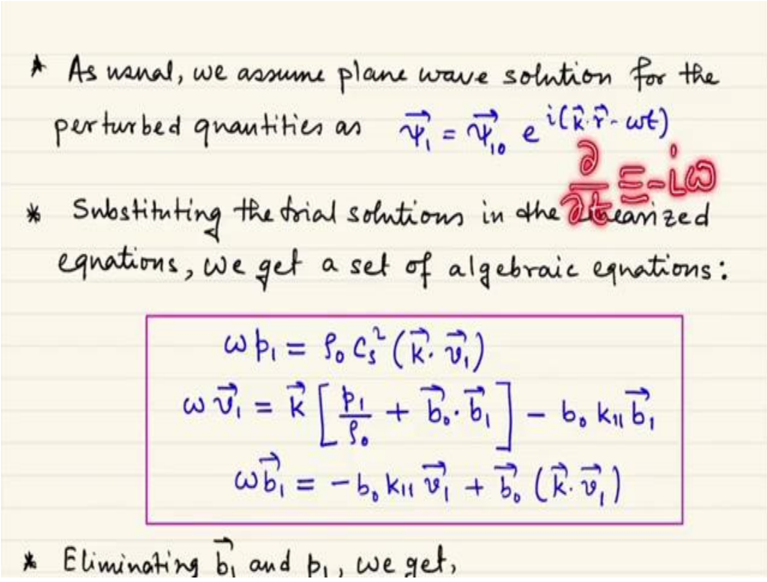

So, now what we do? As a next step we simply assume plane wave solutions for the perturbed quantities. So, for an arbitrary perturbed quantity $\boldsymbol{\psi}$, we can write the perturbed quantity as $\boldsymbol{\psi}_{10}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$.

So, you can also reason this in this part that as the equations are linear in nature for all $\boldsymbol{b}_1$, $\boldsymbol{v}_1$, p_1, ρ_1, so for all these quantities simply we can assume the solution as a sum of the Fourier components. And there we just choose one single Fourier component and we will study the relation between the frequency and the wave vector for that particular Fourier component. So, that is the philosophy of this treatment.

So, then finally, you can just substitute all this type of plane wave solutions in the previous equations. And finally, you will be given $\omega p_1 = \rho_0 c_s^2(\boldsymbol{k}\cdot\boldsymbol{v}_1)$. Now, this one is not exactly obtained from the continuity equation.

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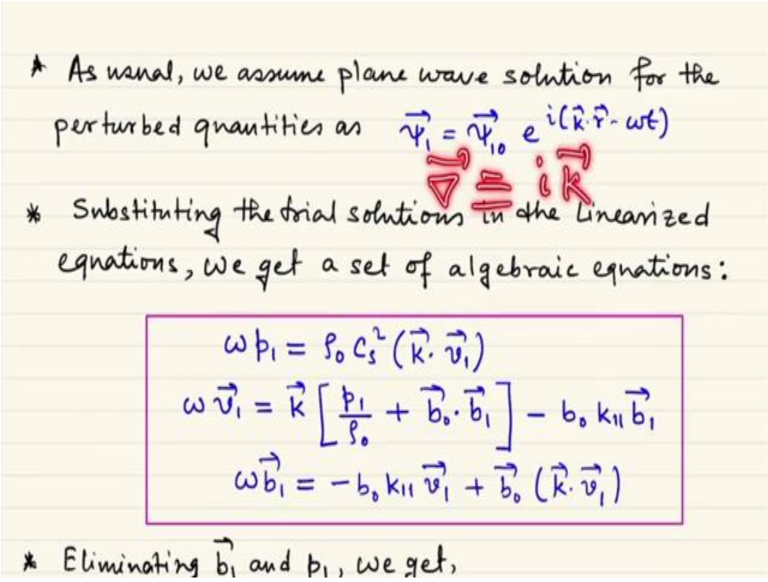

So, if you just assume plane wave type of solution this one then any $\frac{\partial}{\partial t}$ will be simply converted into $-i\omega$ and any $\boldsymbol{\nabla}$ will be converted with $i\boldsymbol{k}$. So, $\frac{\partial \rho_1}{\partial t} = -i\omega\rho_{10}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ and $\rho_0(\boldsymbol{\nabla}\cdot\boldsymbol{v}_1) = i\rho_{10}(\boldsymbol{k}\cdot\boldsymbol{v}_1)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$.

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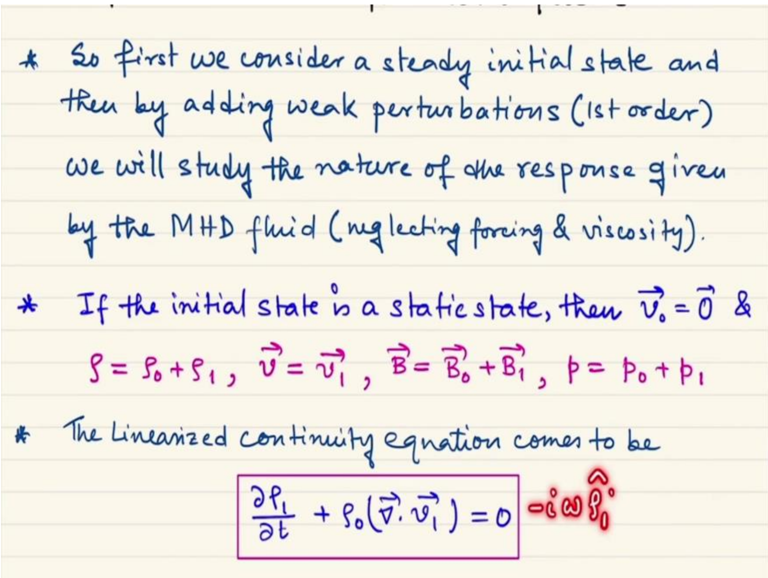

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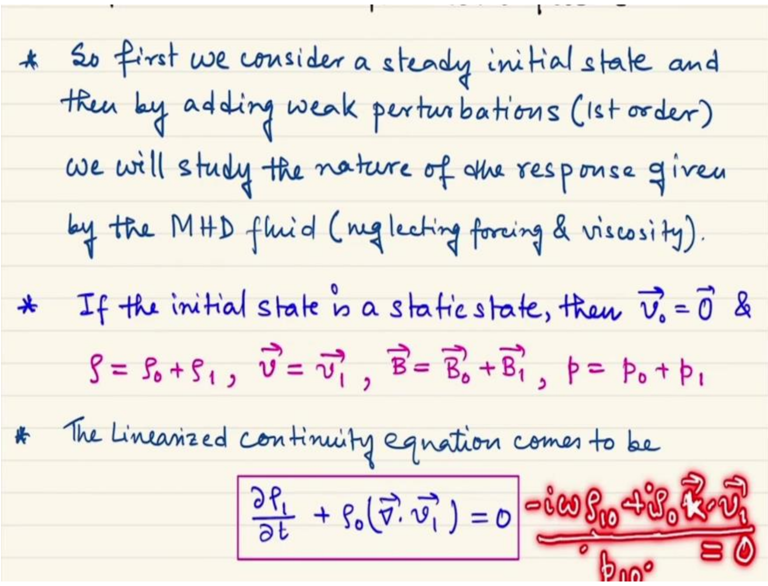

So the continuity equation becomes $-i\omega\rho_{10} + i\rho_{10}(\boldsymbol{k} \cdot \boldsymbol{v}_1) = 0$. Now, you just multiply c_s^2 throughout. So, $c_s^2\rho_{10}$ will give you p_{10}. And then you just take i out of the equation, and you have

$$\omega p_1 = \rho_0 c_s^2(\boldsymbol{k} \cdot \boldsymbol{v}_1).$$

So, once again, this is very easy to understand, you just have to multiply c_s^2 to the both sides of the linearized continuity equation.

And the linearized momentum evolution equation gives you simply

$$\omega \boldsymbol{v}_1 = \boldsymbol{k}\left[\frac{p_1}{\rho_0} + \boldsymbol{b}_0 \cdot \boldsymbol{b}_1\right] - b_0 k_\parallel \boldsymbol{b}_1.$$

Now, what is $k_\parallel$? $k_\parallel$ is nothing but the projection of the propagation vector in the direction of $\boldsymbol{b}_0$. So, if you want, this is nothing but $(\boldsymbol{b}_0 \cdot \boldsymbol{k})$.

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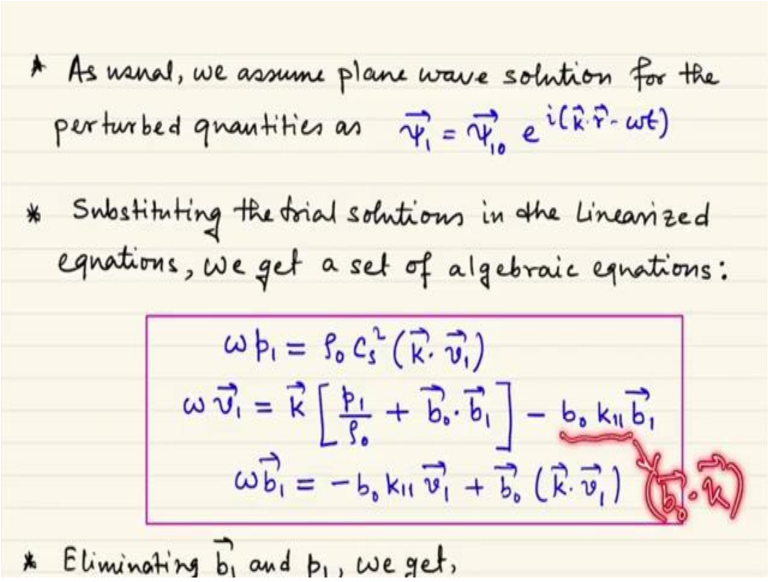

And the induction equation, as one can easily guess, will be equal to

$$\omega \boldsymbol{b}_1 = -b_0 k_{\parallel} \boldsymbol{v}_1 + b_0(\boldsymbol{k}\cdot\boldsymbol{v}_1).$$

So, once again $b_0 k_{\parallel}$ is nothing but $(\boldsymbol{b}_0 \cdot \boldsymbol{k})$.

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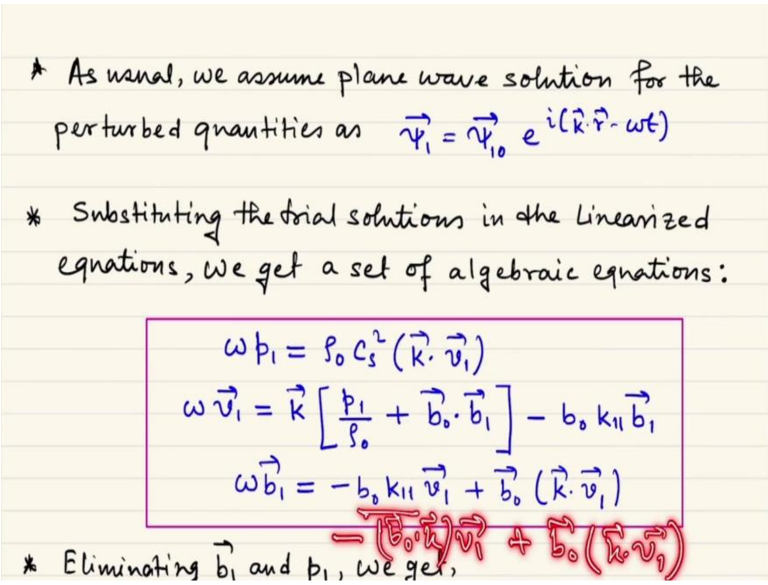

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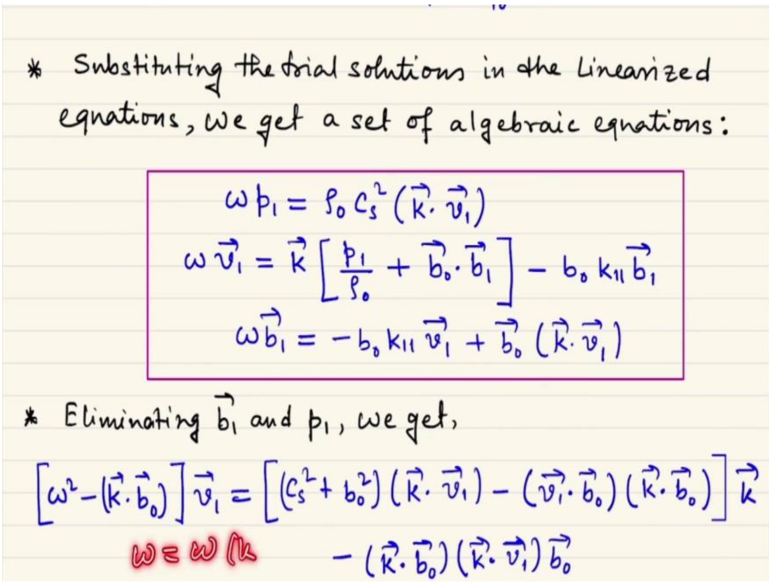

So, you already can see that this gives us the complete set of equations which are no longer differential equations, but they are set of algebraic equations.

Now, we have three algebraic equations and we have actually three unknowns. Which are those three unknowns? $\boldsymbol{v}_1$, $\boldsymbol{b}_1$, p_1. And our basic objective is to get rid of all these three to get the relation between ω and $\boldsymbol{k}$, which is known as the dispersion relation. And if that dispersion relation can give us real frequency, then we will confirm that there will be linear wave modes, if there will be an imaginary part in the ω then we have the possibility of a linear instability.

So, if we just eliminate at the first step $\boldsymbol{b}_1$ and p_1, we simply get

$$[\omega^2 - (\boldsymbol{k}\cdot\boldsymbol{b}_0)]\boldsymbol{v}_1 = [(c_s^2 + b_0^2)(\boldsymbol{k}\cdot\boldsymbol{v}_1) - (\boldsymbol{v}_1\cdot\boldsymbol{b}_0)(\boldsymbol{k}\cdot\boldsymbol{b}_0)]\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{b}_0)(\boldsymbol{k}\cdot\boldsymbol{v}_1)\boldsymbol{b}_0.$$

So, this is the whole thing where you cannot see p_1, you cannot see $\boldsymbol{b}_1$, the only variable is $\boldsymbol{v}_1$.

Now, so this is another technique when you have an equation like this, so directly you cannot find from here, ω as a function of k. So, first of all you have to get rid of two unknowns and then you just reduce everything in only one vectorial unknown and here which is the $\boldsymbol{v}_1$.

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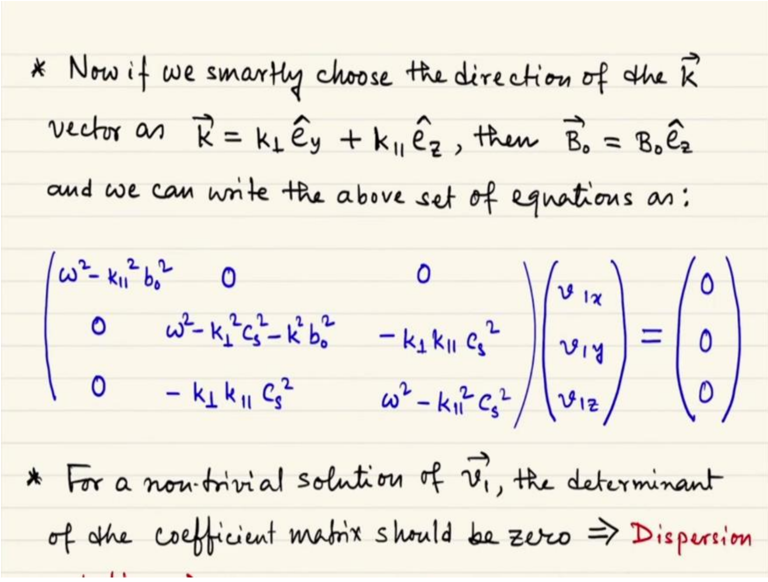

And then what you have to do? This is a classical trick that you have to learn. So, now, first of all we would do some smart choice of the direction of the propagation vector, and you will see why this is very good. We just choose without losing any generality that our $\boldsymbol{k}$ is having two components, $\boldsymbol{k} = k_\perp \hat{\boldsymbol{e}}_{\boldsymbol{y}} + k_\parallel \hat{\boldsymbol{e}}_{\boldsymbol{z}}$.

So, along x-direction there is no component for $\boldsymbol{k}$, only $k_\perp$ is along y-direction and $k_\parallel$ is about z-direction, the parallel sign comes when it is parallel to $\boldsymbol{B}_0$ or $\boldsymbol{b}_0$. So, it is simply written as $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{e}}_{\boldsymbol{z}}$ and then you can simply say that the y-direction will be perpendicular to $\boldsymbol{B}_0$.

If we project those equations, in this choice of k and $\boldsymbol{B}_0$, then the total equation, as you can easily understand, it is a vector equation in $\boldsymbol{v}_1$. So, there should be three component equations, and we can write the whole set of three component scalar equations corresponding to every component in matrix form and this matrix is now written in a block form

$$\begin{pmatrix} \omega^2 - k_\parallel^2 b_0^2 & 0 & 0 \\ 0 & \omega^2 - k_\perp^2 c_s^2 - k^2 b_0^2 & -k_\perp k_\parallel c_s^2 \\ 0 & -k_\perp k_\parallel c_s^2 & \omega^2 - k_\parallel^2 c_s^2 \end{pmatrix} \begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The ideal would be if that would be totally diagonalize; that means, you have terms only on diagonal positions and otherwise nothing anywhere. But this is a bit worse than that, but still we should be happy because at least we can decouple one mode totally from the other two modes.

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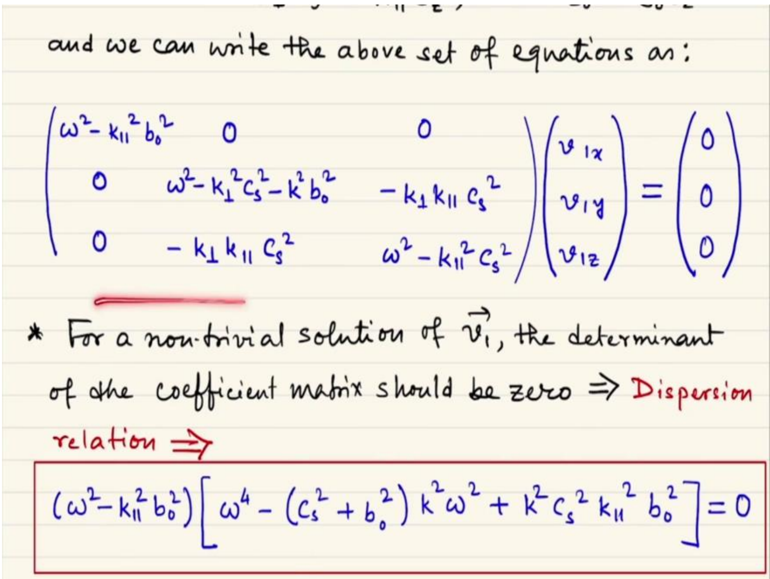

So, if you now write this in this matrix form then one can easily say that from the linear algebra that for nontrivial solution of $\boldsymbol{v}_1$, what is the necessary and the required condition? that the determinant of this whole matrix must vanish. And this when you calculate the determinant of the matrix and will equate it to 0 that will give you the dispersion relation. So, just remember this trick that trick is actually is useful for many general systems.

And then the dispersion relation finally is free of $\boldsymbol{v}_1$, $\boldsymbol{b}_1$ and p_1 will simply carry relation between ω, k and other constants of the system. So, what is that relation?

$$\left(\omega^2 - k_\parallel^2 b_0^2\right)\left[\omega^4 - (c_s^2 + b_0^2)k^2\omega^2 + k^2 c_s^2 k_\parallel^2 b_0^2\right] = 0.$$

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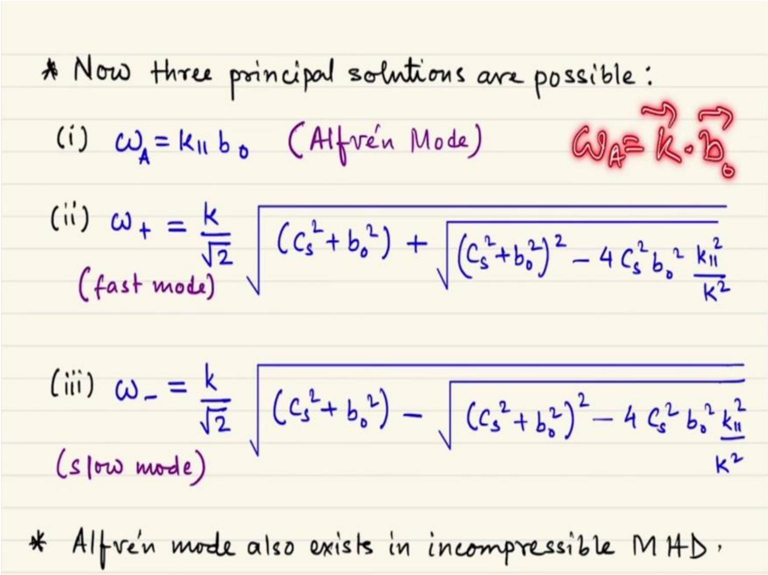

Now, we can easily understand that from this one since we are not talking about the negative solutions of ω. So, just considering positive solutions of ω, we can see that there are three solutions which are possible.

So, the first one is of course $\omega_A = k_\parallel b_0$. So, another way of writing this is, ω_A is nothing but $\boldsymbol{k} \cdot \boldsymbol{b}_0$, that is another point way of writing. And this mode is called Alfvén mode. This mode is of super importance in different aspects of MHD, including MHD turbulence, other phenomena like heating, high correlation between $\boldsymbol{v}$ and $\boldsymbol{b}$ etc.

Now, the other two solutions which can be obtained from this equation,

$$\omega_+ = \frac{k}{\sqrt{2}}\sqrt{(c_s^2 + b_0^2) + \sqrt{(c_s^2 + b_0^2)^2 - 4c_s^2 b_0^2 \frac{k_\parallel^2}{k^2}}},$$

this is called the fast mode. And we also have

$$\omega_- = \frac{k}{\sqrt{2}}\sqrt{(c_s^2 + b_0^2) - \sqrt{(c_s^2 + b_0^2)^2 - 4c_s^2 b_0^2 \frac{k_\parallel^2}{k^2}}},$$

called the slow mode.

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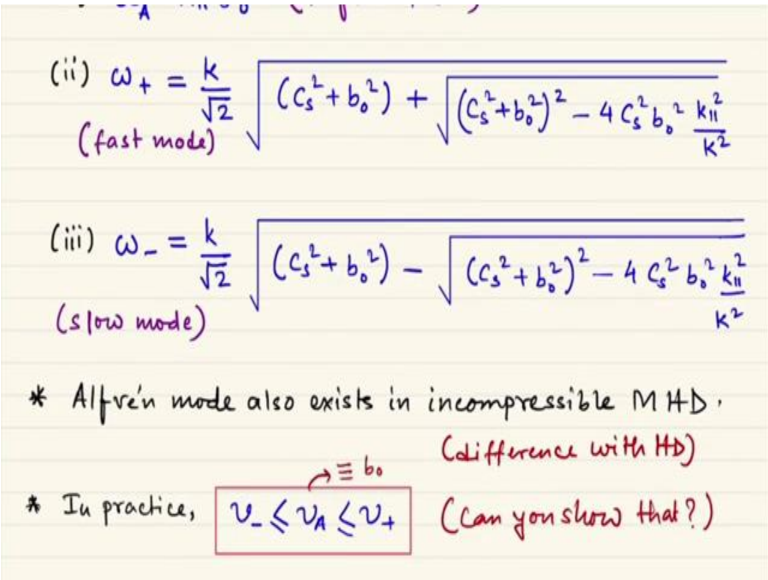

Now, it is true that when your equations are incompressible MHDs equation, that means, all the velocity vector is solenoidal, all the $\boldsymbol{k} \cdot \boldsymbol{v}$ type of things are 0, because divergence of $\boldsymbol{v}$ is equal to 0.

So, in this type of case you can simply understand that Alfvén mode is an incompressible MHD mode because at this condition only one mode survives and that is the Alfvén mode. That you can actually check. So, my suggestion will be to start directly from the linearized equations and try to understand which thing should be changed if you have incompressible MHD. So, actually you will see that this is somehow very easy to show that only one wave mode will then be retained and should have a dispersion relation like $(\omega^2 - k_\parallel^2 b_0^2) = 0$.

Then another point is that actually one can show, that the phase velocity of the slow mode is actually less than or equal to the phase velocity of the Alfvén mode and that will be less than equal to the phase velocity of the fast mode i.e., $v_- \leq v_A \leq v_+$.

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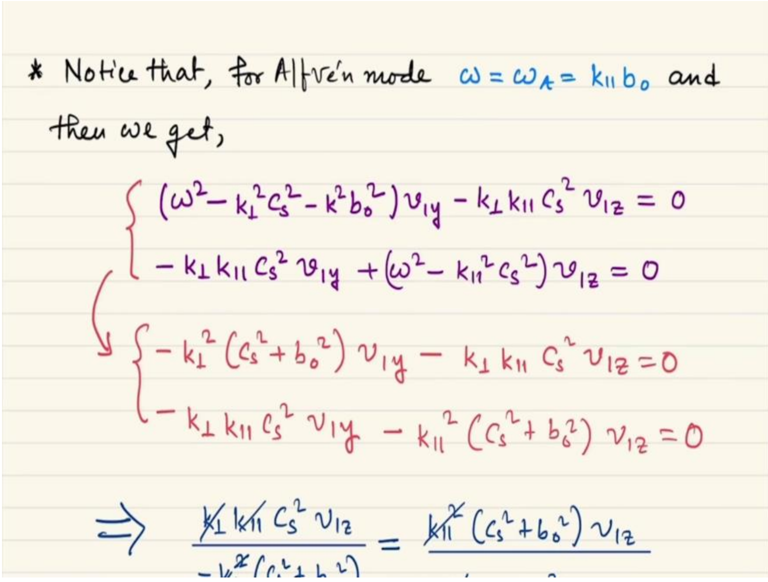

But before that let us do something very interesting. First of all, I said that if your set of equations are of incompressible nature, then I said that just you can check and verify that the only dispersion relation which you get is that of the Alfvén modes. But we can do something very easily. We can just say that if we just take the dispersion relation of the Alfvén mode which is $\omega = \omega_A = k_\parallel b_0$, and then we simply write the expressions v_{1x}, v_{1y} and v_{1z}, obtained from the matrix above, we have for v_{1x}, $(\omega^2 - k_\parallel b_0^2) v_{1x} = 0$ and which simply says that as $(\omega^2 - k_\parallel b_0^2)$ is 0 then this equation will be true even when v_{1x} is nonzero.

Now, what happens for other two equations? So, for example, if you write those two equations in a set of linear equations then you will see that it is

$$(\omega^2 - k_\perp^2 c_s^2 - k^2 b_0^2) v_{1y} - k_\perp k_\parallel c_s^2 v_{1z} = 0,$$

$$-k_\perp k_\parallel c_s^2 v_{1y} + \left(\omega^2 - k_\parallel^2 c_s^2\right) v_{1z} = 0.$$

Now, I am just checking how does it behave when $\omega = k_\parallel b_0$.

Now, if I just use this value for ω, then I will simply get

$$-k_\perp^2 (c_s^2 + b_0^2) v_{1y} - k_\perp k_\parallel c_s^2 v_{1z} = 0,$$

$$-k_\perp k_\parallel c_s^2 v_{1y} - k_\parallel^2 (c_s^2 + b_0^2) v_{1z} = 0.$$

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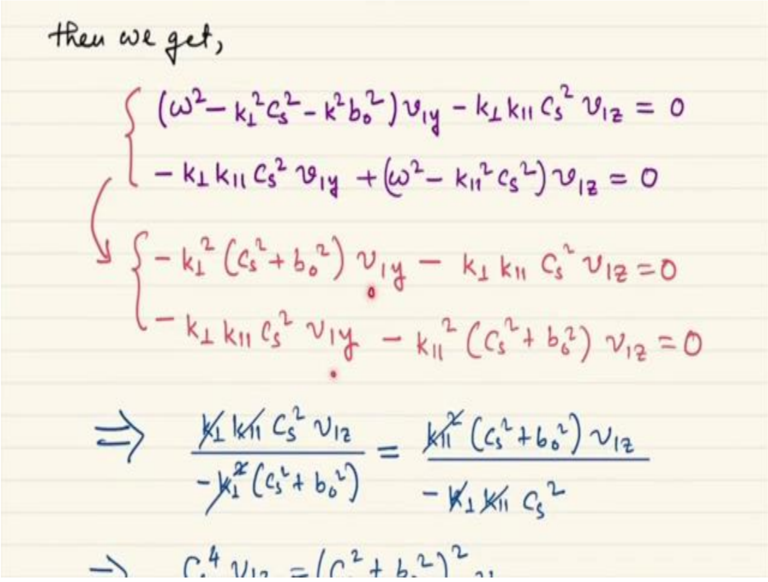

So, I just eliminate v_{1y} from both these equations and after necessary cancellations, you will have the relation, $c_s^4 v_{1z} = (c_s^2 + b_0^2)^2 v_{1z}$.

So, one possibility is of course, v_{1z} is 0, the other possibility is $c_s^4 = (c_s^2 + b_0^2)^2$. As c_s^2 should be positive, then $c_s^2 = (c_s^2 + b_0^2)$.

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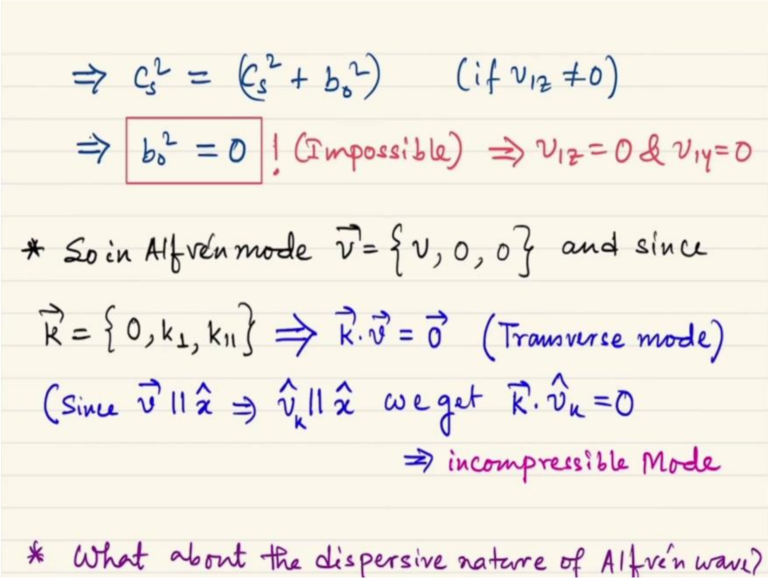

As $v_{1z} \neq 0$, that simply gives us $b_0^2 = 0$, and which is not possible, which simply violates our main assumption that there is a magnetic field. Otherwise, this is no longer an MHD fluid. So, this is impossible. So, v_{1z} must be equal to 0. And similarly, you can show v_{1y} is also 0.

So, the only nonzero component of $\boldsymbol{v}$ is the x-component. And so, our $\boldsymbol{v}$ can be written as $\boldsymbol{v} = \{v, 0,0\}$, but our propagation vector is $\boldsymbol{k} = \{0, k_\perp, k_\parallel\}$. So, we can simply write that $\boldsymbol{k} \cdot \boldsymbol{v} = 0$. So, the particle velocity is perpendicular to the wave propagation and that is the definition of a transverse mode.

So, when you simply pluck at one point of a tense string and then release it, then basically the string vibrates and the vibration is propagating in the direction perpendicular to the particles' vibration. So, this is nothing but a classical example of our transverse wave. And you all know that light, heat, all these electromagnetic waves in vacuum are always perfectly transverse waves. So, transverse simply means that the particle velocity and the propagation direction they are mutually perpendicular. Now, simply once again, so as this is the case $\boldsymbol{k} \cdot \boldsymbol{v}$, so in our case $\boldsymbol{v}$ is parallel to x, and so the Fourier component of $\boldsymbol{v}$ is also parallel to x. And then we can actually get $\boldsymbol{k} \cdot \boldsymbol{v_k} = 0$. So, it simply says, this is nothing but the incompressibility condition. So, this wave mode is also incompressible. And that is why even in this wave mode appears as a totally decoupled mode from the other two modes in the compressible MHD, the nature of this mode is in incompressible in nature.

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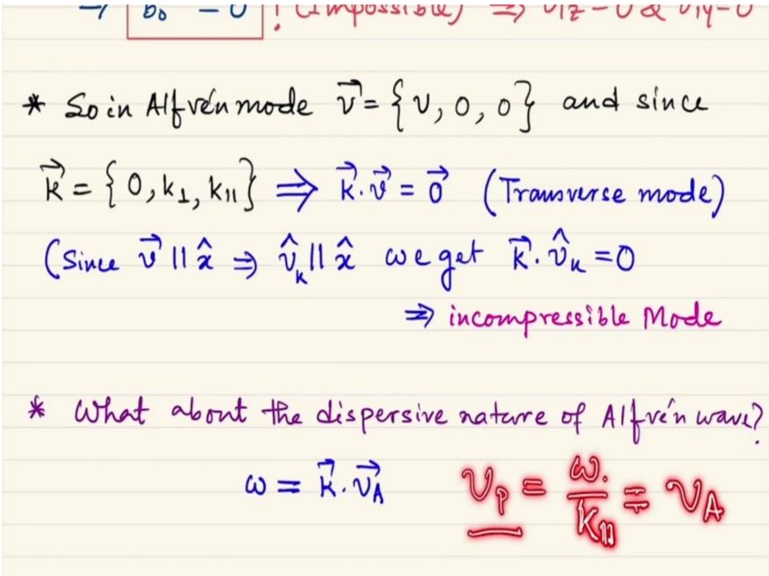

Now, my question is, what about the dispersive nature of Alfvén wave? Well, the dispersion relation is $\omega = \boldsymbol{k} \cdot \boldsymbol{v}_A$ and here the question is how to define the phase velocity?

So, it's ω by the wave vector parallel to the direction of propagation ($k_{\parallel}$), and that is exactly the proper definition of phase velocity and that is simply given here just by v_A.

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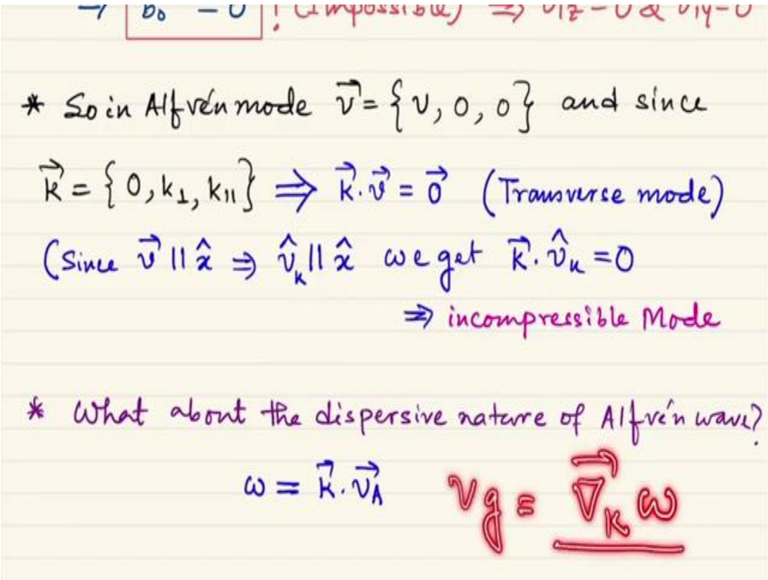

On the other hand, if you do group velocity, it is nothing but $v_g = |\boldsymbol{\nabla}_{\boldsymbol{k}}\omega|$. And if you do this one rightly, you will see that is also equal to v_A.

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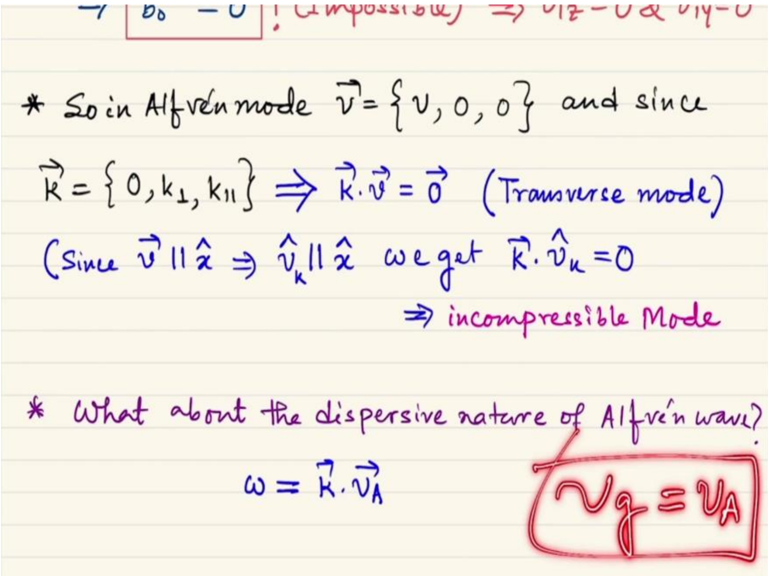

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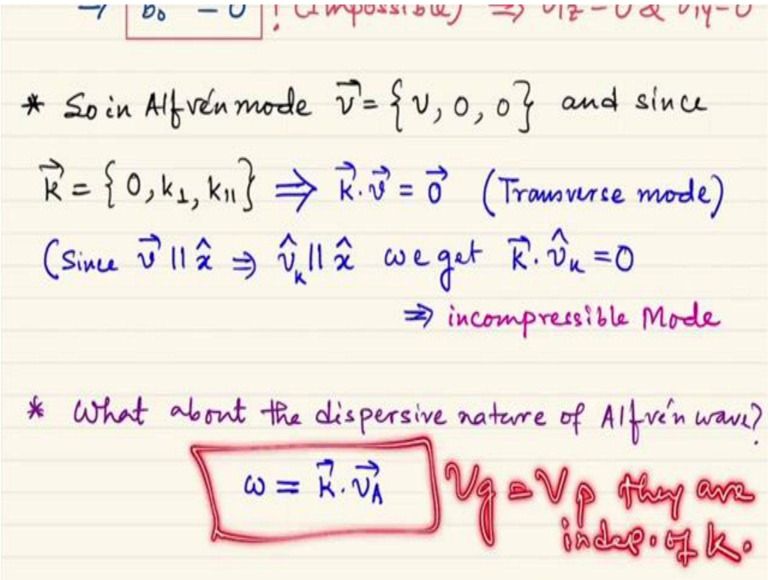

So, v_g is equal to v_p and they are they are independent of $\boldsymbol{k}$. So, this is a non-dispersive mode. Non-dispersive mode as $v_p = \frac{\omega}{k_\parallel} = v_g = |\boldsymbol{\nabla_k} \omega|$.

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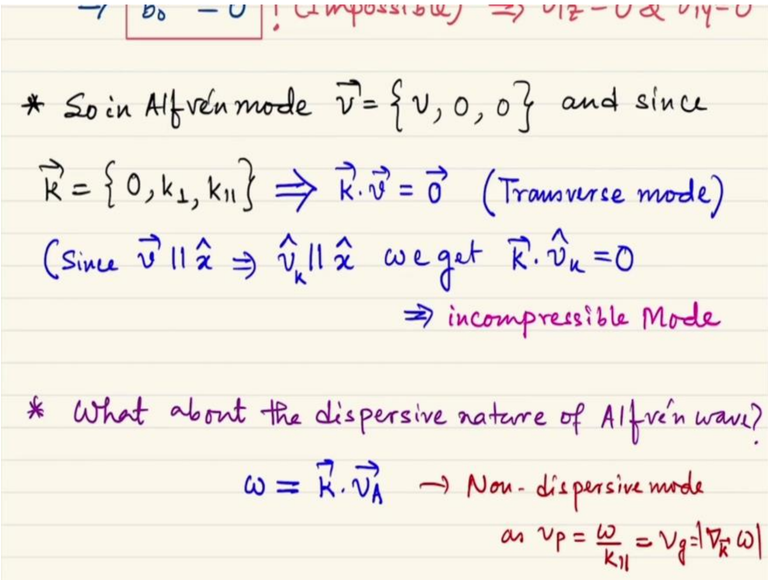

So, this is the brief story about the Alfvén mode. What about the other two?

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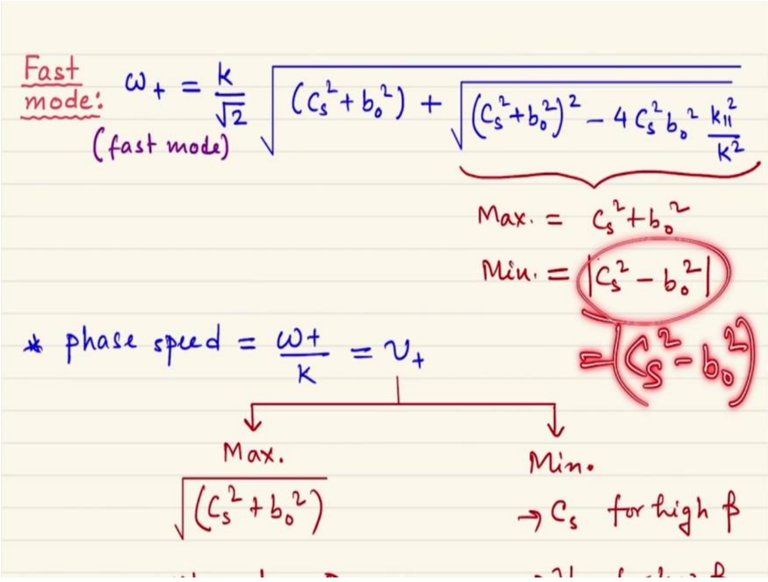

So, there are two other modes, one is called the fast mode, and other, the slow mode. So, the fast mode has the dispersion relation

$$\omega_+ = \frac{k}{\sqrt{2}} \sqrt{(c_s^2 + b_0^2) + \sqrt{(c_s^2 + b_0^2)^2 - 4c_s^2 b_0^2 \frac{k_\parallel^2}{k^2}}}.$$

So, just try to understand the factor, $\sqrt{(c_s^2 + b_0^2)^2 - 4c_s^2 b_0^2 \frac{k_\parallel^2}{k^2}}$, can have a maximum value when $4c_s^2 b_0^2 \frac{k_\parallel^2}{k^2}$ is minimum, because this is getting subtracted and this term is minimum when $k_\parallel$ is equal to 0.

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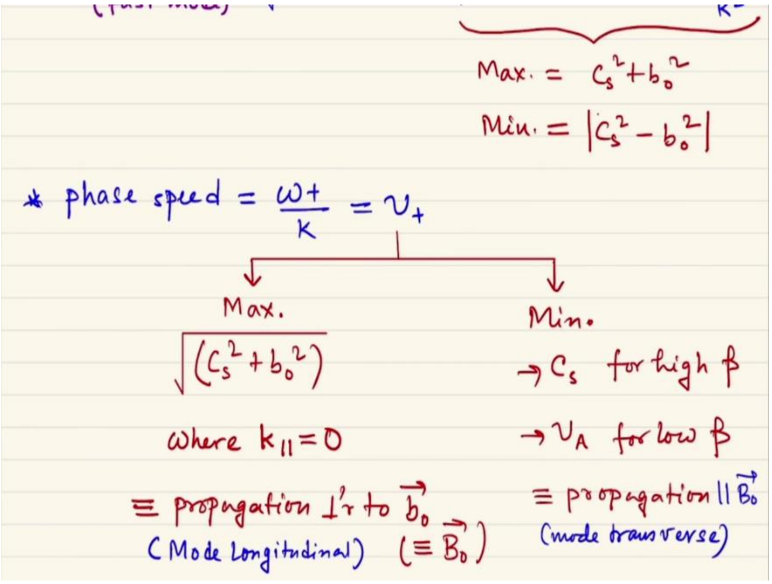

So, that means, when the propagation is perpendicular to $\boldsymbol{b}_0$. That means, the propagation vector has no parallel component.

Now, so that is the maximum case and the minimum case is when $k_{\parallel}$ is almost equal to k. And then what happens? The factor inside the second square root is $|c_s^2 - b_0^2|$. Now, we do not know which one is greater.

So, if your β is greater than 1, then $c_s > b_0$, so it should be then this one should be equal to $(c_s^2 - b_0^2)$. And if we are considering the case of a low β, then this one will be simply $(b_0^2 - c_s^2)$.

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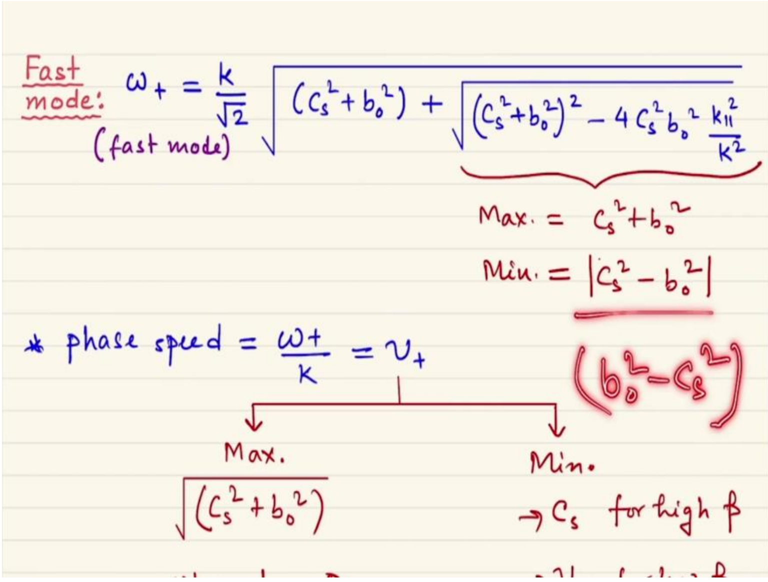

So, in the first case, what happens?

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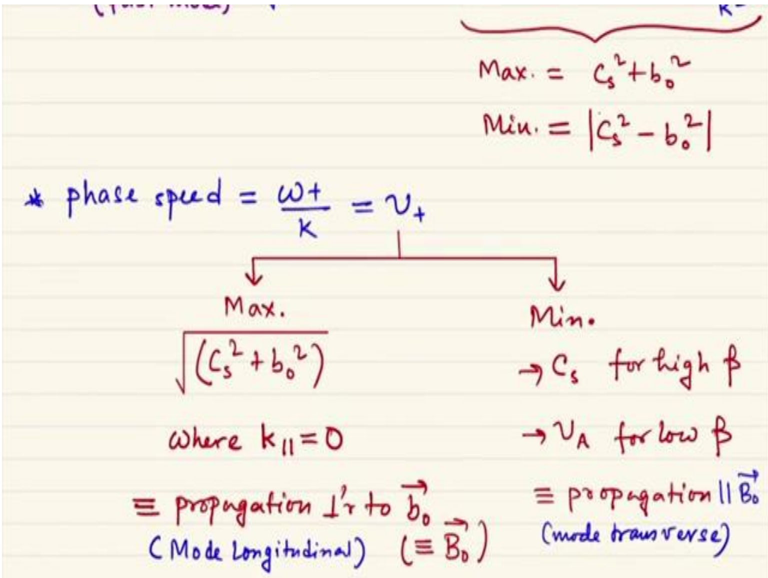

So, phase speed for the maximum speed is given by $v_+ = \frac{\omega_+}{k} = \sqrt{c_s^2 + b_0^2}$.

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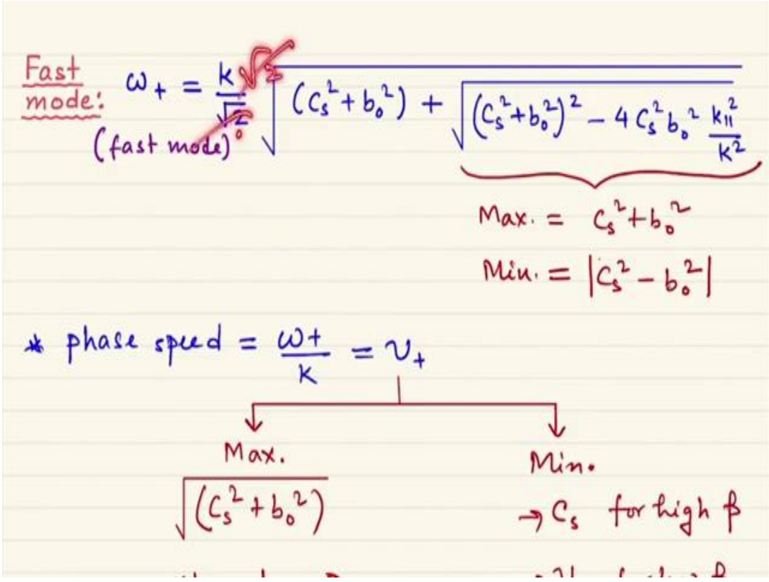

Now, what will be the minimum phase speed?

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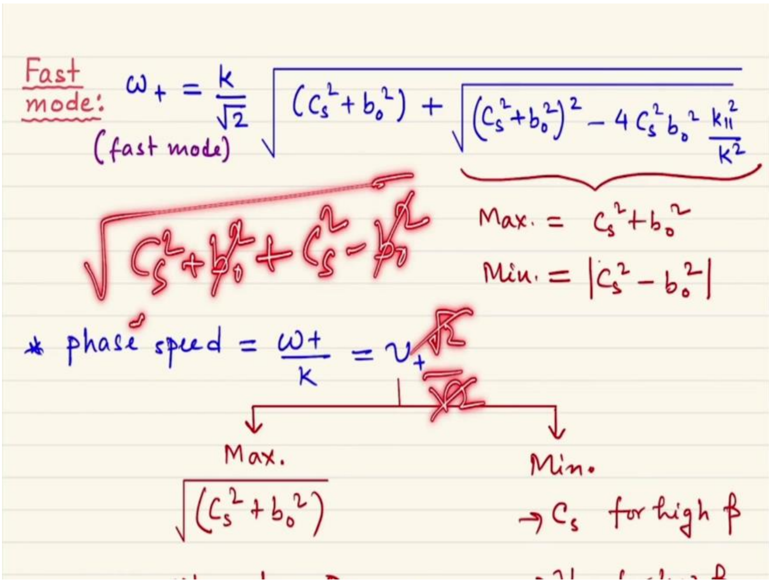

For high β, we have $v_p = c_s$. So, for high β, the fast mode reduces to nothing but an acoustic mode, sound wave mode.

And for low β the fast mode reduces to an Alfvén mode. And that is actually when its propagation is parallel (error in lecture) to $\boldsymbol{b}_0$, because this is minimum when $k_\parallel$ is almost equal to the k.

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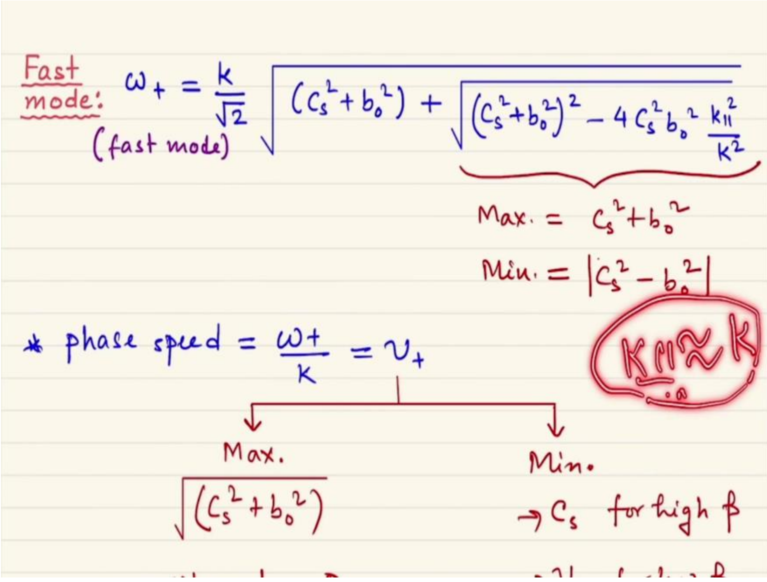

So, that means, the propagation vector is almost aligned with respect to the $\boldsymbol{b}_0$. But this is also the transverse mode.

So, you see that the minimum phase speed corresponds to the sound speed for high β and corresponds to Alfvén speed for low β. I must also mention that b_0 is also the Alfvén speed for us i.e., $b_0 = v_A$.

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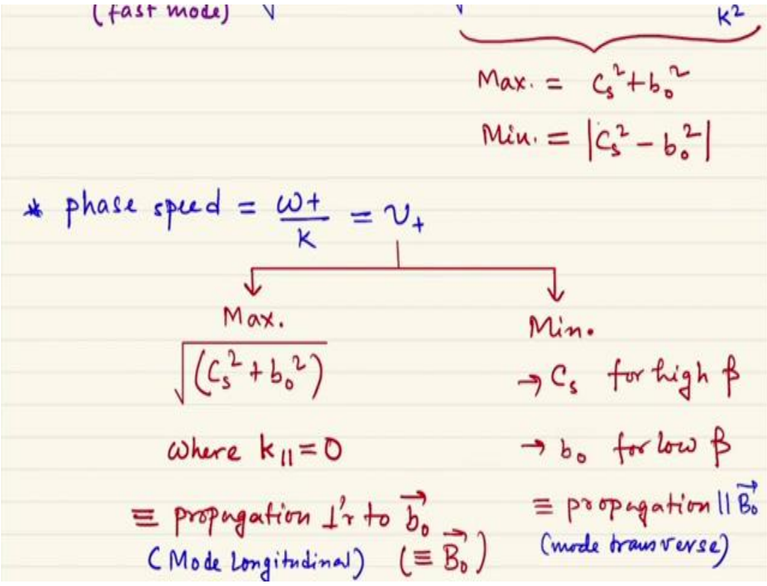

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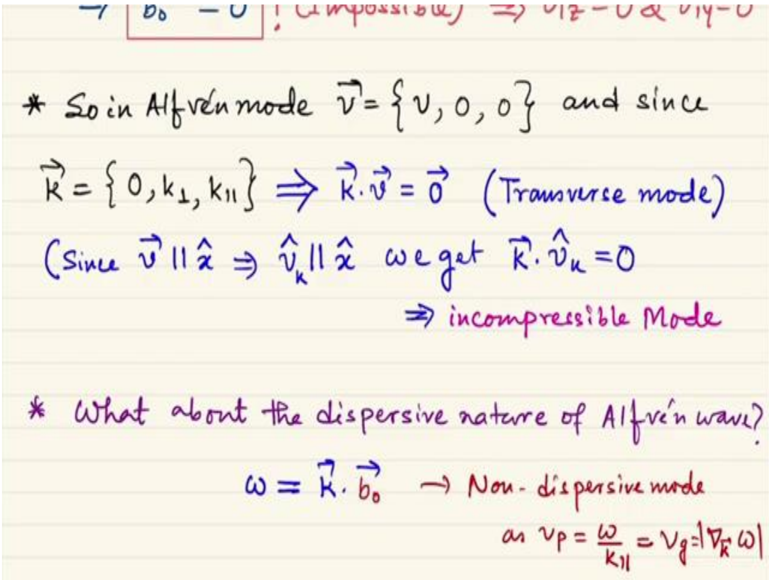

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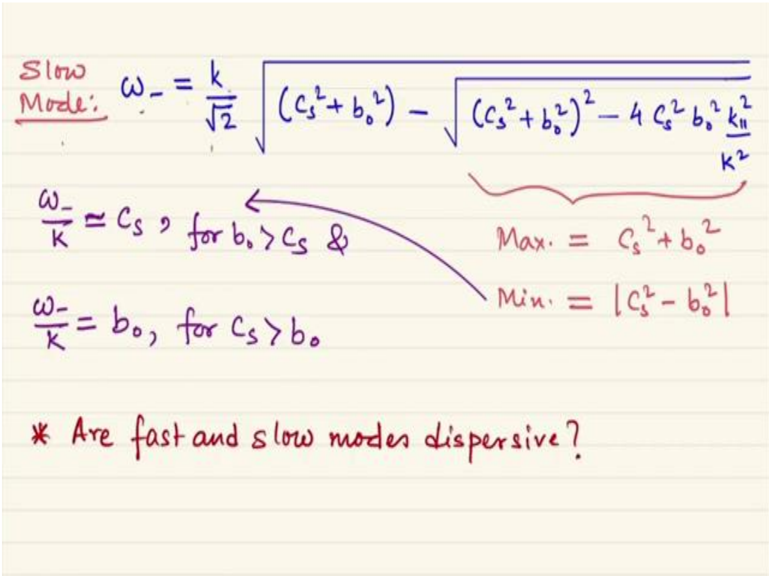

Now what is the story for slow mode? So, for slow modes what happens? So,

$$\omega_- = \frac{k}{\sqrt{2}}\sqrt{(c_s^2 + b_0^2) - \sqrt{(c_s^2 + b_0^2)^2 - 4c_s^2 b_0^2 \frac{k_\parallel^2}{k^2}}}.$$

Proceeding as above, we obtain $\frac{\omega_-}{k} = c_s$ for $b_0 > c_s$ and $\frac{\omega_-}{k} = b_0$ for $c_s > b_0$.

So, if your system has a high β, the fast mode becomes acoustic, but the slow mode becomes Alfvénic. When your system has a low β, the fast mode becomes Alfvénic, the slow mode becomes acoustic. Now, my question to you to think, are fast and slow modes dispersive? And actually if you think very deeply you will see that none of these three modes are dispersive in nature in general.

So, here actually at this point I am just putting an end to the discussion of the waves in MHD. And it is a very vast chapter, there are several books which are only written on the waves in MHD. So, this is a vast subject of research and analysis.

So, now, in the context of space and astrophysics these wave modes are very very important because this leads to several phenomena and this controls different type of behavior of a MHD fluid. And that is why very good knowledge, even if not much mathematical knowledge, but at least a very good knowledge about those three modes are very much important.

So, one thing to be said that unlike the incompressible Alfvén mode the last two modes which are the fast mode and the slow mode, they are only appearing when your system is compressible, when you are talking about compressible MHD. So, that is why these two modes are known as sonic modes and they are called fast magnetosonic mode and slow magnetosonic mode.

Since, there are the two contributions in general, one is from magnetic field another is from the acoustic wave, that is why they are called fast and slow magnetosonic or magnetoacoustic modes, all these vocabularies are there.

So, from the next lecture we will start discussing two or three interesting applications of MHD and then I will pass to the discussion of turbulence as well.

Thank you very much.