Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 48
Elsasser variables in MHD

Hello, and welcome to another session of Introduction to Astrophysical Fluids. In this session, we will discuss a very small topic, all though very interesting that is the topic of Elsasser variables. So, in the year 1950, Elsasser wrote the whole set of MHD equations actually incompressible MHD equations in terms of Elsasser variables.

That specific style of writing actually revealed a number of very interesting facts and features of MHD which you we cannot see in general when we write the equation in terms of v and B.

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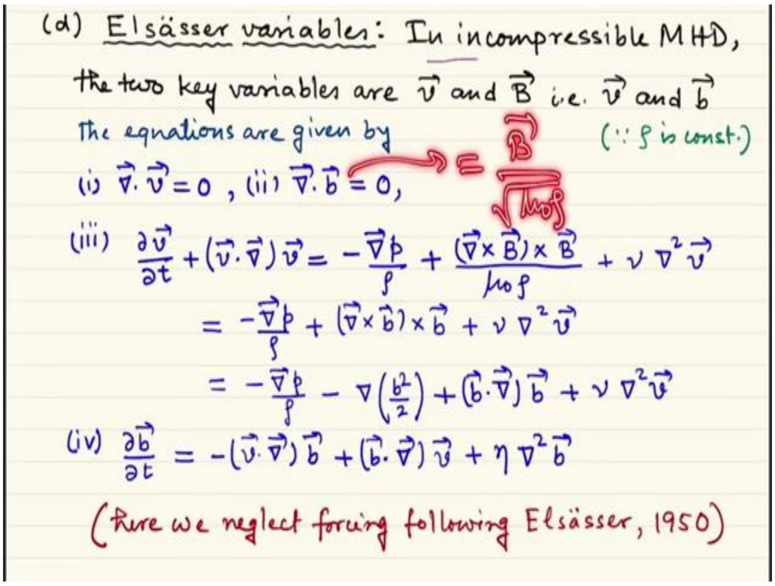

So, if you remember that the incompressible MHD, for example, the key variables are simply v and B, because ρ is a constant the fluid density is constant. So, simply we can write the whole thing in terms of v and B. Of course, we have pressure I mean we have a pressure like variable, but we also know from our previous discussion that finally pressure can be solved in terms of velocity, and finally, the message is that in order to construct a dynamical theory we simply need only v and b.

And when the fluid is incompressible, so incompressible MHD means that the fluid is incompressible, that means that is the divergence of the fluid velocity is 0, and we also know that the divergence of b, where b is nothing but $\frac{\vec{B}_0}{\sqrt{\mu_0 \rho}}$.

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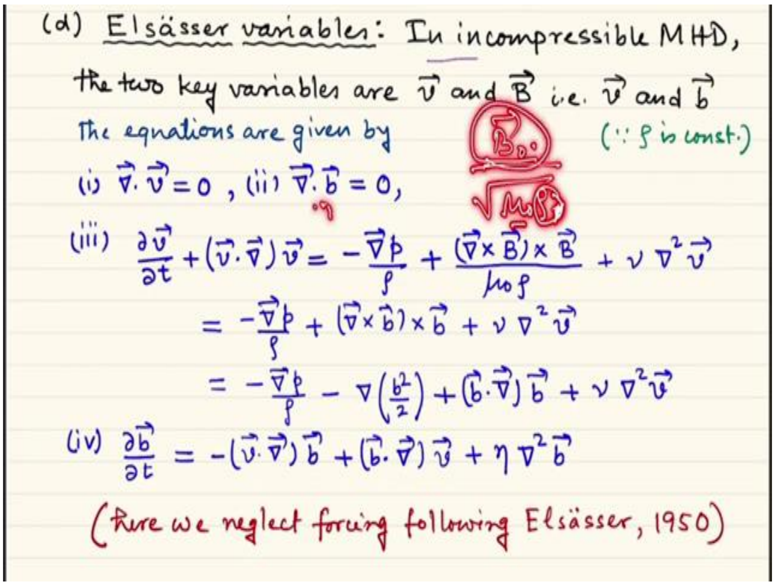

So, ρ is a constant. So, the divergence of b, small b is nothing but equivalent to some constant times divergence of $\vec{B}_0$, and divergence of $\vec{B}_0$ is always 0. So, this will also be divergence less vector, small v, and small b.

Now, we can write the two basic equations of incompressible MHD as this. The first one is the momentum collision equation. So,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = \frac{\vec{\nabla} p}{\rho} + \frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{\mu_0 \rho} + \vartheta \nabla^2 \vec{v} \qquad \text{(iii)}$$

So, that is the Lorentz force term we have already seen.

Here we are just neglecting the forcing term because in the original work of Elsasser in 1950, he also neglected the forcing term, but I mean if you include the forcing term that does not change any big thing because forcing term is in general, I mean independent of the force I mean of the fluid fields like v and b, so because this is an external forcing term.

However, here we just omit that, and we write this the right-hand side as the pressure gradient force, Lorentz force and the viscous force which is $\vartheta \nabla^2 \vec{v}$. Now, so this term is kept intact this term $\frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{\mu_0\rho}$ can be now written as $(\vec{\nabla}\times\vec{b})\times\vec{b}$ because every B is just divided by $\sqrt{\mu_0\rho}$, it gives one b.

So, this total equation will have the simple expression as $(\vec{\nabla}\times\vec{b})\times\vec{b}$ plus again the viscous term, and then this one $(\vec{\nabla}\times\vec{b})\times\vec{b}$ by using the famous vector identity you can write this one is equal to $-\nabla\left(\frac{b^2}{2}\right)+(\vec{b}.\vec{\nabla})\vec{b}$. So, this treatment we have already seen when we introduce magnetic pressure and magnetic tension right from Lorentz force.

So, now if we simply see the equation, it should look like

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}.\vec{\nabla})\vec{v}=\frac{\vec{\nabla}p}{\rho}+-\nabla\left(\frac{b^2}{2}\right)+(\vec{b}.\vec{\nabla})\vec{b}+\vartheta\nabla^2\vec{v}. \qquad \text{(iii)}$$

What about the equation for the magnetic field evolution?

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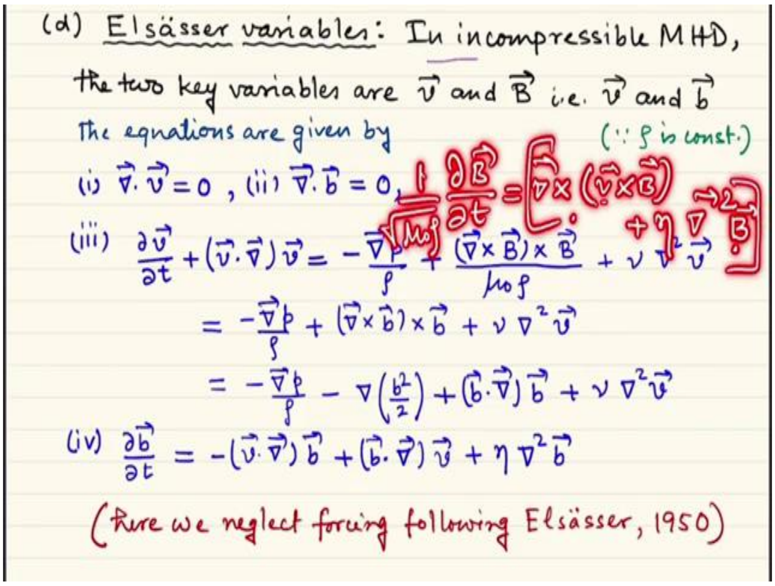

So, that we can actually write

$$\frac{\partial\vec{b}}{\partial t}=-(\vec{v}.\vec{\nabla})\vec{b}+(\vec{b}.\vec{\nabla})\vec{v}+\eta\nabla^2\vec{b}. \qquad \text{(iv)}$$

Now, we just divide both sides by $\sqrt{\mu_0\rho}$. So, this side as well.

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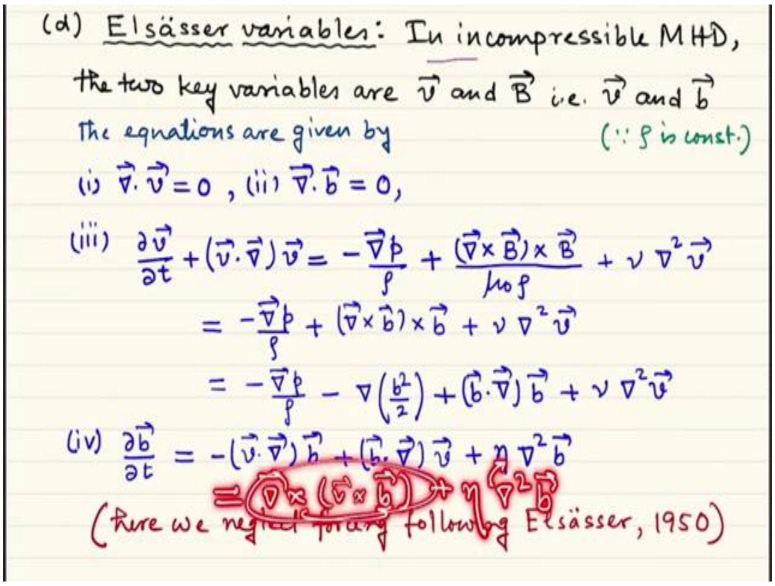

So, and then finally, you have this

$$\frac{\partial\vec{b}}{\partial t} = (\vec{\nabla}\times\vec{v})\times\vec{b}+\eta\nabla^2\vec{b}.$$

So, you see here we have the equation for both v and b, both are divergence less, and the evolution equation of both includes only terms like $(\vec{b}.\vec{\nabla})\vec{b}$, $(\vec{b}.\vec{\nabla})\vec{v}$, $(\vec{v}.\vec{\nabla})\vec{v}$, $(\vec{v}.\vec{\nabla})\vec{b}$, and gradient type of term. This $\frac{\vec{\nabla}p}{\rho}$ is also a pure gradient here, because ρ is a constant.

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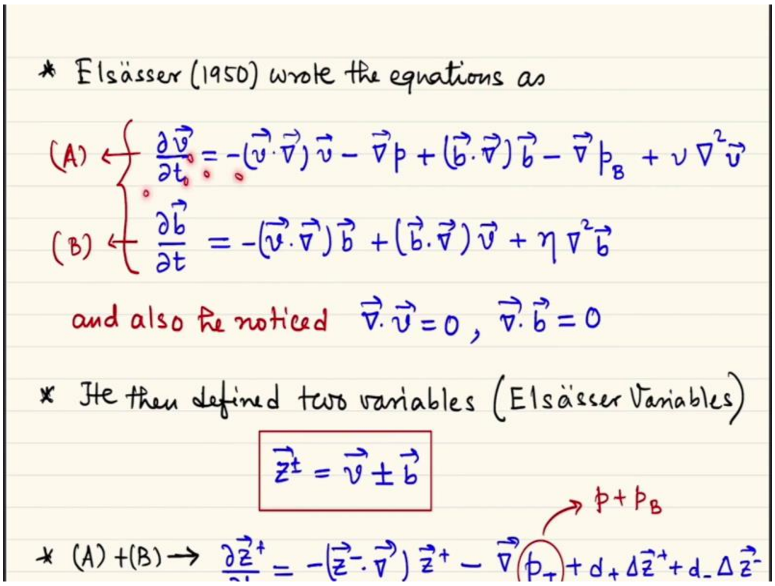

Elsasser just observed this thing and he just once again wrote this equation as this

$$\frac{\partial \vec{v}}{\partial t} = -(\vec{v}.\vec{\nabla})\vec{v} - \vec{\nabla}p + (\vec{b}.\vec{\nabla})\vec{b} - \vec{\nabla}p_B + \vartheta\nabla^2\vec{v} \qquad \text{(A)}$$

p_B is the magnetic pressure, and here he just said that the constant density is now normalized to unity. So, we just do not write any longer the ρ over here. We do not need, and then we write

$$\frac{\partial \vec{b}}{\partial t} = -(\vec{v}.\vec{\nabla})\vec{b} + (\vec{b}.\vec{\nabla})\vec{v} + \eta\nabla^2\vec{b} \qquad \text{(B)}$$

So, finally, he had these two equations.

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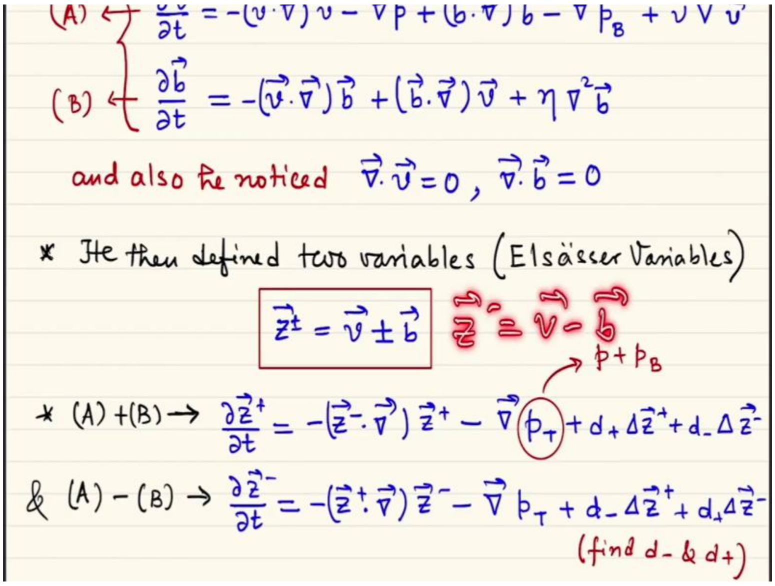

Then he actually wrote the whole equation I mean these set of two equations by redefining a new set of variables called the Elsasser variables which are nothing but $\vec{v} \pm \vec{b}$, right. So, both of them are of dimension of velocity. So, Elsasser variables are of the dimension of velocity. So, $\vec{Z}^+ = \vec{v} + \vec{b}$, and $\vec{Z}^- = \vec{v} - \vec{b}$.

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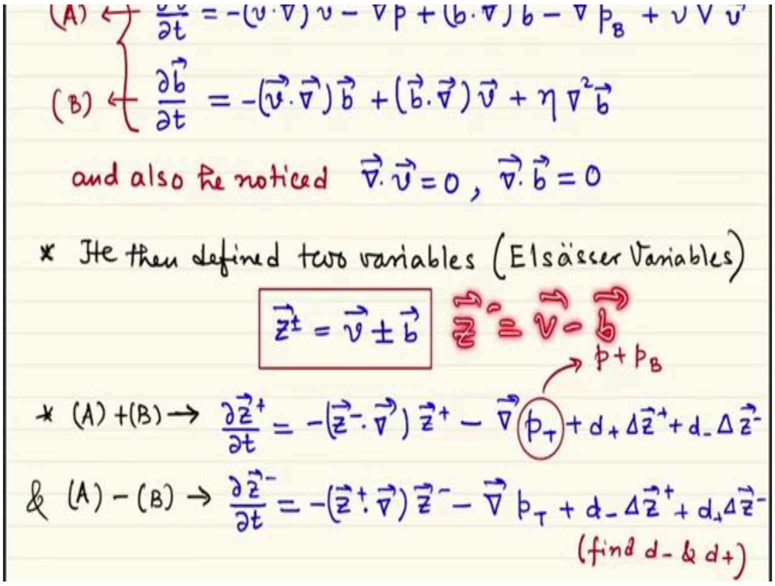

Now, if you just add (A) and (B), you will get the equation for $\vec{Z}^{+}$ and you will simply write

$$\frac{\partial \vec{Z}^{+}}{\partial t} = -(\vec{Z}^{-}.\vec{\nabla})\vec{Z}^{+} - \vec{\nabla} p_T + d_+ \Delta \vec{Z}^{+} + d_- \Delta \vec{Z}^{-},$$

p_T which is $p + p_B$, and

$$\frac{\partial \vec{Z}^{-}}{\partial t} = -(\vec{Z}^{+}.\vec{\nabla})\vec{Z}^{-} + \vec{\nabla} p_T + d_- \Delta \vec{Z}^{+} + d_+ \Delta \vec{Z}^{-}.$$

Now, this dissipation terms are written in a way so that this carries some symmetry.

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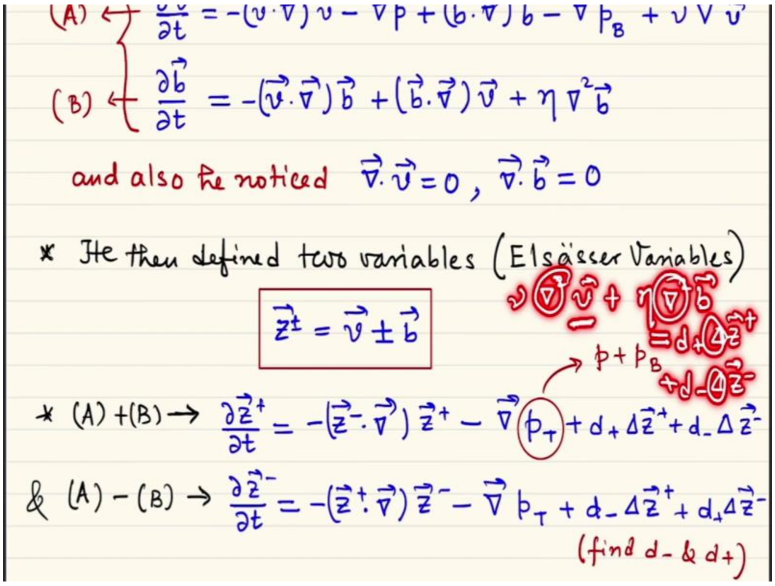

You will have the dissipation contribution like this $\vartheta\nabla^2\vec{v} + \eta\nabla^2\vec{b}$. So, we just say this one is equal to $d_+\Delta\vec{Z}^+ + d_-\Delta\vec{Z}^-$, of course here this sign they are exactly equal to ∇^2.

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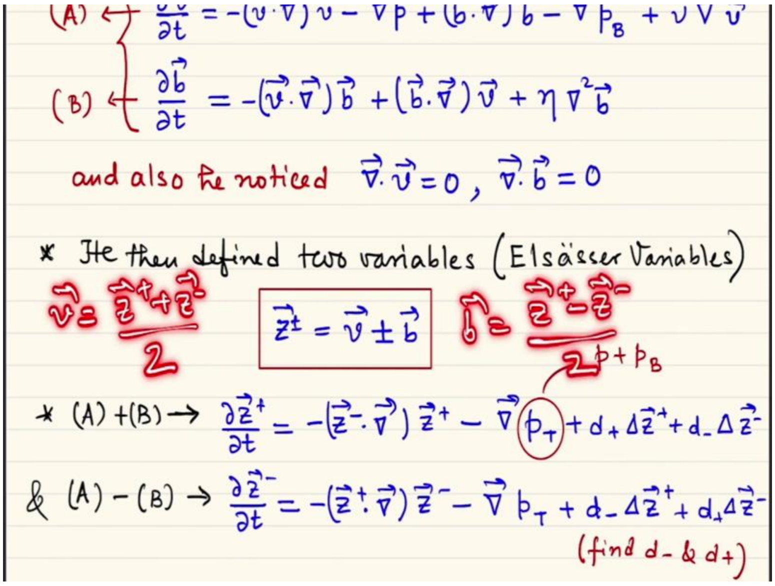

Then you just write $\vec{v}$ is equal to $\frac{\vec{Z}^+ + \vec{Z}^-}{2}$, and $\vec{b}$ will be equal to $\frac{\vec{Z}^+ - \vec{Z}^-}{2}$.

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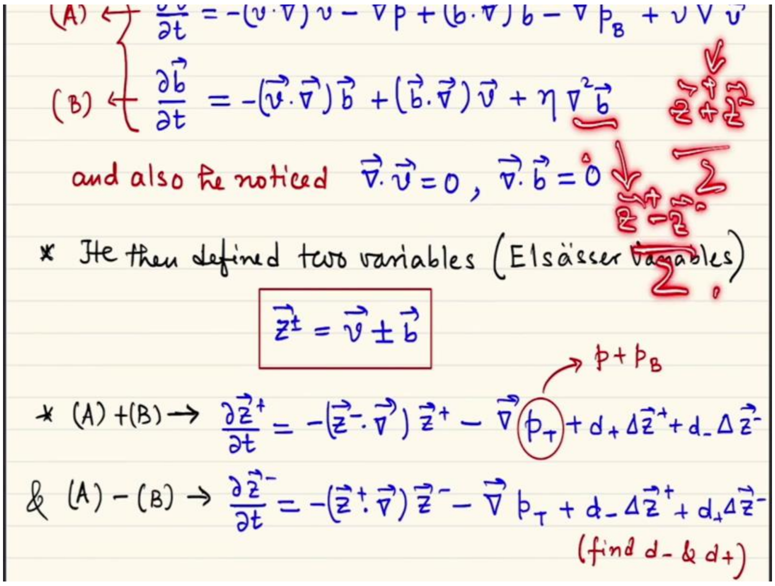

Then you can actually find d_+ and d_- in terms of ϑ and η, and I mean this is a simply a two-step calculation. So, I rather ask you to do.

Now, the interesting thing is that now we have the evolution equation for two variables which are the Elsasser variables $\vec{Z}^-$ and $\vec{Z}^+$. So, for $\vec{Z}^+$, you see that the advective term contains $\vec{Z}^-$, and for $\vec{Z}^-$, the advective term contains $\vec{Z}^+$.

So, $\vec{Z}^+$ is advected by $\vec{Z}^-$, and $\vec{Z}^-$ is advected by $\vec{Z}^+$, and fluid pressure is nothing but the total pressure which is the sum of the fluid pressure and the magnetic pressure. These specific writing is I mean particularly interesting when we will talk about MHD turbulence, and the phenomenology. (Refer Time: 12:06).

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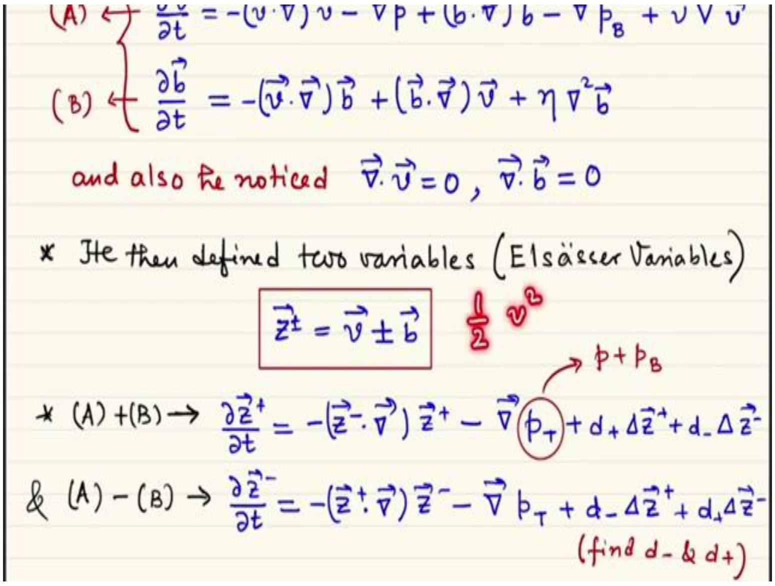

So, can you now write these two quantities like let us say I can just write $\frac{1}{2}v^2$.

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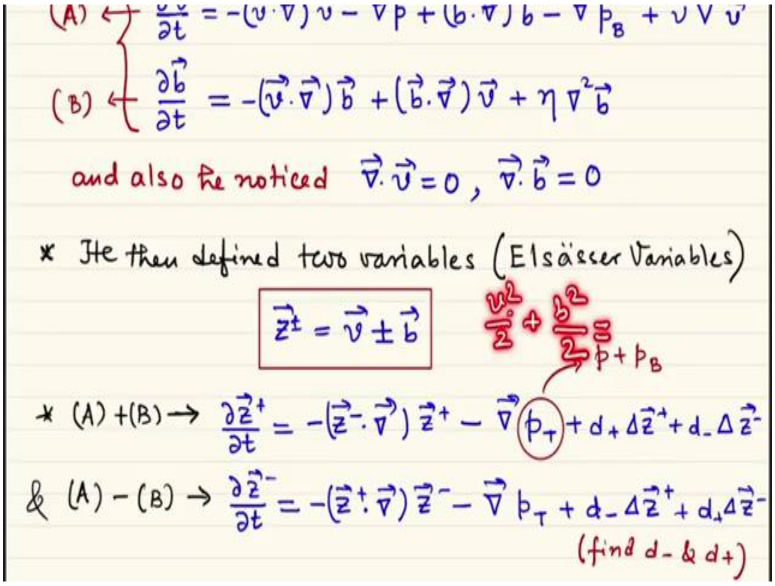

So, if it is incompressible MHD and ρ is 1, then the total energy density is nothing but $\frac{1}{2}v^2 + \frac{1}{2}b^2$, and then what will be this in terms of the Elsasser variables. Well, can you write that?

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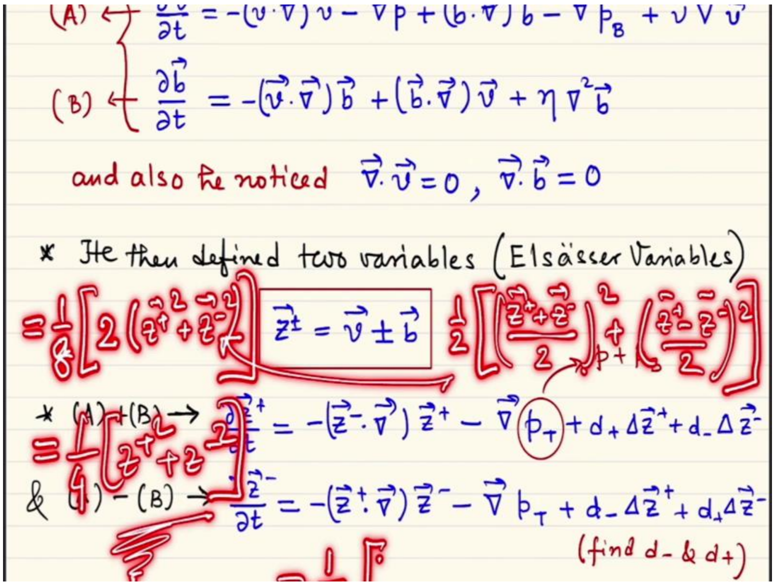

So, simply it will be $\frac{1}{2}v^2 + \frac{1}{2}b^2$, and this $v^2 + b^2$ is nothing but $\left(\frac{\vec{Z}^+ + \vec{Z}^-}{2}\right)^2 + \left(\frac{\vec{Z}^+ - \vec{Z}^-}{2}\right)^2$, and what is this?

So, it will be simply $\frac{1}{4}[\vec{Z}^{+^2} + \vec{Z}^{-^2}]$. So, that is the expression of the total energy density in terms of $\vec{Z}^+$ and $\vec{Z}^-$.

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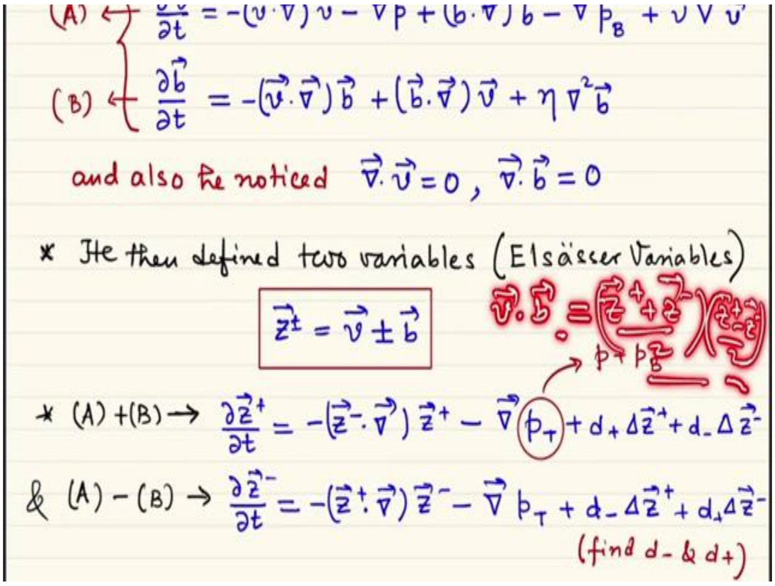

In the same way, you can also express the cross-velocity density in terms of the Elsasser variables. How? So, $\vec{v}$ is equal to $\frac{\vec{Z}^{+}+\vec{Z}^{-}}{2}$, and $\vec{b}$ will be equal to $\frac{\vec{Z}^{+}-\vec{Z}^{-}}{2}$, and then you multiply and you know better than me what would be the expression. So, it will be $\frac{1}{4}[\vec{Z}^{+^2} - \vec{Z}^{-^2}]$.

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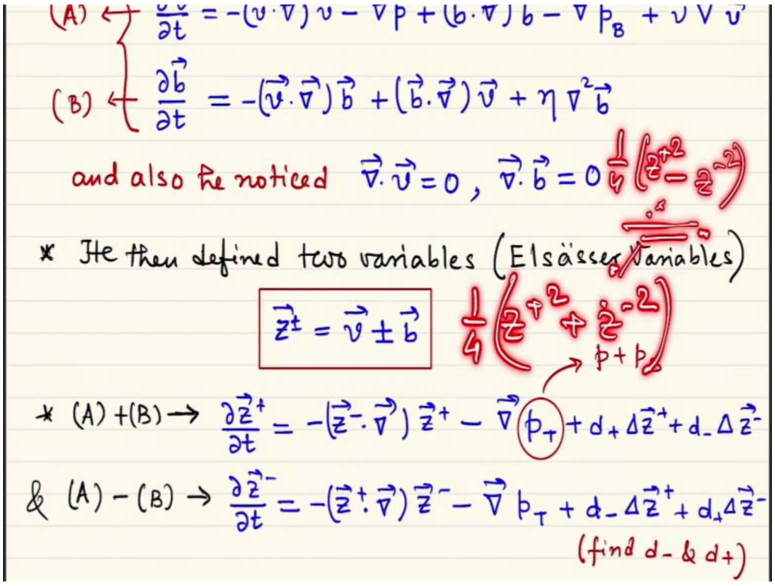

So, now, the thing is that you have seen that $\frac{1}{4}[\vec{Z}^{+^2} + \vec{Z}^{-^2}]$ is something which is the density of a conserved quantity, and also you have seen that $\frac{1}{4}[\vec{Z}^{+^2} - \vec{Z}^{-^2}]$ is something which is the density of something which is conserved that is the cross-velocity part. If both are conserved, then if you I mean the sum of the two will be conserved and the subtraction of the two will conserved right, I mean will be the density of some conserved quantities.

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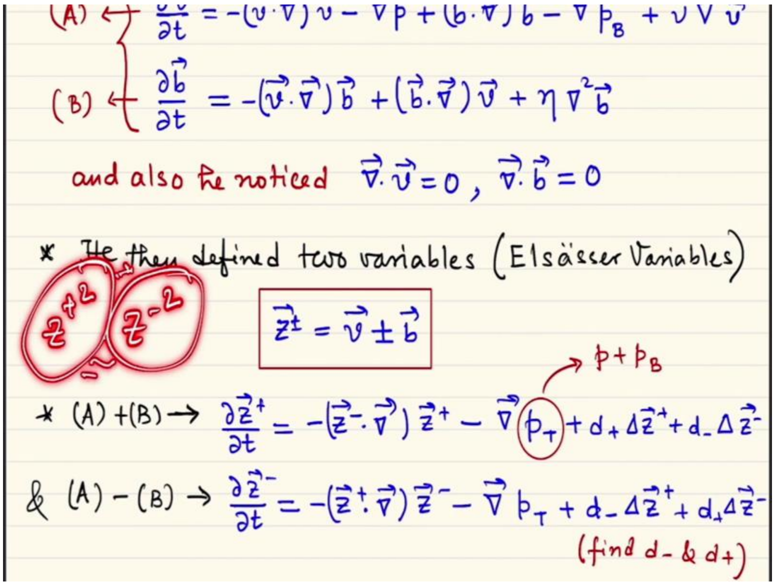

That is why you can actually say that both the $\vec{Z}^{+^2}$ and $\vec{Z}^{-^2}$ will correspond to the densities of two conserved quantities, and they are known as pseudo energies. So, all these things are there. So, Elsasser variables are very interesting for several cases.

So, historically what Elsasser did was to write the incompressible MHD equations in terms of these variables. But after that later manjoney in the year 1987 which is 37 years later, they use this Elsasser variables. I mean they tried to write the compressible MHD equations for a polytropic MHD fluid in terms of compressible Elsasser variables.

That means so, the Elsasser variables are having the same definition but your ρ is no longer changing now. So, it does not have a very symmetric form, but still it can be written, and this writing specifically is very interesting for compressible MHD turbulence as well. So, from the next lecture, we will start discussing another interesting aspect of MHD that is the linear wave modes.

Thank you very much.