Introduction to Astrophysical Fluids

Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 47
Inviscid Invariants in MHD (contd.)

Hello, and welcome to another session of Introduction to Astrophysical Fluids. In this session, we just continue our discussion from the previous session on the inviscid invariants for MHD fluids. So, last time, we discussed the conservation of mass, we discussed the subtle issues related to the conservation of linear momentum, and also finally, we discussed very thoroughly the invariance or the conservation of the total energy.

Now, as I just said that unlike the ordinary hydrodynamic case, here we have three components of the total energy one is the kinetic energy, another is the magnetic energy, the third one is the compressible thermodynamic energy, or sometimes we also say compressible potential type of energy. So, we also saw that the form of that energy part, that compressible energy part actually changes from I mean depending on the nature of the closure.

Here in our cases, we just used two I mean we just showed that two very simple closures – one was the isothermal closure, another was the polytropic closure, and then we also showed that basically the $\frac{d}{dt}$ of the total energy basically boils down to a divergence of some quantity.

Then also there is a subtle point that whether the divergence when gets integrated would vanish, or not for that we have to assure that at every point on the surface enclosing the volume of the flow of the whole flow is such that I mean or rather how to say that at every point of the enclosing surface, the dot product of certain quantities let us say the velocity or the magnetic field must vanish, when it gets scalarly contracted with the surface area vector, or the surface vector.

So, that was all about the energy conservation. Now, in this discussion, we will first introduce some new quantities and then we also discuss the invariance.

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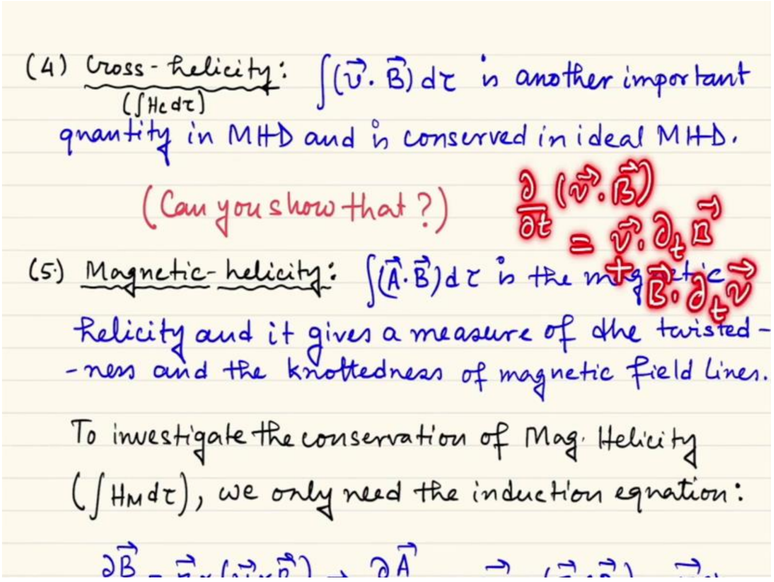

So, the first one is the cross-helicity. So, cross-helicity is nothing but a quantity whose density is given by the scalar product of velocity and magnetic field.

So, if you just integrate $\vec{v} \cdot \vec{B}$ over the whole space you will get the total cross-helicity, and this is very important quantity in MHD because this somehow gives a measure of the interlinkedness of the velocity and the magnetic lines of force or rather the stream lines and the magnetic lines of force, and people actually have succeeded to show that this quantity is an inviscid invariant in ideal MHD.

Now, what exactly is the implication of that? Maybe we can discuss at very briefly when we will discuss the turbulence in a plasma or in an MHD fluid just and there can be other implications as well. But in the scope of this course, at least as of now it will be just an information this cross-helicity is an Inviscid invariant of ideal MHD, both compressible and incompressible.

Now, my question to you is that can you show that? So, in the last lecture, I showed you very carefully step by step how to derive the energy conservation equation. So, as you can easily understand for to show cross-helicity conservation, you have to start with the equations like this $\vec{v} \cdot \vec{B}$, and then you will have two terms $\vec{v} \cdot \partial_t \vec{B} + \vec{B} \cdot \partial_t \vec{v}$, and then you have to use the governing equations to write the whole thing in such a manner that either that should be 0, or that should boil down to a global divergence term.

Now, remember once very interesting thing that when we discussed normal hydrodynamic fluids, then some quantity we defined ω which is the vorticity vector you know $\vec{\nabla} \times \vec{v}$, and there actually we can also define another quantity which is $\vec{v} \cdot \vec{\omega}$.

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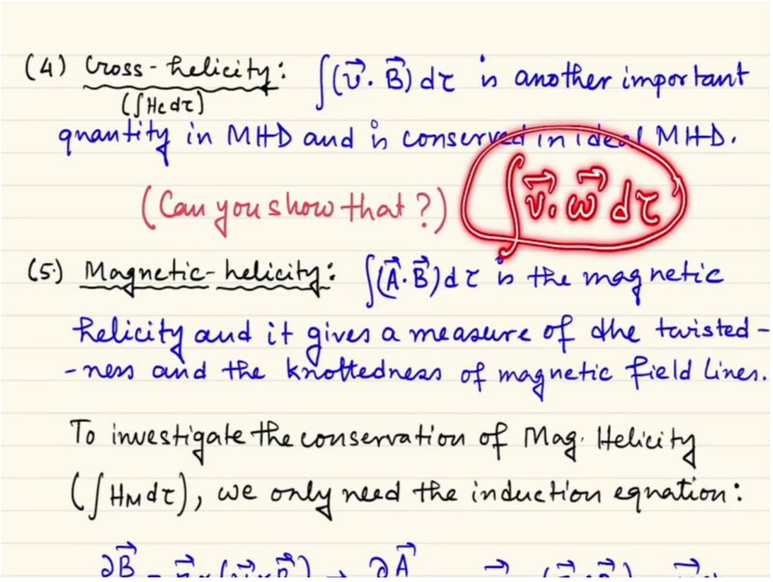

If you do that this thing $\vec{v} \cdot \vec{\omega}$ is integrated over the whole space, you actually get another quantity which is called the kinetic helicity.

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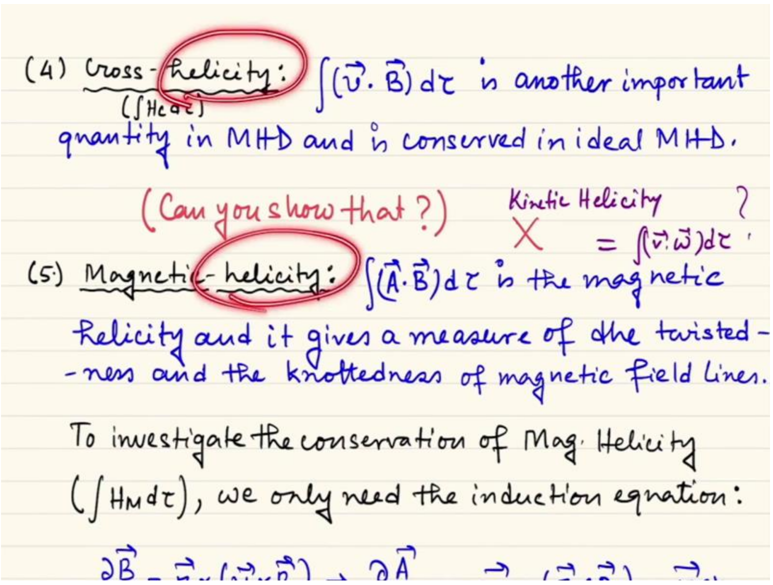

So, my question to you is, can you actually also investigate kinetic helicity, so kinetic helicity which looks like this $\vec{v}\cdot\vec{\omega}d\tau$ is an inviscid invariant or not? Actually, you will see that in case of normal MHD kinetic helicity is no longer inviscid invariant. So, these two things you have to understand.

Now, finally, we are talking about another quantity, this is the magnetic-helicity. What is that? So, magnetic-helicity is that you take the scalar product with the magnetic field to its vector potential A and then you take the volume integral that will give you the total magnetic-helicity, and this magnetic-helicity topologically gives a measure of the twistedness and the knottedness of the magnetic field lines.

So, Mufat, in the year 1970, he published a very, very important work on magnetic-helicity, and different type of topology depending on the magnetic-helicity. So, the question comes naturally whether this quantity is an inviscid invariant for MHD or not? Well, the good news is yes. Magnetic-helicity is an inviscid invariant, and that we will actually investigate step by step thoroughly.

Before doing this derivation, let me just mention one simple thing is that you see that we have given a name helicity, and maybe in physics different type of context, you can come across the word helicity. So, if you remember, for example, in theoretical physics sometimes, helicity is given by the dot product of the linear momentum vector to the spin angular momentum vector. So, this type of thing.

So, in any case it is true that the linear momentum vector is a true vector, and the spin angular momentum is a pseudo vector or an axial vector. So, when true vector is contracted with the pseudo vector, the scalar which we will get is a pseudo scalar. It is not a true scalar right because I mean by the change of the handedness of the coordinate system this or by the action of the parity, this pseudo scalar actually changes its sign, whereas a true scalar never changes its sign, for example, mass. Mass is a true scalar, or the distance, the modulus of a vector that is a true scalar in general.

But when you are talking about this type of scalar product, I mean dot product of one polar vector and one pseudo vector, then simply you have this inversion of sign of that product under the parity.

Then you can say that these things are known as like pseudo scalers. So, all the helicities which we will talk in this case there I mean actually all the helicities, they are traditionally pseudo scalar in nature. So, here you can simply see that $\vec{v}$ is a true vector, $\vec{B}$ is a pseudo vector, $\vec{A}$ is a true vector. As I said kinetic helicity, $\vec{v}$ is a true vector, $\vec{\omega}$ is a pseudo vector.

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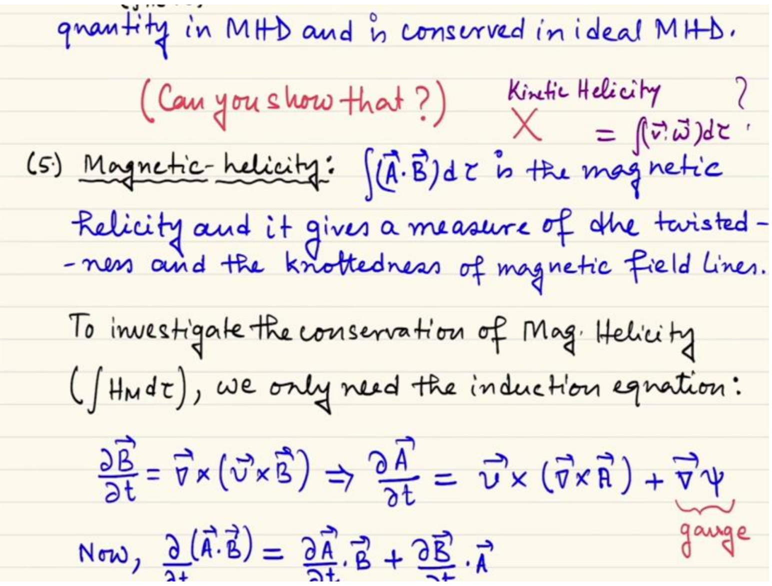

I said that this type of helicity, they are giving in general topological views. So, that means, for example, for velocity you have stream lines, for magnetic field we have magnetic field lines.

So, helicity, I mean sort of keeps measure of the knottedness or the interlinkedness of the different type of lines of force I mean between the lines of magnetic lines of force or the stream lines that is given by some of the cross-helicity.

So, this is quite a non-trivial subject. If you are interested, you can search over internet, and in the both plasma turbulence and in normal fluid turbulence, turbulence I have not yet introduced, but actually this is very interesting to see that helicity has a very important role.

One thing is true that magnetic-helicity whenever we talk about, for example, the conservation of cross-helicity or kinetic helicity, we somehow have to think that whether the system is like compressible or incompressible, because we always have to consider this $\frac{\partial \vec{v}}{\partial t}$ type of equation and then the question of the compressibility of the fluid comes into play.

But when we are talking about or talking of the invariance of the magnetic-helicity, then simply you can easily see that magnetic-helicity, but when we consider the conservation or the inviscid invariance of magnetic-helicity simply you can easily see that it does not depend on the velocity or the momentum evolution equation. So, whether the fluid is compressible or not, it does not play a big role I mean actually it does not play any role here.

So, if we can show that this is an inviscid invariant for compressible MHD and vice versa because we do not need the fluid momentum evolution equation or the continuity equation.

So, in any case, there is maybe one and actually as you go through the proof you will see that we actually also do not need in that sense the divergence of v is equal to 0 explicitly. So, exactly that is why I thought to show this proof step by step, so that you can understand different steps of the demonstration or the proof.

So, here we are just trying to investigate the conservation of magnetic-helicity which is integration over $H_M d\tau$ where H_M is nothing but $\vec{A}\cdot\vec{B}$ that is just a way of writing to that symbol, and just following that sometimes we also denote $\vec{v}\cdot\vec{B}$ as H_c.

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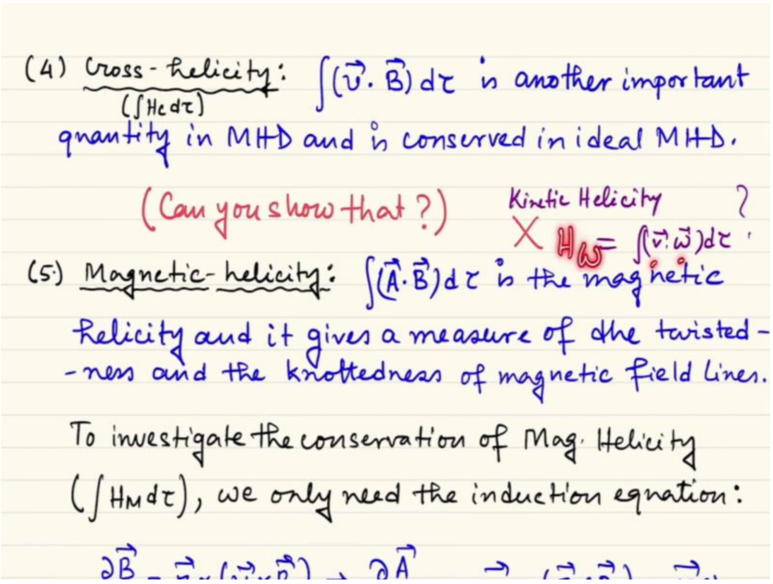

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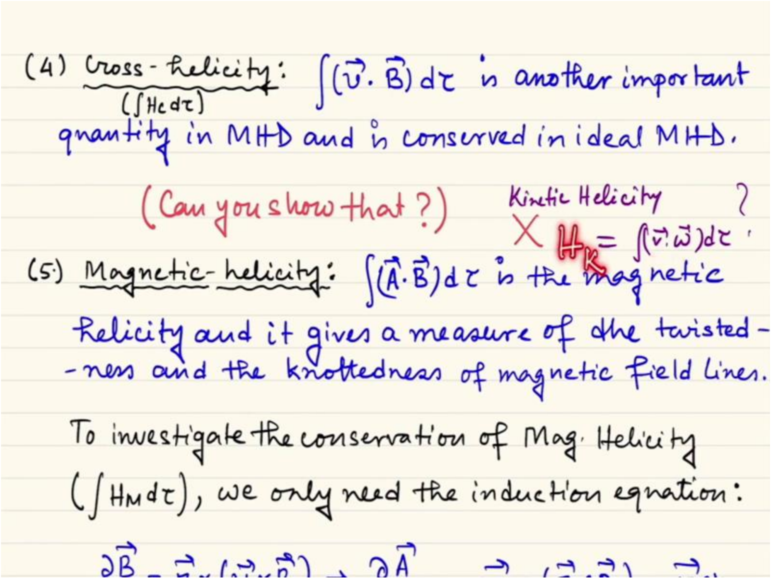

And we also denote $\vec{v}\cdot\vec{\omega}$ as H_{ω} or sometimes we also say H_k kinetic helicity there are different type of symbols.

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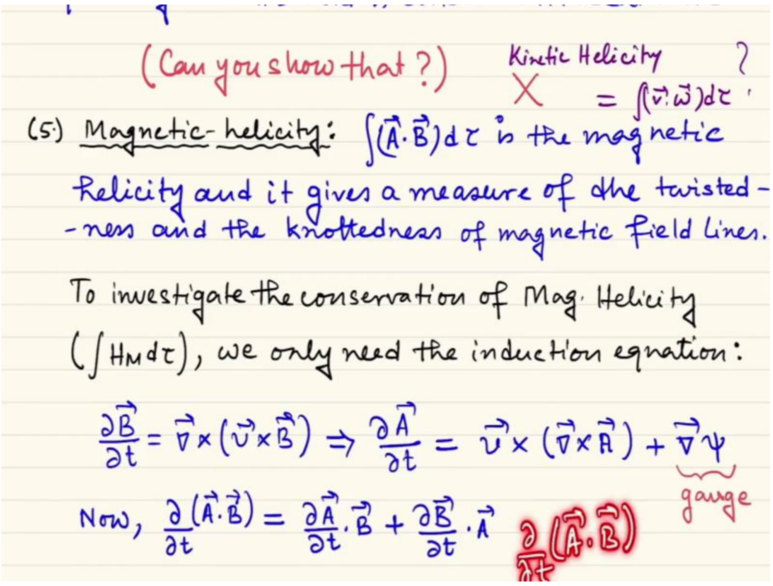

Then what we have to do we first have to derive or I mean like obtain the relations for this right, and for that we need both $\frac{\partial\vec{B}}{\partial t}$ and $\frac{\partial\vec{A}}{\partial t}$. So, $\frac{\partial\vec{B}}{\partial t}$ is given for ideal MHD which is simply equal to $\vec{\nabla}\times(\vec{v}\times\vec{B})$.

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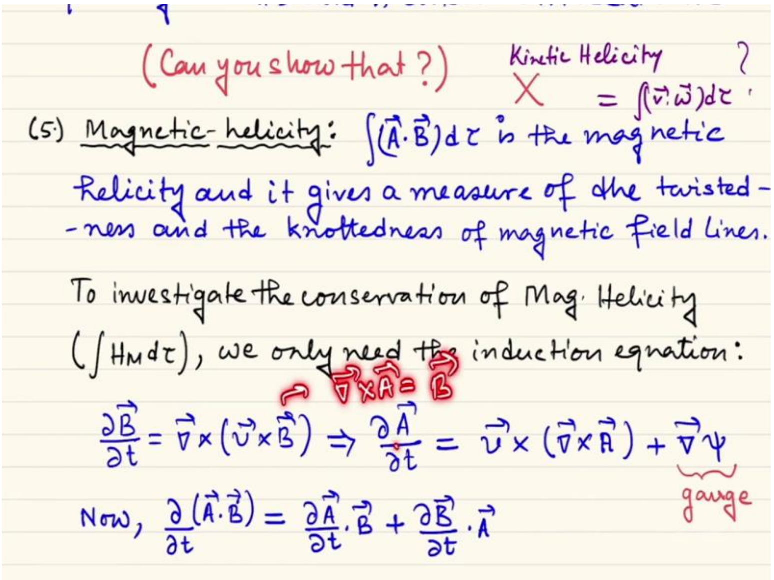

From that if you write $\vec{B}$ is equal to your $\vec{\nabla}\times\vec{A}$ because that is the definition for the vector potential, then you can simply write that $\vec{\nabla}\times\frac{\partial\vec{A}}{\partial t}$ is equal to $\vec{\nabla}\times(\vec{v}\times\vec{B})$. So, $\frac{\partial\vec{A}}{\partial t}$ will simply be equal to $\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right)+\vec{\nabla}\psi$.

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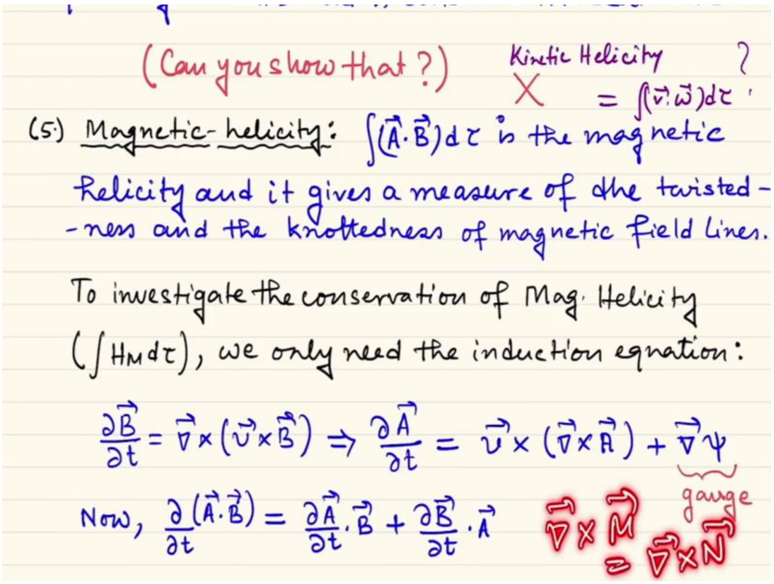

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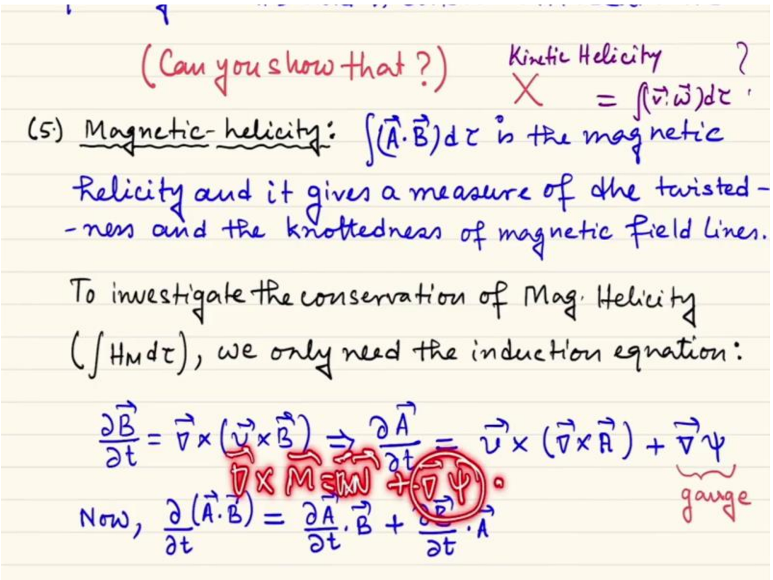

Because they are curl of something a $\left(\vec{\nabla} \times \vec{M}\right)$ is equal to $\left(\vec{\nabla} \times \vec{N}\right)$ simply says that M is equal to N plus some gradient. Because if you take the curl of both sides, then this will vanish identically under a curl, so because curl of grad is 0, and that is why we have written $\frac{\partial \vec{A}}{\partial t}$ be equal to $\vec{v} \times \left(\vec{\nabla} \times \vec{A}\right) + \vec{\nabla}\psi$.

So, this $\vec{\nabla}\psi$ is a gauge basically. In case you do not know what gauge is do not worry much I mean, for our current framework, this $\vec{\nabla}\psi$ is an arbitrary quantity which should come because we have from here to reach here, we have used the equality under a curl. So, finally, $\frac{\partial}{\partial t}(\vec{A} \cdot \vec{B})$ is nothing but $\frac{\partial \vec{A}}{\partial t} \cdot \vec{B} + \frac{\partial \vec{B}}{\partial t} \cdot \vec{A}$, that we agree?

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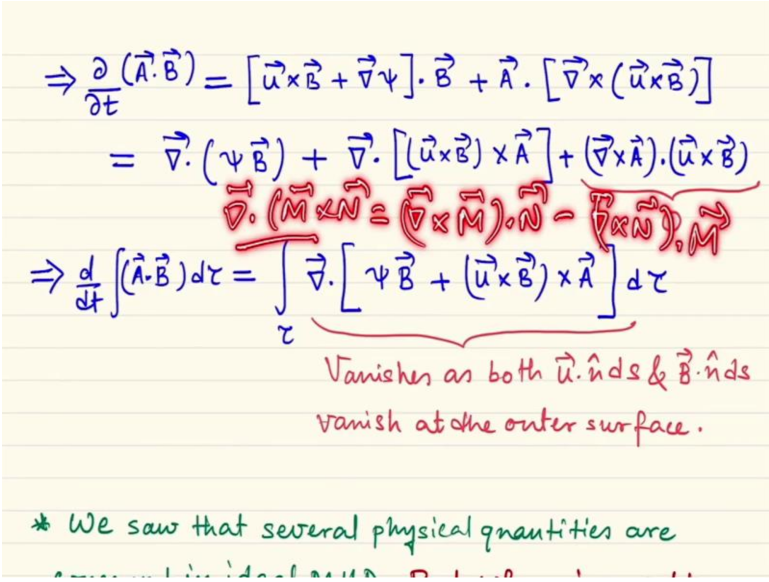

Then

$$\frac{\partial}{\partial t}\left(\vec{A}\cdot\vec{B}\right) = \left[\left(\vec{\mathrm{u}}\times\vec{B}\right) + \vec{\nabla}\psi\right]\cdot\vec{B} + \vec{A}\cdot\left[\vec{\nabla}\times\left(\vec{\mathrm{u}}\times\vec{B}\right)\right].$$

Now, first of all you will see that this $\left[\left(\vec{\mathrm{u}}\times\vec{B}\right) + \vec{\nabla}\psi\right]$ part comes really from the $\frac{\partial\vec{A}}{\partial t}$ part, and this term $\left[\vec{\nabla}\times\left(\vec{\mathrm{u}}\times\vec{B}\right)\right]$ just come from $\frac{\partial\vec{B}}{\partial t}$ part. So, this one I have taken here to contract into a divergence because divergence of B is 0.

So, I can take B inside this, and we can write $\vec{\nabla}\cdot(\psi\vec{B})$, then you can see there are two things one is you have one term $\left(\vec{\mathrm{u}}\times\vec{B}\right)\cdot\vec{B}$ which is 0 because in a scalar triple product if two elements are equal minimum, then this is 0. So, this $\left(\vec{\nabla}\times\vec{A}\right)\cdot\left(\vec{\mathrm{u}}\times\vec{B}\right)$ is 0.

We can again use the identity that $\vec{\nabla}\times\left(\vec{\mathrm{M}}\times\vec{N}\right)$ is equal to $\left(\vec{\nabla}\times\vec{M}\right)\cdot\vec{N} - \left(\vec{\nabla}\times\vec{N}\right)\cdot\vec{M}$. So, just using that identity, we can write this one as $\vec{\nabla}\cdot\left[\vec{\mathrm{u}}\times\left(\vec{\mathrm{B}}\times\vec{A}\right)\right] + \left(\vec{\nabla}\times\vec{A}\right)\cdot\left(\vec{\mathrm{u}}\times\vec{B}\right)$. Now, what is $\left(\vec{\nabla}\times\vec{A}\right)$? It is again B. So, $\left(\vec{\mathrm{u}}\times\vec{B}\right)\cdot\vec{B}$ is 0. So, finally, I have this equation.

So, you see in the whole proof, $\frac{\partial\vec{A}}{\partial t}\cdot\vec{B}$ is nothing but a divergence of a whole thing where this thing under this divergence has no problem to be vanished at infinity because either it will contain terms like $\vec{B}\cdot\hat{n}ds$ or it will contain terms $\vec{u}\cdot\hat{n}ds$. So, in any case, you will see that this will ascertain you the inviscid invariance of the magnetic-helicity. However, as I said that we do not need any compressibility or incompressibility to show that.

As you have also remarked that here in this whole demonstration, we have also not used any closure property that means whether this is a polytropic fluid, or this is a pedotropic fluid, or this is an isothermal fluid, we do not worry.

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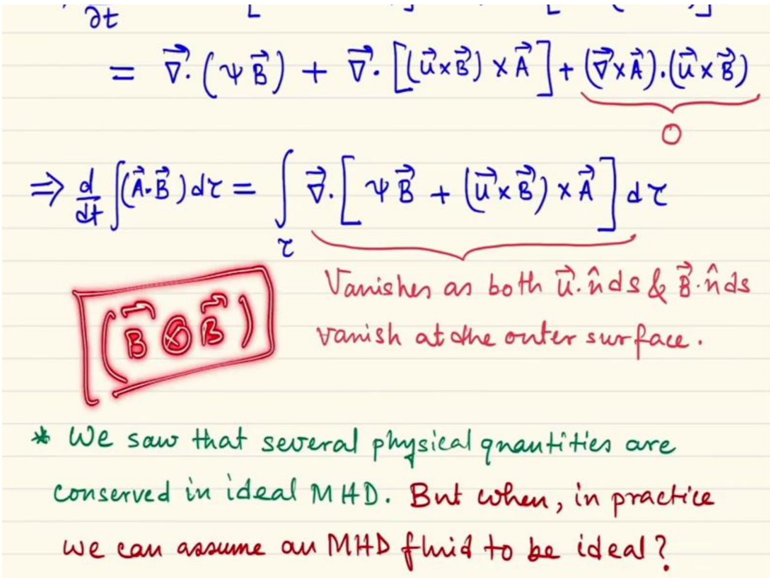

So, that is why the conservation of magnetic-helicity is a very robust property of a fluid irrespective of its closure, irrespective of its compressibility, and so that is why this is very, very interesting for physicists who are doing several types of things like in space physics including plasma turbulence, several type of dynamos. So, dynamo I have not yet introduced. So, those two things turbulence and dynamo, I will introduce very briefly in the last two weeks.

So, in the whole discussion till now we have discussed the inviscid invariance of several variables. Most of them are scalar in nature, of course, a mass and energy they are true scalars in nature, and all the helicity they are pseudo scaler in nature, and also, we discuss the only vector quantity and the conservation of this that is the linear momentum, and we said that this is very subtle and this is not in general conserved due to the quantity like $\vec{B}\otimes\vec{B}$.

So, in a nutshell, we actually saw that several physical quantities are conserved in ideal MHD, but these types of conservations are of prior importance in modelling different type of or rather in explaining different phenomena in space and astrophysics.

But the question is that whether the plasma or the MHD fluid which we are trying to describe in a certain specific astrophysical or space physical framework, what is the guarantee that the ideal MHD assumption is a good assumption?

Because finally, if we can say that our fluid is such that this is of MHD type and all the viscous effects like the kinematic viscosity, the magnetic diffusivity, they are negligibly smaller they are non-existing, well, non-existing is not possible, but at least they are negligibly small, then we can use this conservation of this, helicity energy, this type of thing mass. But if this of course, for mass we do not need viscosity that is true that is just a very normal thing.

But, for example, for magnetic-helicity conservation, it is true that if we have ϑ for example, the kinematic viscosity, but no magnetic-helicity is conserved that is true. But if ϑ is there, the magnetic diffusivity magnetic-helicity is no longer conserved. So, the question is that in practice under which circumstances we can assume the MHD fluid which we are treating is an ideal one?

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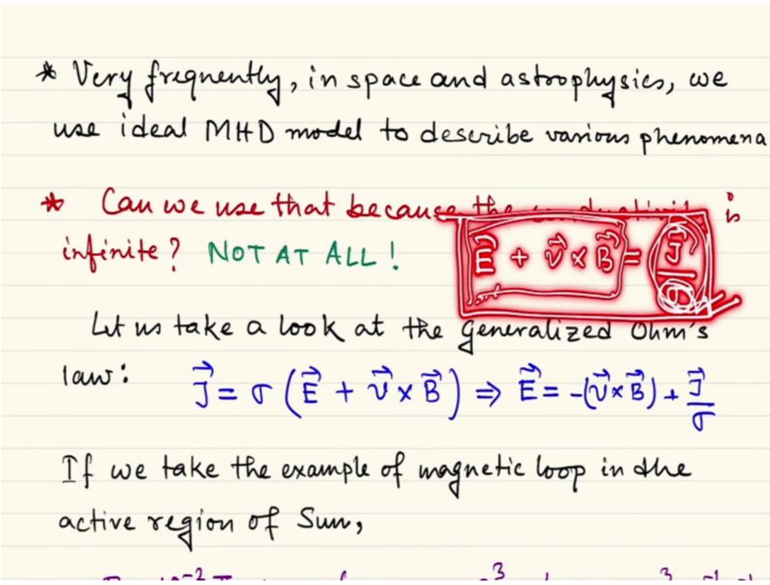

Well, of course, the answer or the first tendency of answer is that well for the cases where the fluid has a very negligible viscosity, very negligible like magnetic diffusivity, and then you can say well this is a plasma.

So, if it has a very high conductivity, then maybe because if you remember the generalized Ohm's law this is given by $\vec{E} + \left(\vec{v} \times \vec{B}\right)$, I mean it is a very simplistic form, of course, after neglecting a lot of terms this is nothing but the conductivity of the fluid.

So, if σ is tending to infinity, then this $\frac{J}{\sigma}$ term is 0, and then basically your ideal MHD assumption is valid. Now, the question is that is it really necessary that always this should be very, very greater? Well, of course, not. A smart answer is well this ratio should be very small with respect to the other terms, then that is something very important, and that is exactly what is exploited in the framework of space and astrophysics.

So, now that exactly what I said that, can we use the fact that because the conductivity is infinite, we can use ideal MHD? Not at all. So, let us just look at the generalized Ohm's law which should look like this $\frac{J}{\sigma} = \vec{E} + \left(\vec{v} \times \vec{B}\right)$, where σ is the conductivity, J is the current density, E is the electric field, v is the fluid velocity, B is the magnetic field. Then E is equal to $\frac{J}{\sigma} - \left(\vec{v} \times \vec{B}\right)$.

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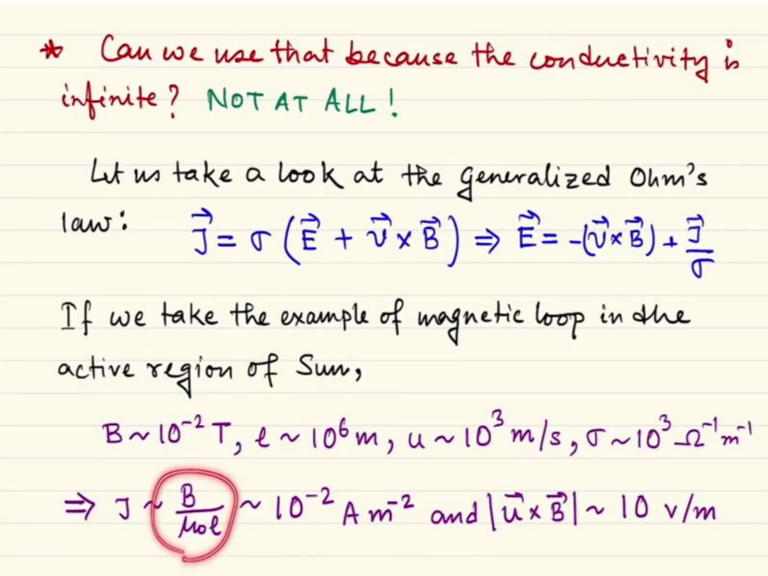

Now, we take a very simple example where we actually use in practice ideal MHD and we will see really that this one is very far from being infinity. So, we just take the example of a magnetic loop in the active region of the Sun.

So, if we take such a loop, then the magnetic field is very, very strong in that region, strong with respect to the magnetic field what we actually observe in solar wind or something. So, in solar wind at 1 astronomical unit, the magnetic field is of the order of 10^5 nano Tesla.

Now, here this is 10^2 Tesla. Well, this is quite high. Then the length scale at which we are interested is 10^6 meter that is the typical size of the magnetic loops, for example and u is 10^3 meter per second, and actually a bit larger than, so, that is of the order of the kilometers per second.

Here, ρ is $10^3\ \Omega^{-1}m^{-1}$, and actually you can see that is high, but that is not infinitely large, 10^3 is moderately large number. So, J which is $\frac{B}{\mu_0 l}$ because it comes like $\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)$, so just by order analysis we can say this will be of the order of 10^{-2} ampere per meter square.

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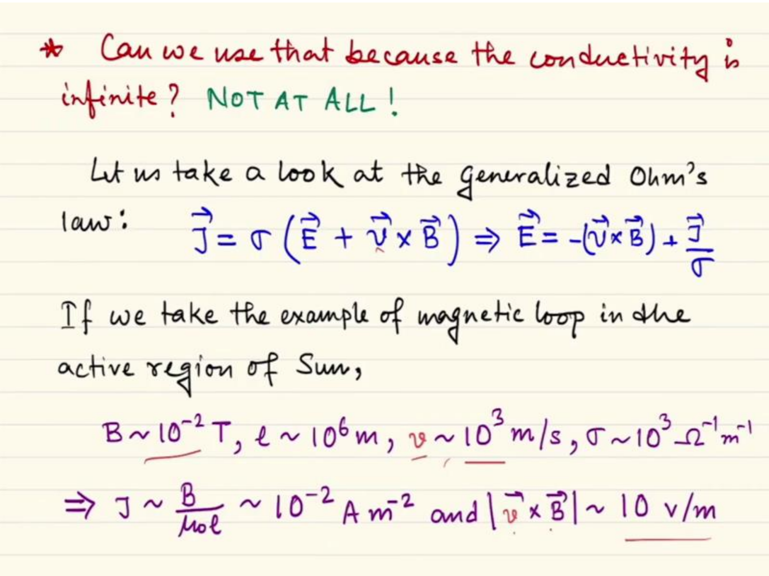

Then what happens, I can actually again change this to v. So, this is the velocity, this not u, and $\vec{v}\times\vec{B}$ just the order will simply be given by 10 volt per meter roughly, we are just doing the order analysis.

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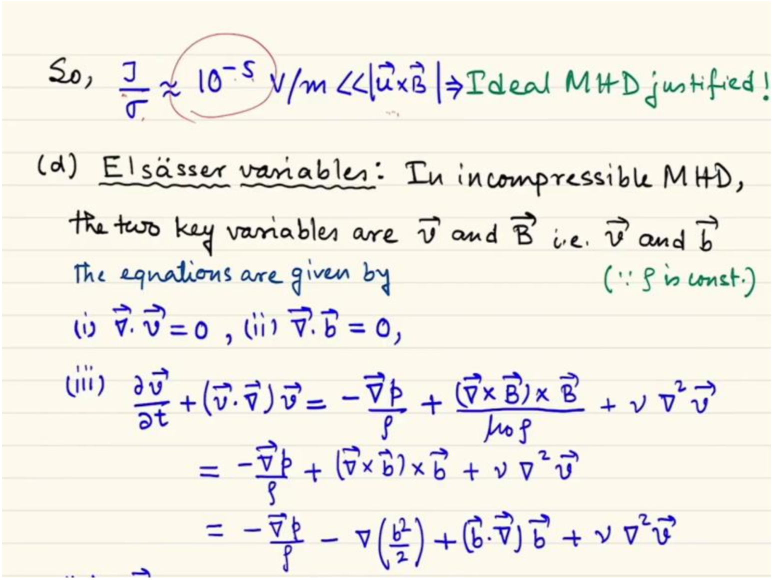

Here, $\frac{J}{\sigma}$ will be simply 10^{-5} volt per meter, and you see if J is not very, very small let us say J is of the order of 10 Ampere per meter square, then $\frac{J}{\sigma}$ will simply be 0.1, and then we cannot say that $\frac{J}{\sigma}$ ratio is negligibly small.

So, finally, with respect to this one $|\vec{v}\times\vec{B}|$, this one $\frac{J}{\sigma}=10^{-5}$ is very, very small. So, there is an order 6. So, this one $|\vec{v}\times\vec{B}|$ is larger than this one $\frac{J}{\sigma}$ is larger than this one by a factor of 10^6, and that is why you can say that this term $\frac{J}{\sigma}$ can actually be neglected with respect to this term $|\vec{v}\times\vec{B}|$. So, you can say E is equal to $-(\vec{v}\times\vec{B})$. So, your ideal MHD situation is retrieved.

So, that is something very, very interesting that whenever doing some analytical relations like some quantities are conserved, some quantities are not conserved. So, this type of inferences is very good, but on the other hand it is also important to have a practical knowledge, so that you know the real order of magnitude of the parameters on the fields in several cases, for example, I mean the magnetic field, the density.

If you just do analytics using B, v, ρ, p, t, without really knowing that whether you are I mean what would be the practical values for v, B, ρ, E, then actually this is just playing a violin to a deaf person right.

You cannot enjoy the real essence of the subject. So, the essential thing is to have a real knowledge, and at least for not the exact values, but at least the order of magnitudes of the variables of different situations, different means in the different frameworks of space and astrophysics.

So, here we just talked about the active region of the Sun and the corresponding magnetic loops. Then another situation may be possible for the solar wind at 1 Astronomical unit, at 10 Astronomical unit, the plasma inside the solar corona that is also possible. The plasma at the very, very large scale of a galaxy that is possible right.

So, all these things will only be meaningful if you know that and actually you can understand that which model is much more appropriate than the other only when you have a proper idea of the order of magnitudes of different quantities, different fluid variables, in different situations.

In the next lecture, I will discuss very small, but interesting topic that is an alternative formulation or an alternative writing style introduced by Elsasser in the year 1950, to write the MHD equations when these equations are incompressible. So, that is a small topic, but quite interesting. So, here for this invariance and this ideal MHD part I am just ending here. So, see you in the next session.

Thank you.