Introduction to Astrophysical Fluids
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Lecture - 46
Inviscid Invariants in MHD

Hello, and welcome to another session of Introduction to Astrophysical Fluids. In this lecture, we will continue our discussion on the different type of properties or features of MHD fluids and specifically we will discuss the Inviscid Invariants of several fluid variables or fluid quantities in the case of MHD.

In general, we will assume that the MHD fluid is compressible, but it has a simplistic closure for example, either isothermal or polytropic. So, for most of the cases we will use the polytropic closure.

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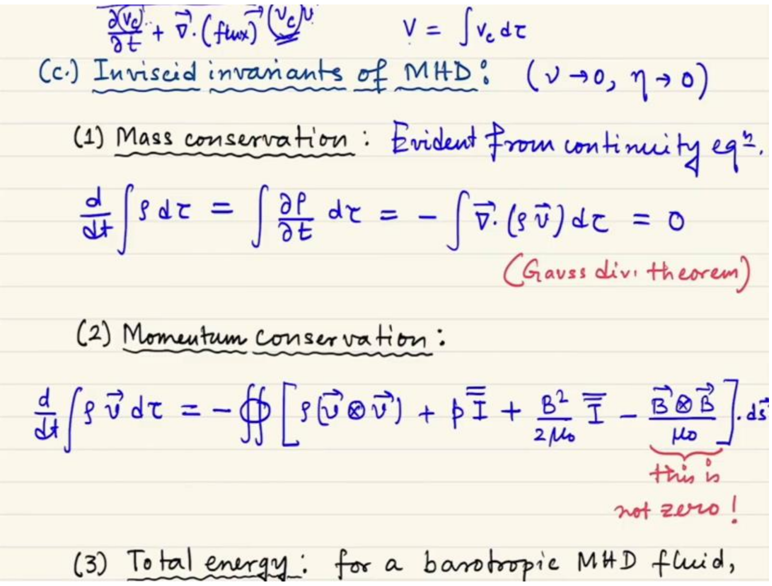

So, first we start with the very fundamental inviscid invariant or the conservation of mass. So, when we talk about first of all inviscid invariants it simply says that we will try to see that which quantities are conserved in time if the fluid has negligible viscous effects.

So, for our case it will just simply give that ϑ tends to 0, I mean the viscosity and the magnetic diffusivity also tends to 0. So, under this condition we will investigate which quantities are conserved. Now once again when we are talking about conservation as we discussed in one of the previous lectures while discussing the inviscid invariants for normal hydrodynamic fluid, we said that for the invariants of the extensive quantities.

That in density corresponding to the extensive quantity should follow a specific I mean the evolution of the density should follow such type of equation which are known as the conservative forms. So, if you really remember then the condition for certain extensive quantity v to be conserved to be an inviscid invariant if we write v is equal to some $v_c d\tau$ I mean I can also write like this.

So, this is a vector or just I am taking scalar, for simplicity. So, then v_c is the density of that quantity v then in order that v is an inviscid invariant we have to have this type of equation plus divergence of some flux, and that flux simply says that it should be exactly equal to either the density v_c times the fluid velocity v or some other quantity which is not exactly equal to v_c, but some other quantity times v.

So, we sometimes call the current associated with that quantity. So, for example, for mass we just simply identify ρ is the density of mass and $\rho\vec{v}$ is the mass current. So, this type of things is there. So, that this a simple form that is the $\frac{d}{dt}$ of the that density v_c should be equal to 0.

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$$\frac{\partial (v_c)}{\partial t} + \vec{\nabla}\cdot(\overrightarrow{\text{Current}}) = 0 \Rightarrow \frac{\partial \int v_c d\tau}{\partial t} = \boxed{-\int \vec{\nabla}\cdot(\text{Current})\, d\tau} = \oint \text{Current}\cdot\hat{n}\, d\vec{s}$$

(c.) Inviscid invariants of MHD: $(\nu \to 0, \eta \to 0)$

(1) Mass conservation: Evident from continuity eq^{n}.

$$\frac{d}{dt}\int \rho\, d\tau = \int \frac{\partial \rho}{\partial t}\, d\tau = -\int \vec{\nabla}\cdot(\rho\vec{v})\, d\tau = 0$$

(Gauss div. theorem)

(2) Momentum conservation:

$$\frac{d}{dt}\int \rho\vec{v}\, d\tau = -\oint\!\!\oint \Big[\rho(\vec{v}\otimes\vec{v}) + p\,\overline{\overline{I}} + \underbrace{\frac{B^2}{2\mu_0}\overline{\overline{I}} - \frac{\vec{B}\otimes\vec{B}}{\mu_0}}_{\text{this is not zero!}}\Big]\cdot d\vec{s}$$

(3) Total energy: for a barotropic MHD fluid,

So, in that case, you can simply say that $\frac{dv_c}{dt}$ will be equal to minus divergence of that current and just writing current instead of flux because flux can be misleading and if I can write then simply $\frac{dv_c}{dt}$ is equal to the divergence of some quantity then actually we are done.

But it is preferable, of course, as you understand that the divergence should be multiplied with some quantity some field quantity with which it is then easy to use the Gauss's divergence theorem that means finally, we have to prove that the volume integral of this thing is 0 and for that if you really understand the Gauss's divergence theorem this is nothing, but the closed surface integral of this current dot $\hat{n}ds$.

If that current is proportional to the velocity or in MHD we will see that it can also be magnetic field. So, that in the surface enclosing that volume at every point of the surface that $\vec{v}\cdot\hat{n}ds$ or $\vec{b}\cdot\hat{n}ds$ should vanish then you can actually justify that this divergence will lead to a vanishing contribution to the $\frac{d}{dt}\int v_c d\tau$ and thereby confirming the conservation.

So, just following the same ethics I mean we will finally, try to show different type of conservation principles. So, let me first it is all those things I will come into that and this part I will do actually very thoroughly. So, that you can actually know how to do that.

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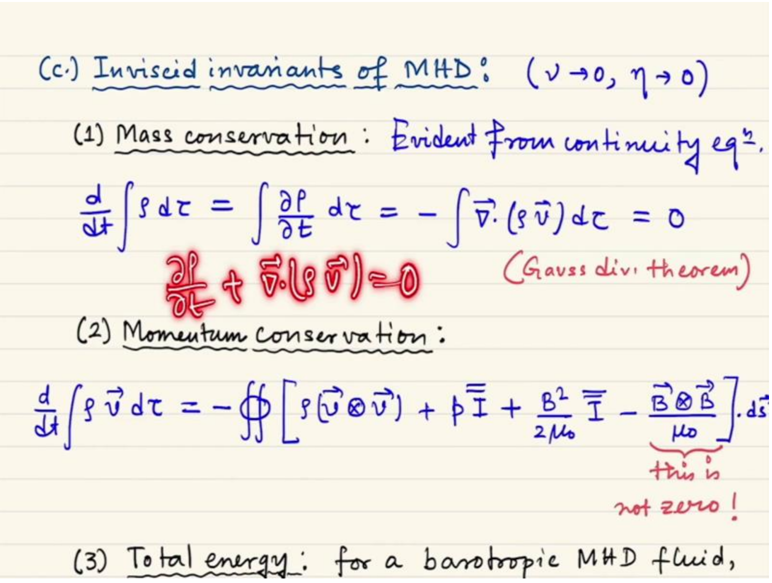

So, the first one is once again so, we start by discussing the inviscid invariants of mass, and this is evident from the basic continuity equation which simply says that $\frac{d\rho}{dt} + \vec{\nabla}\cdot(\rho\vec{v})$ is equal to 0. So, then $\frac{d}{dt}\int \rho d\tau$ which is the total mass, of course, then $\frac{d}{dt}$ will enter inside the integration that you know from mathematics,

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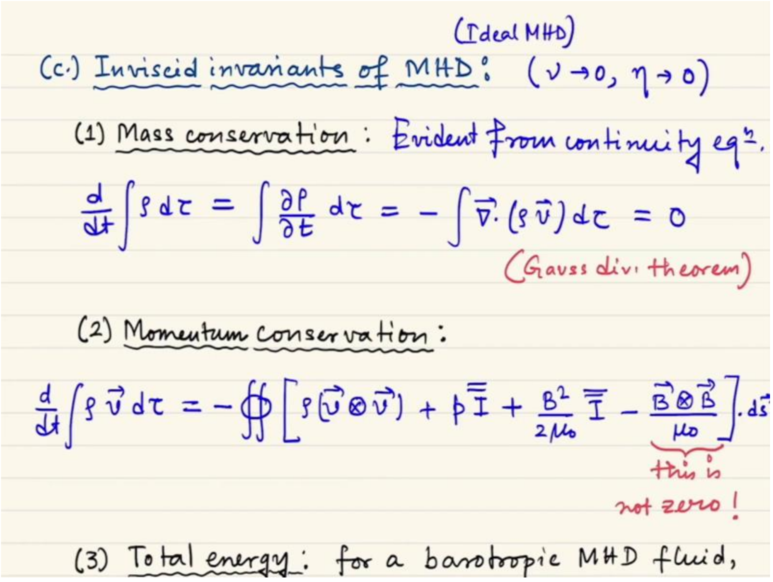

and that becomes a $\frac{d}{dt}$ inside this integration and then this $\frac{d\rho}{dt}$ by continuity equation is nothing but $-\vec{\nabla}\cdot(\rho\vec{v})$. So, $\int \vec{\nabla}\cdot(\rho\vec{v})\, \mathrm{d}\tau$ is nothing, but equal to closed surface integral $\rho\vec{v}.d\vec{s}$ right.

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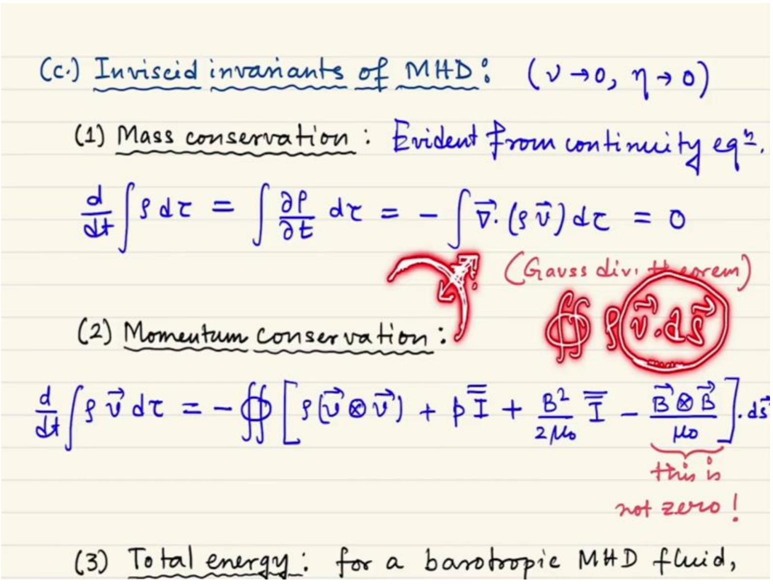

Now, $\rho\vec{v}.d\vec{s}$ at the surface if you remember sorry what I said that at the surface $\vec{v}.d\vec{s}$ should vanish because this is the extreme surface. If $\vec{v}.d\vec{s}$ is non zero that means we have some component of the velocity in this direction and it simply says that this cannot be the surface or the boundary surface enclosing the whole fluid because it simply says that the fluid can come out or can go in right.

So, then this is no longer a boundary surface so, in order that this is a boundary surface $\vec{v}.d\vec{s}$ simply should be 0. So, we just use that and we will get finally, this is equal to 0, so, $\frac{d}{dt}\int \rho d\tau$ is equal to 0 and mass is conserved. Then we have another type of conservation principle and that is a bit subtle and that is the momentum conservation which is trivially true for hydrodynamic fluid here, there is a trick.

Because the momentum first of all this is a vector conservation so, we will somehow write $\frac{d}{dt}\int \rho\vec{v}d\tau$ and that will be simply equal to if you do the calculation just by using the Navier-Stokes's equation including the low range force you will see that will give you divergence of this quantity. and by using Gauss's divergence law you can simply write minus closed surface integral $\left[\rho(\vec{v}\otimes\vec{v}) + p\bar{\bar{I}} + \frac{B^2}{2\mu_0}\bar{\bar{I}} - \frac{\vec{B}\otimes\vec{B}}{\mu_0}\right] \cdot d\vec{s}$. Now, the problem is that at this point you can say that this $\rho(\vec{v}\otimes\vec{v}) \cdot d\vec{s}$ would be 0, this $p\bar{\bar{I}} \cdot d\vec{s}$ will also be 0, this $\frac{B^2}{2\mu_0}\bar{\bar{I}} \cdot d\vec{s}$ will also be 0, but not this $\frac{\vec{B}\otimes\vec{B}}{\mu_0} \cdot d\vec{s}$.

So, the $\frac{\vec{B}\otimes\vec{B}}{\mu_0} \cdot d\vec{s}$ is not necessarily equal to 0. So, for that I am not going into the very detail of this part so, you can see any standard book of magneto hydrodynamics here the problem is described. So, this basically, causes the momentum linear momentum in MHD equations not to be conserved. So, that is something we have to understand.

However, for hydrodynamic case this term $\frac{\vec{B}\otimes\vec{B}}{\mu_0}$ is no longer present, but it is true that you can have this impression that if this is $\frac{\vec{B}\otimes\vec{B}}{\mu_0}$ causing problem then this $\rho(\vec{v}\otimes\vec{v})$ can also cause problem, but I mean fortunately this $\rho(\vec{v}\otimes\vec{v})$ does not cause any problem.

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$$\frac{d}{dt}\int \rho \, d\tau = \int \frac{\partial \rho}{\partial t} d\tau = -\int \vec{\nabla}\cdot(\rho \vec{v})\, d\tau = 0$$

(Gauss div. theorem)

(2) Momentum conservation:

$$\frac{d}{dt}\int \rho \vec{v}\, d\tau = -\oiint \left[\rho(\vec{v}\otimes\vec{v}) + p\overline{\overline{I}} + \frac{B^2}{2\mu_0}\overline{\overline{I}} - \underbrace{\frac{\vec{B}\otimes\vec{B}}{\mu_0}}_{\text{this is not zero!}}\right]\cdot d\vec{s}$$

$$p = p(\rho)$$

(3) Total energy: for a barotropic MHD fluid, one can show that (Edτ)

$$\int \underbrace{\left(\rho\frac{u^2}{2} + \frac{B^2}{2\mu_0} - \int_{\rho_0}^{\rho} p\, d\left(\frac{1}{\rho}\right)\right) d\tau}_{\text{Total Energy}}$$

is conserved in ideal MHD

Now, momentum conservation is not a guarantee in magneto hydrodynamics. I mean rather in ideal magneto hydrodynamics because in this whole equation whole discussion we are talking about inviscid invariants. So, the basic equations are the background equations. That will simply be the equations of ideal MHD. So, I can then write this as ideal MHD.

Now, I am going very slowly to the most important conservation of magneto hydrodynamics that is the conservation of total energy. Now, we also discussed the conservation of total energy may be in the case of compressible fluids, but not really in detail here I will just try to do it at least once thoroughly step by step. So, that you can have the idea not only for total energy, but also for other type of quantities, how to prove or how to investigate whether the quantity is an inviscid invariant or not.

So, we take a barotropic MHD fluid which is a bit more general than polytropic so, barotropic simply says that the fluid pressure is simply a function of ρ only fluid density.

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$$\frac{d}{dt}\int \rho \, d\tau = \int \frac{\partial \rho}{\partial t} d\tau = -\int \vec{\nabla}\cdot(\rho\vec{v})\, d\tau = 0$$

(Gauss div. theorem)

(2) Momentum conservation:

$$\frac{d}{dt}\int \rho\vec{v}\, d\tau = -\oiint \Big[\rho(\vec{v}\otimes\vec{v}) + p\bar{\bar{I}} + \frac{B^2}{2\mu_0}\bar{\bar{I}} - \underbrace{\frac{\vec{B}\otimes\vec{B}}{\mu_0}}_{\text{this is not zero!}}\Big]\cdot d\vec{s}$$

(3) Total energy: for a barotropic MHD fluid, one can show that $(\int E d\tau)$

$$\underbrace{\int\left(\rho\frac{u^2}{2} + \frac{B^2}{2\mu_0} - \underbrace{\rho\int_{\rho_0}^{\rho} p\, d\left(\frac{1}{\rho}\right)}_{\rho e}\right) d\tau}_{\text{Total Energy}}$$

is conserved in ideal MHD

So, the total energy is given by this $\int E d\tau$ and E is the volume density of energy, and for a barotropic MHD fluid we will try to show that the total energy is conserved. Because I mean conserved in the absence of any net viscosity or magnetic resistivity or magnetic diffusivity. So, that is something very intuitive, but we now have to give the proper expression for the total energy.

So, one we know that is the kinetic energy part and which is quite evident rho u square by 2 that is the kinetic energy density that is the magnetic energy density B square by 2 mu 0 minus there is another quantity which comes due to the compressibility of the fluid and for a so, there is already an integration.

So, inside integration, the total energy density function and which will be then integrated over $d\tau$. But this compressible energy or the compressible potential energy function is itself derived from another integration if for knowing the functional form of that compressible energy we have to use the corresponding closure.

Because we have p is equal to $p(\rho)$ so, we have to know the exact functional dependence of p on ρ, and then if we do from ρ_0, ρ_0 is some reference value for the density to some current and I mean current means like the instantaneous ρ, and this integration is done on $p\, d\left(\frac{1}{\rho}\right)$, and the whole thing will then be again integrated on the whole space so, will be multiplied by $d\tau$ and this will be integrated.

So, these will be giving us total energy and people have already succeeded to show that this is an inviscid invariant. Now in our lecture we will try to show that.

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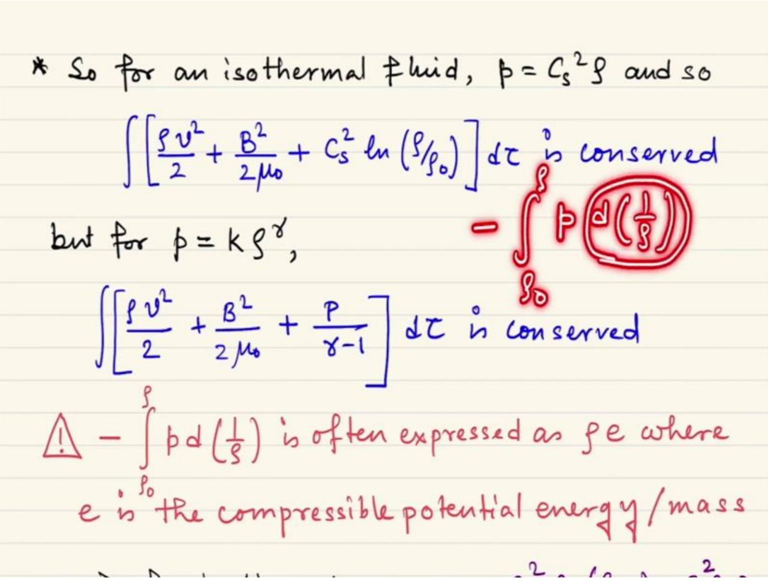

Before showing this conservation or the invariants we discuss two small points one is that what will be the compressible potential energy function for a simple isothermal fluid having this very simple closure p is equal to some constant square times ρ and this constant is nothing, but the sound speed square.

So, p is equal to $C_s^2 \rho$. So, for that case this one $\frac{\rho v^2}{2}$ is the kinetic energy density, this one $\frac{B^2}{2\mu_0}$ is the magnetic energy density now we have to do something like that $-\int_{\rho_0}^{\rho} p \, d\left(\frac{1}{\rho}\right)$ and this $d\left(\frac{1}{\rho}\right)$ is nothing, but $-\frac{d\rho}{\rho^2}$. So, finally, that will be becoming $\int_{\rho_0}^{\rho} p \, \frac{d\rho}{\rho^2}$. So, this p is nothing but $C_s^2 \rho$.

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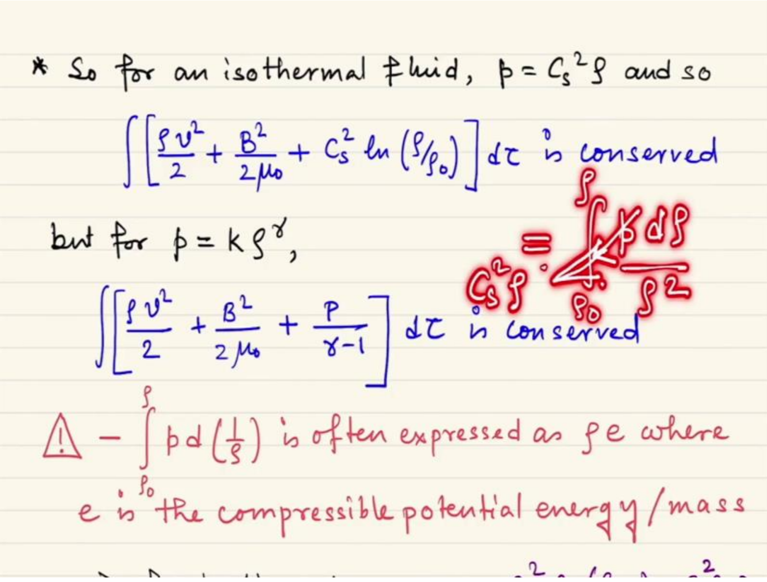

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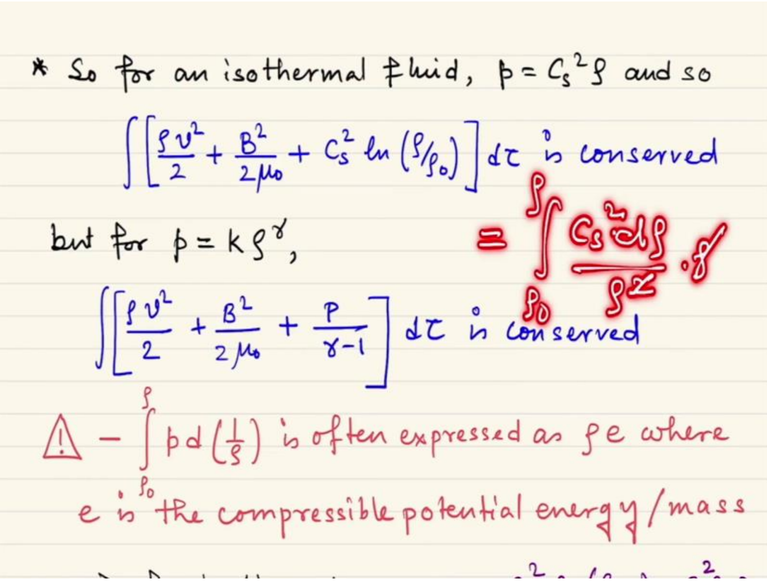

So, finally, this will become $\int_{\rho_0}^{\rho} C_s^2 \rho \, \frac{d\rho}{\rho^2}$. So, ρ, ρ^2 will cancel one density and you have $\int_{\rho_0}^{\rho} C_s^2 \, \frac{d\rho}{\rho}$. So, finally, the functional form will be $C_s^2 \ln\left(\frac{\rho}{\rho_0}\right)$.

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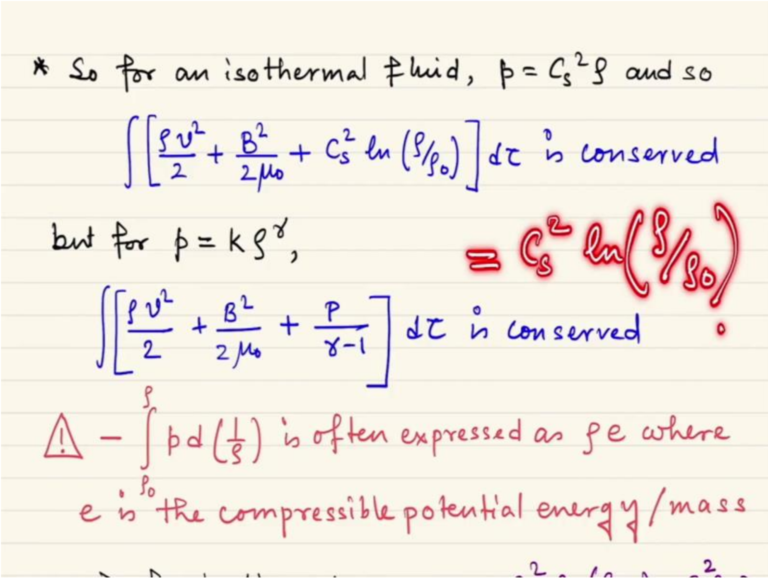

That is exactly what we will use here, and for a polytropic case where p is equal to some constant times ρ^γ then if you follow the same formula and it's you to do now.

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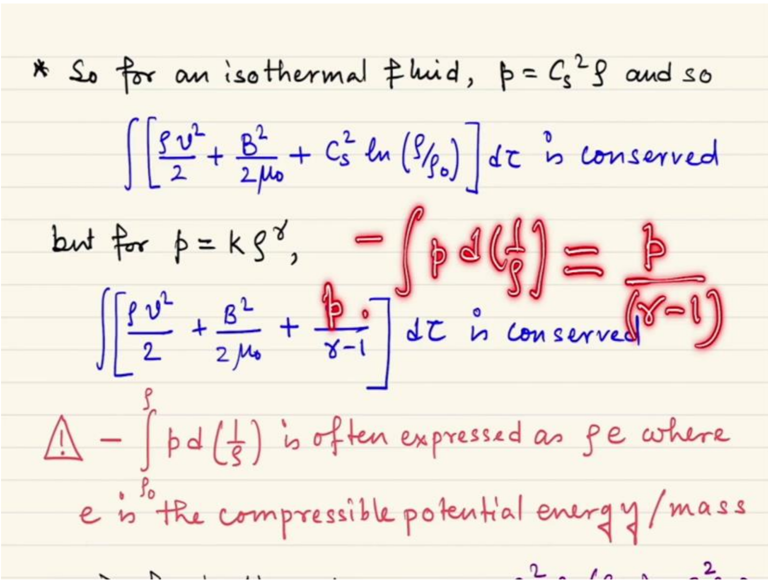

So, do it yourself then you will see that $-\int p \, d\left(\frac{1}{\rho}\right)$ is simply equal to $\frac{p}{\gamma-1}$.

There should be a ρ multiplied. So, ρ is the density times you have to have some function e. So, basically the whole thing should be written as ρe.

Now, how to understand this so, p has a dimension of the velocity square times a density because its like $C_s^2\rho$ and $d\left(\frac{1}{\rho}\right)$ has a dimension of $\left(\frac{1}{\rho}\right)$. So, p times $\left(\frac{1}{\rho}\right)$ finally, gives you a dimension of velocity square. So, it is simply this e part and because the whole thing has a dimension of density times velocity square.

So, by this I have just detected the error in my form I mean writing. So, finally, I think this is not a big problem. So, I am just writing once again so, ρ will be there. So, for polytropic case where p is equal to $k\,\rho^{\gamma}$ then this is the correct expression and actually e will be then $\left(\frac{p}{\rho(\gamma-1)}\right)$ and then ρe will be simply $\left(\frac{p}{(\gamma-1)}\right)$.

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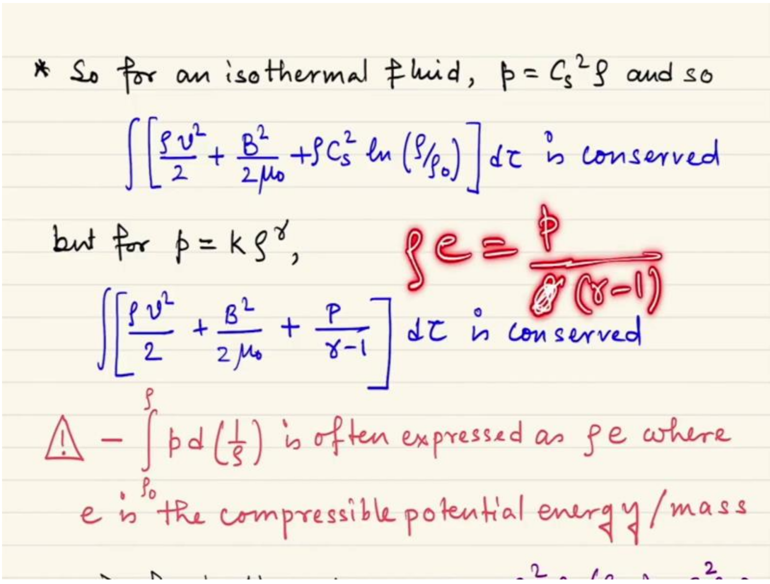

Let me just write once again one thing lets p. That is the thing we use throughout. Now, we have to check for polytropic case that $\int[\frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} + \frac{p}{(\gamma-1)}]\, d\tau$ is conserved.

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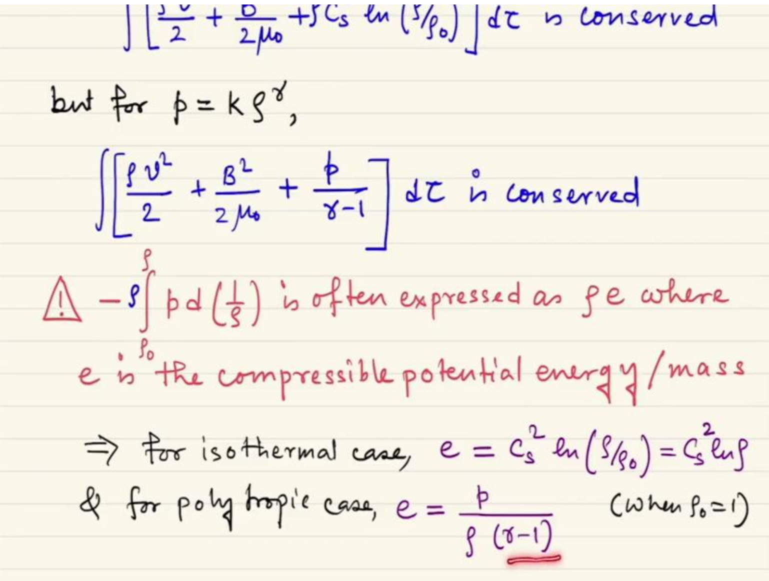

So, as I just said that this $-\rho \int_{\rho_0}^{\rho} p \, d\left(\frac{1}{\rho}\right)$ is often expressed as ρe where e is the compressible potential energy per unit mass. So, then we are done.

Now, once again, for isothermal case, we agree e is equal to $C_s^2 \ln\left(\frac{\rho}{\rho_0}\right)$ and in case you just choose just for simplicity your ρ_0 or the reference density is 1 then your $ln(\rho_0)$ is 0, and then simply e is equal to $C_s^2 \ln(\rho)$ that is exactly what we do when we talk about an isothermal fluid.

And for polytropic case e is equal to $\frac{p}{\rho(\gamma-1)}$. So, here in this lecture what I will do? I will derive the step by step the conservation or the inviscid invariants of the total energy. So, as you know that we have to show that E the total energy density is $\frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} + \rho e$, and we have to show that $E \, d\tau$ is conserved in ideal MHD then we actually have to show that $\frac{d}{dt} \int E \, d\tau$ is equal to 0 that we have to show.

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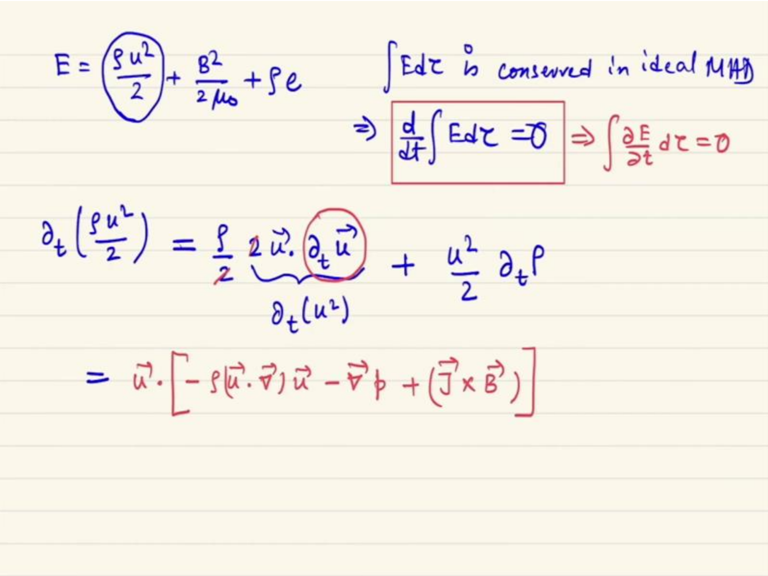

So, we start by doing or we have to show $\int \frac{\partial E}{\partial t}\, d\tau$ is equal to 0. So, we start by different piece of E. First one is this the kinetic energy term that is $\frac{d}{dt}\left(\frac{\rho v^2}{2}\right)$. So, now $\frac{d}{dt}(v^2)$ is nothing, but 2 times $\vec{v}\cdot\partial_t\vec{v}$. Then I have another term which is $\frac{v^2}{2}\partial_t\rho$.

So, I am just writing the expression for $\rho\frac{\partial v}{\partial t}$, and that is $-\rho(\vec{v}\cdot\vec{\nabla})\partial_t\vec{v} - \vec{\nabla}p + (\vec{J}\times\vec{B})$ and we are not considering any dissipation. So, here we have just used this equation in case you have forgotten that we have used this equation.

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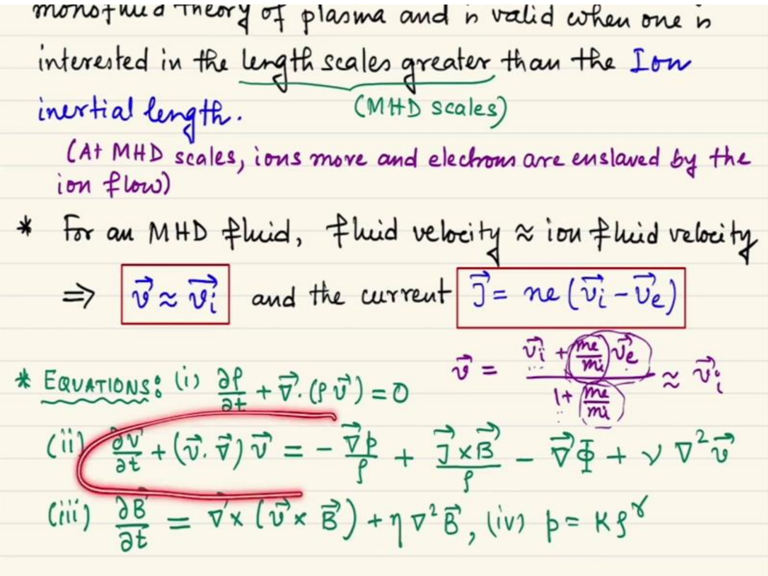

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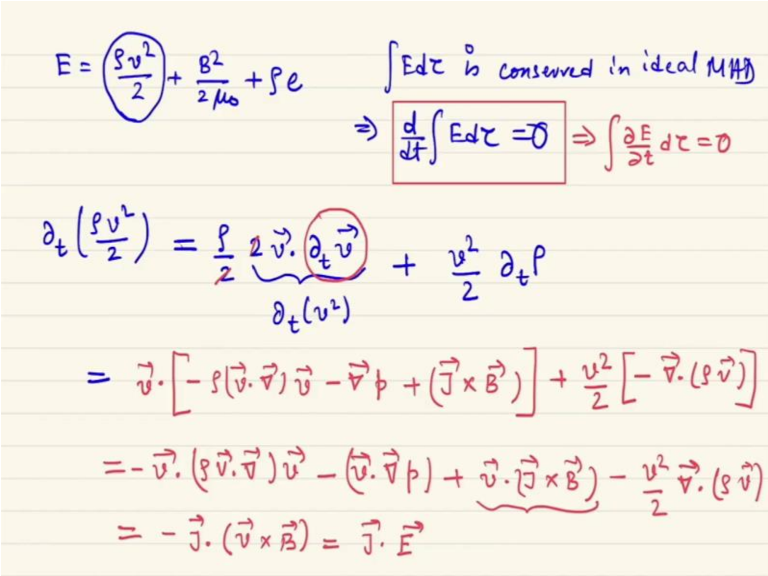

I have another term which is $\frac{v^2}{2}\partial_t \rho$ and $\frac{\partial \rho}{\partial t}$ by continuity equation is nothing, but equal to $-\vec{\nabla}\cdot(\rho\vec{v})$. So, finally, what you have is equal to

$$= \vec{v}\cdot\left[-\rho(\vec{v}\cdot\vec{\nabla})\vec{v} - (\vec{v}\cdot\vec{\nabla}p) + \vec{v}\cdot(\vec{J}\times\vec{B}) - \frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})\right].$$

Now you see that this is nothing, but $-\vec{J}\cdot(\vec{v}\times\vec{B})$, and you know that for ideal MHD case, this is equal to $\vec{J}\cdot\vec{E}$.

Now, that part is solved. So, I can simply write now $\vec{J}\cdot\vec{E}$ for this part $\vec{v}\cdot(\vec{J}\times\vec{B})$. Now, what about these 2 this one $\vec{v}\cdot(\rho\vec{v}\cdot\vec{\nabla})\vec{v}$ and this one $\frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})$ what to do with them.

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$$E = \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} + \rho e$$

$\int E\, d\tau$ is conserved in ideal MHD

$$\Rightarrow \boxed{\frac{d}{dt}\int E\, d\tau = 0} \Rightarrow \int \frac{\partial E}{\partial t}\, d\tau = 0$$

$$\partial_t\left(\frac{\rho v^2}{2}\right) = \frac{\rho}{2}\underbrace{2\vec{v}\cdot\partial_t\vec{v}}_{\partial_t(v^2)} + \frac{v^2}{2}\partial_t\rho$$

$$= \vec{v}\cdot\left[-\rho(\vec{v}\cdot\vec{\nabla})\vec{v} - \vec{\nabla}p + (\vec{J}\times\vec{B})\right] + \frac{v^2}{2}\left[-\vec{\nabla}\cdot(\rho\vec{v})\right]$$

$$= -\vec{v}\cdot(\rho\vec{v}\cdot\vec{\nabla})\vec{v} - (\vec{v}\cdot\vec{\nabla}p) + (\vec{v}\cdot(\vec{J}\times\vec{B})) - \frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})$$

$$= \vec{J}\cdot\vec{E} - (\vec{v}\cdot\vec{\nabla}p) - \rho\vec{v}\cdot\left[\nabla\left(\frac{v^2}{2}\right) - (\vec{v}\times\vec{\omega})\right] - \frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})$$

So, I mean what we can do? That we will simply try to understand that what this thing is so, I mean I can do one simple thing that from this one $\vec{v}\cdot\left(\rho\vec{v}\cdot\vec{\nabla}\right)\vec{v}$, I can write like this $-\rho\vec{v}\cdot\left[\left(\vec{\nabla}\left(\frac{v^2}{2}\right) - (\vec{v}\times\vec{\omega})\right)\right] - \frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})$. So, $\vec{v}\cdot(\vec{v}\times\vec{\omega})$ that is 0 because 2 elements in a scalar triple product are identical so, this is 0.

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$$\partial_t\left(\frac{\rho v^2}{2}\right) = \frac{\rho}{2}\underbrace{2\vec{v}\cdot\partial_t\vec{v}}_{\partial_t(v^2)} + \frac{v^2}{2}\partial_t\rho$$

$$= \vec{v}\cdot\left[-\rho(\vec{v}\cdot\vec{\nabla})\vec{v} - \vec{\nabla}p + (\vec{J}\times\vec{B})\right] + \frac{v^2}{2}\left[-\vec{\nabla}\cdot(\rho\vec{v})\right]$$

$$= -\vec{v}\cdot(\rho\vec{v}\cdot\vec{\nabla})\vec{v} - (\vec{v}\cdot\vec{\nabla}p) + (\vec{v}\cdot(\vec{J}\times\vec{B})) - \frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})$$

$$= \vec{J}\cdot\vec{E} - (\vec{v}\cdot\vec{\nabla}p) - \rho\vec{v}\cdot\left[\nabla\left(\frac{v^2}{2}\right) - (\vec{v}\times\vec{\omega})\right] - \frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})$$

$$= \vec{J}\cdot\vec{E} - (\vec{v}\cdot\vec{\nabla})p - \underbrace{\left[\rho\vec{v}\cdot\nabla\left(\frac{v^2}{2}\right) + \frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})\right]}_{\vec{\nabla}\cdot\left(\rho\frac{v^2}{2}\vec{v}\right)}$$

So, we finally, have $\vec{J}\cdot\vec{E} - \left(\vec{v}\cdot\vec{\nabla}p\right) - \left[\rho\vec{v}\cdot\vec{\nabla}\left(\frac{v^2}{2}\right) + \frac{v^2}{2}\vec{\nabla}\cdot(\rho\vec{v})\right]$.

So, if we now check the whole thing so, we have this $\vec{J}\cdot\vec{E}$ component we have this $\left(\vec{v}\cdot\vec{\nabla}p\right)$ component and we have one divergence $\vec{\nabla}\cdot\left(\rho\frac{v^2}{2}\vec{v}\right)$. So, that is already a good thing. So, we have some divergence. Then we go to the other expressions.

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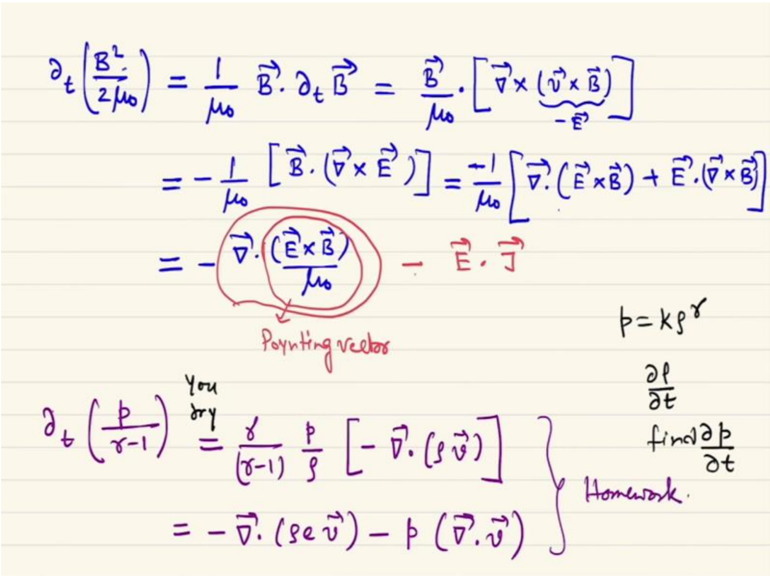

So, first we have to take the magnetic energy and the evolution of the magnetic energy density which gives this expression $\partial_t\left(\frac{B^2}{2\mu_0}\right)$ and here we have simply $\frac{1}{\mu_0}\vec{B}\cdot\partial_t\vec{B}$ and this $\partial_t\vec{B}$ is nothing, but from ideal equation ideal MHD equation this is $\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)$.

So, we can write in another compact form just by using a divergence. You can simply write as $\vec{\mathrm{B}}\cdot\left(\vec{\nabla}\times\vec{E}\right)$ simply and then you can write this is equal to $-\frac{1}{\mu_0}\left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)+\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)\right]$.

So, you have $-\vec{\nabla}\cdot\frac{(\vec{E}\times\vec{B})}{\mu_0}$, which you can simply identify as very popular pointing vector right, and the next term simply becomes $\vec{E}\cdot\vec{J}$. So, I have a $-\vec{E}\cdot\vec{J}$ and simply see in previous equation you have a $\vec{E}\cdot\vec{J}$. So, already this is good that these two things will cancel each other and we have one divergence so, nothing to worry just this part $\partial_t\left(\frac{p}{(\gamma-1)}\right)$ can raise some concern.

So, how to get rid of this term $\partial_t\left(\frac{p}{(\gamma-1)}\right)$ either we have to make it cancel by other term or it can combine with other term to give a resultant divergence. So, for that finally, if we do calculations so, that will be given by this $\partial_t\left(\frac{p}{(\gamma-1)}\right)$ and for that the trick is I have to write $\frac{\gamma}{(\gamma-1)}\frac{p}{\rho}[-\vec{\nabla}\cdot(\rho\vec{v})]$. This is a very small and elegant homework which you can do so, I just write you try. So, you have this relation p is equal to $k\rho^{\gamma}$ and you know $\frac{\partial\rho}{\partial t}$. So, from that you try to find $\frac{\partial\rho}{\partial t}$, and if you do that very carefully you will see this equation $\frac{\gamma}{(\gamma-1)}\frac{p}{\rho}[-\vec{\nabla}\cdot(\rho\vec{v})]$.

If you further just decompose all these things, you will see that this equation will be simply equal to $-\vec{\nabla}\cdot(\rho e\vec{v})-p\vec{\nabla}\cdot\vec{v}$. So, this is also a very good homework please do that homework.

Then you see that this has a divergence part so, we do not have any problem and this has a $-p\vec{\nabla}\cdot\vec{v}$. So, it has a $-p\vec{\nabla}\cdot\vec{v}$ and the previous part had $-(\vec{v}\cdot\vec{\nabla})p$. So, finally, if they are added you will simply see that we have now two terms other than the divergences.

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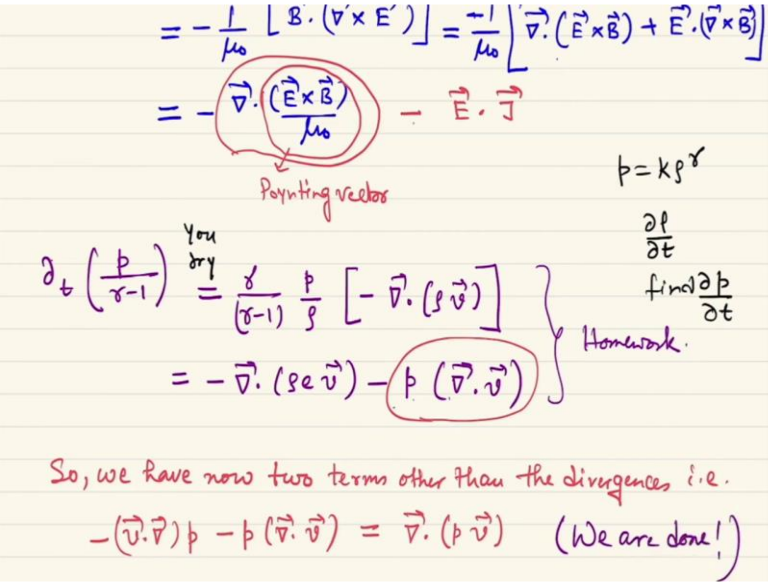

So, the combination $-p\vec{\nabla}\cdot\vec{v}-\left(\vec{v}\cdot\vec{\nabla}\right)p$ would give us $\vec{\nabla}\cdot p\vec{v}$. So, we are done. So, this is also a divergence. So, finally, $\frac{\partial}{\partial t}\left[\frac{\rho v^2}{2}+\frac{B^2}{2\mu_0}+\rho e\right]$ will be simply equal to $-\vec{\nabla}\cdot\left[\frac{\rho v^2}{2}+p+\rho e\right]\vec{v}$ plus pointing vector.

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$$\partial_t \int \left[\frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} + \rho e \right] d\tau$$

$$= -\int \vec{\nabla} \cdot \left[\left(\frac{\rho v^2}{2} + p + \rho e \right) \vec{v} + \frac{\vec{E} \times \vec{B}}{\mu_0} \right] d\tau$$

$$= -\oiint \left[\underline{\left(\frac{\rho v^2}{2} + p + \rho e \right) \vec{v}} + \frac{\vec{E} \times \vec{B}}{\mu_0} \right] \cdot d\vec{s}$$

$$= 0$$

$$-(\vec{v} \times \vec{B}) \times \vec{B}$$

$$\vec{v} \cdot d\vec{s} \qquad \vec{B} \cdot d\vec{s}$$

So, once again finally, it is true that we have to I mean like certify or we have to ascertain that we will do the integration over $d\tau$.

Now, as this $\left[\frac{\rho v^2}{2} + p + \rho e\right]\vec{v}$ total part is multiplied with $\vec{v}$, we do not have any problem because $\vec{v}$ is multiplied with $d\vec{s}$ gives you 0 and what about this term $(\vec{E} \times \vec{B})$. So, for that I have to write actually is equal to $-(\vec{v} \times \vec{B}) \times \vec{B}$. So, finally, then you have two types of terms one term like $\vec{v} \cdot d\vec{s}$ which is vanishing and there will be another term $\vec{B} \cdot d\vec{s}$.

That $\vec{B} \cdot d\vec{s}$ will also vanish otherwise the magnetic fields just gets out of the boundary surface that means this is no longer a boundary surface. So, both for velocity lines of force, and the magnetic lines of force both should be at best tangential to the surface of the boundary. Otherwise, if there is no tangential component that means, something is coming out or something is going in. So, this is no longer a boundary so, they should be 0 finally, making the whole thing to be 0 and that finally, confirms the conservation of total energy.

Now you see in the case of magneto hydrodynamic fluid the total energy is consisting of 3 parts. Kinetic energy, magnetic energy and the compressible energy which is a kind of potential energy, and we have seen that in the absence of η and υ that means, the magnetic diffusivity and the kinematic viscosity respectively the total energy of an MHD fluid is conserved.

So, I will continue this discussion just concerning the other conservation laws.

Thank you.