Introduction to Astrophysical Fluids

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Lecture - 41
Effect of Rotation in Stars

Hello, and welcome to another lecture of Introduction to Astrophysical Fluids. So, in this lecture, we continue our discussion on the effect of rotation in astrophysics. Now, in the whole discussion, in the whole series of discussion on the rotational effects in the framework of astrophysics, we started by saying that how the effect of rotation can be included in the equation of dynamics of a fluid.

Then, we also said that the treatment can be very much simple, if we just like place the observer to a co-rotating frame of reference with respect to the rotating fluid, and if the fluid is actually moving with a solid body rotation that means, a constant angular speed. Then if we just place the observer in a co-rotating frame then the fluid is everywhere at rest with respect to the observer.

So, with respect to the observer, then the whole equation becomes just a problem of hydrostatics, but of course, there will be some centrifugal term with respect to the normal hydrostatic problem, and, if the fluid is not really rotating in a solid body rotation, then it will rotate, of course, with different angular velocities that is called the differential rotation and for that we cannot make the velocity, fluid velocity at every point to be 0, but still the problem becomes a much more simpler.

Then we studied some interesting consequences of this rotation. We said that the Kelvin vorticity theorem is modified now, and you do not need an initial vorticity to generate the vorticity in the rotating fluid, the rotation itself actually will create the vorticity.

So, this type of thing we already saw, and then we discussed a very interesting theorem called the Taylor’s Proudman theorem and finally, we discussed in the last lecture, the effect of rotation on self-gravitating mass, which is of enormous practical interest for the astrophysical bodies, mostly the stars.

Now, we explored then two possible cases one is the spheroidal type of solution and one is the ellipsoid type of solution. So, of course, these two possible solutions are laid from our experience of observational experience. What we in general see in nature.

Now, it is also true that for the simplicity of the analytics, we assumed that was very special type of fluid, which is the incompressible fluid and we also said that fluid is rotating with a solid body rotation. If the fluid is no longer incompressible, fairly compressible and if the fluid is not rotating with a solid body rotation, then what happens?

Then, in some cases, we still can conclude something very much interesting that is exactly, what we are going to discuss in this lecture, but not very rigorously. So, sometimes some analytical treatments can be done for special cases, but for a general compressible fluid and general differential rotation problem, solving the effect of rotation on a self-gravitating mass is enormously complicated.

So, one very smart way is to simulate them numerically, and this is also a subject of active research. So, if you are interested you can see over internet, different papers both old and recent papers and you can be updated about the on goings research, corresponding to this subject.

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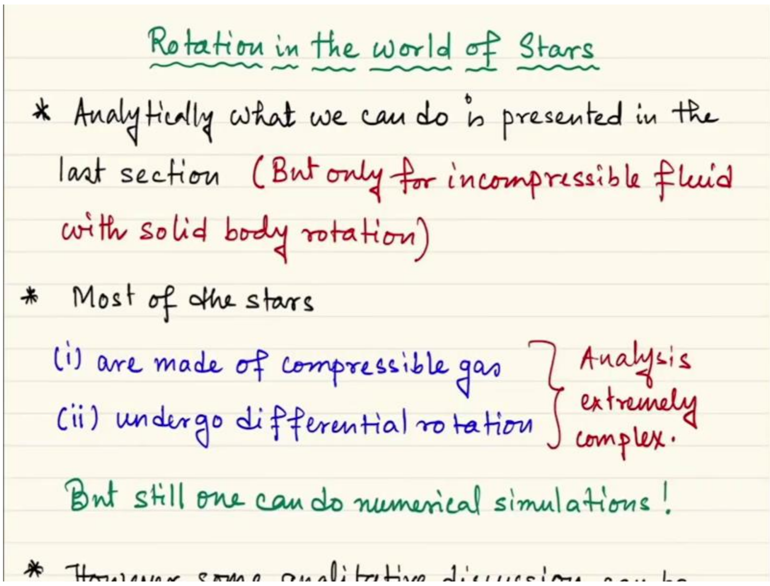

Now, here in this lecture, we will simply try to do some qualitative discussion, but somehow doing some small analytical things, we will try to like explore some qualitative conclusion or some qualitative information about the importance of rotation in different stellar objects.

So, as I just said that analytically what we presented till now, is only for compressible fluids and solid body rotation. But we know that most of the stars are made of compressible gas, and that undergo differential rotation as well. For example, the sun is a very good example, where the star is not undergoing as steady solid body rotation.

Actually, we discussed about this that only for the slowly rotating stars, the establishment of a solid body rotation type of regime can be possible, for a star which is rotating with moderate or fast I mean moderate or large angular speeds, then the possibility of differential rotation cannot be avoided. Then the analysis is extremely complex.

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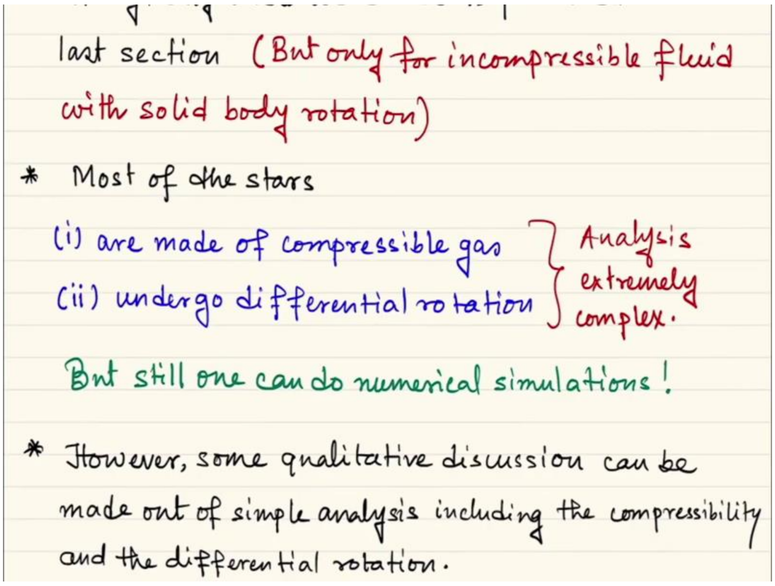

But as I just said that the only way is to do some numerical simulations and of course, for some special cases you can still do some analytical treatments. Now, in this lecture, we will just discuss some qualitative things.

But these qualitative things will be discussed including the compressibility and the differential rotation. May be the differential rotation will not be included in this lecture to be very honest, but only we will take the compressibility into account.

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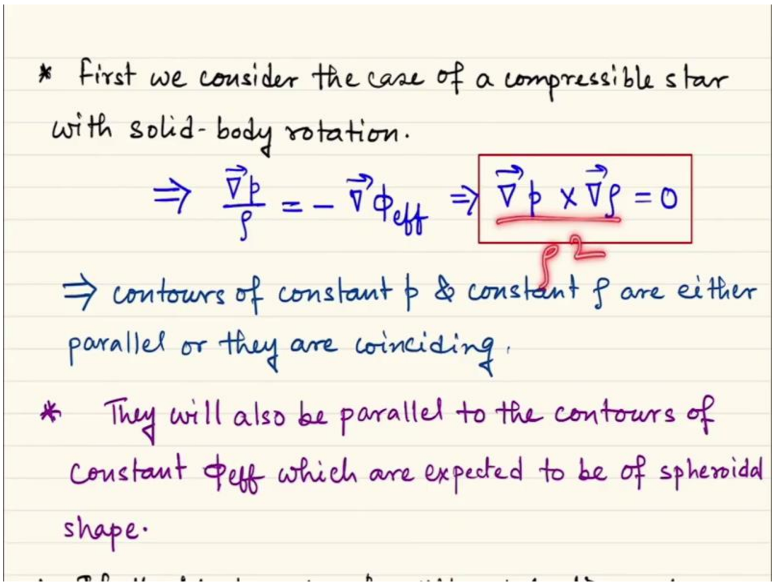

So, if you remember, first we consider the case of a compressible star with solid body rotation. Now, if you remember, the balance equation, from the basic equation of dynamics, then we can actually say that $\frac{\vec{\nabla}p}{\rho}$ will simply be counter balanced by the effective gravitational potential.

Now, if you remember that effective gravitational potential actually includes the true gravitational potential and some equivalent potential which is coming out of the centrifugal term right. So, this balance equation is still valid. Now, of course, when we have an incompressible fluid then ρ also enters into this gradient, so you have $\frac{\vec{\nabla}p}{\rho}$ is equal to $-\vec{\nabla}\varphi_{eff}$.

Now, for compressible case, we simply have this $\frac{\vec{\nabla}p}{\rho}$, ρ is a variable, is equal to $-\vec{\nabla}\varphi_{eff}$ and if we just take the curl on both sides, then we will simply obtain that $\frac{\vec{\nabla}p\times\vec{\nabla}\rho}{\rho^2}$.

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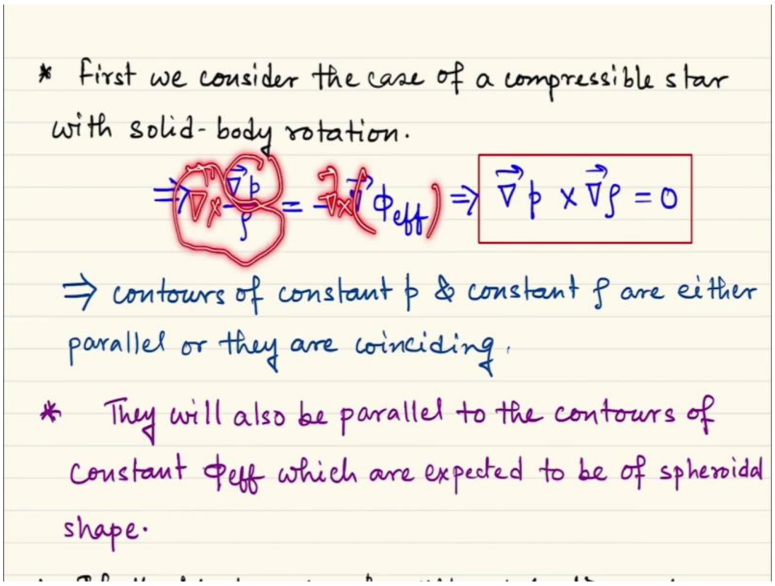

Because, if you take the curl on both sides this will be 0, because this is a pure gradient term and this has a 0 curl. So, basically this will act on this one by ρ term and it will give you $-\frac{\vec{\nabla}\rho}{\rho^2}$, and, so the whole terms will simply be $\frac{\vec{\nabla}p\times\vec{\nabla}\rho}{\rho^2}$ and that will be equal to 0.

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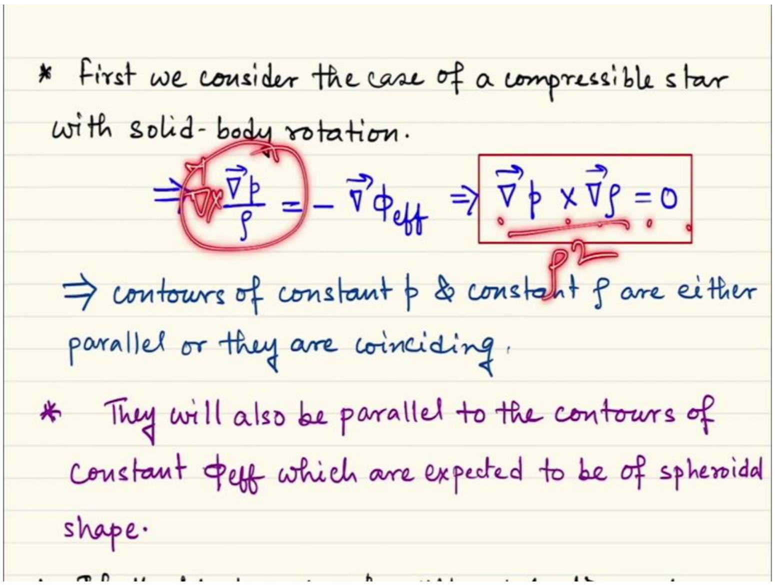

If you remember this term is nothing but the baroclinic term. Now, this one just says that the $\vec{\nabla}p$ and the $\vec{\nabla}\rho$ they are parallel. So, it simply says that as we know that the gradient direction is the normal to the direction of constant pressure and constant density. Then we can say that the contours of constant p and constant ρ are also either parallel or coinciding right. It is simply like this.

If you have a surface with constant p then the gradient direction will be this, normal to the surface, locally normal. Again, here this will be in this direction, here this will be in this direction, local normals. Now, if I simply say that the gradient direction for the ρ is also in these directions, then of course, the possible contour for the constant ρ will be something like that either this or this one itself. So, either those two will coincide or those two will be parallel right.

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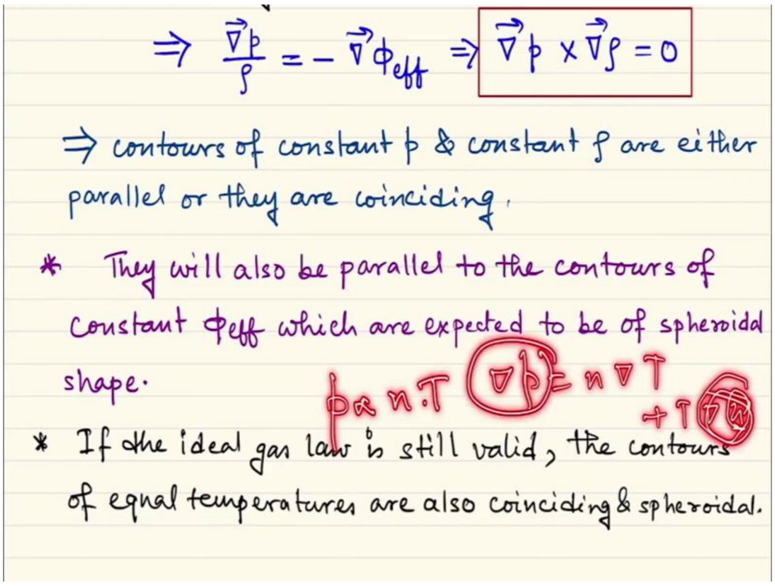

Now, if they are parallel, then actually you can say that does from this equation this $\frac{\vec{\nabla} p}{\rho} = -\vec{\nabla}\varphi_{eff}$, it is true that this equation should always valid. So, this one $\vec{\nabla} p$ and this one $\vec{\nabla}\varphi_{eff}$, they will also have the same direction, because gradient of p vector and gradient of φ_{eff}, they are just connected by a ρ which is a scalar.

We do not care whether this ρ is a constant or a variable. Even if it is a variable these two vectors $\vec{\nabla} p$ and $\vec{\nabla}\varphi_{eff}$ will be simply parallel. So, that is why we say that this will also be parallel to the contours of constant φ_{eff} which are expected to be spheroidal shape and that we actually saw in the last lecture.

That the φ_{eff} is expected to be of the spheroidal nature. So, finally, we have now three contours; one is contour for constant p, contour for constant ρ, contour for constant φ_{eff} and all of them are either parallel or coinciding. Now, if the ideal gas law is still valid here, which is completely a valid approximation here, because we are talking at moderately high temperature, where you can easily assume the gas to be ideal, then the contours of equal temperature are also coinciding and spheroidal.

So, that is true. So, because then p will be proportional to n. So, gradient of p will simply be equal to $n\vec{\nabla}T + T\vec{\nabla}n$. So, if the n is nothing but the number density. So, it will be something very similar to ρ. So, if this one $\vec{\nabla}p$ and this one $\vec{\nabla}n$, they are parallel this one $\vec{\nabla}T$ must be parallel. So, we have four type of contours which are either parallel or coinciding.

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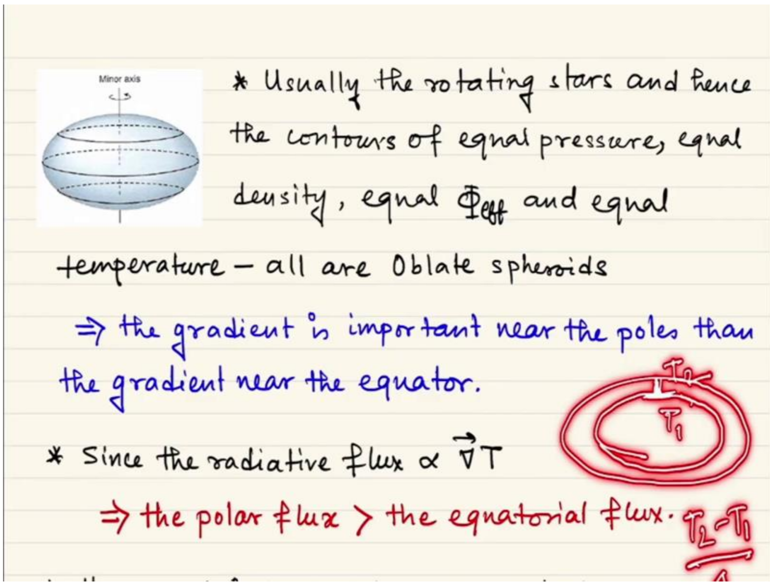

So, finally, the conclusion is that for rotating stars mostly when we are talking about the rotating axis to be vertically upward or something right, so then what we get is an oblate type of spheroid. You know like the spheroidal shapes can be of two types, one is called the prolate, where the vertical elongation and the horizontal contraction occurs, and one is called the oblate, where the vertical compression at the equatorial or the horizontal elongation occurs.

So, in our case, most of the cases, if it is oblate spheroid, the rotating stars themselves are oblate steroids then we know that all the contours of equal φ_{eff}, equal pressure, equal density, equal temperature, all are oblate steroids.

If there are oblate steroids, then, you try to understand that this whole spheroid to be consisting of some spheroid shells like this. Now, it is true that if you just a sphere and can be thought of as a number of concentric spherical shells, then the distance between two successive shells is actually constant independent of the azimuthal directions.

So, you have oblate like this and you have something like this, so, you see that the distance here is greater than the distance over here, but this is isothermal contours. So, let me just draw in a proper way. Now, here the distance between two isothermal contours they are decreasing. So, now let us say here your temperature is T_1 and here your T_2.

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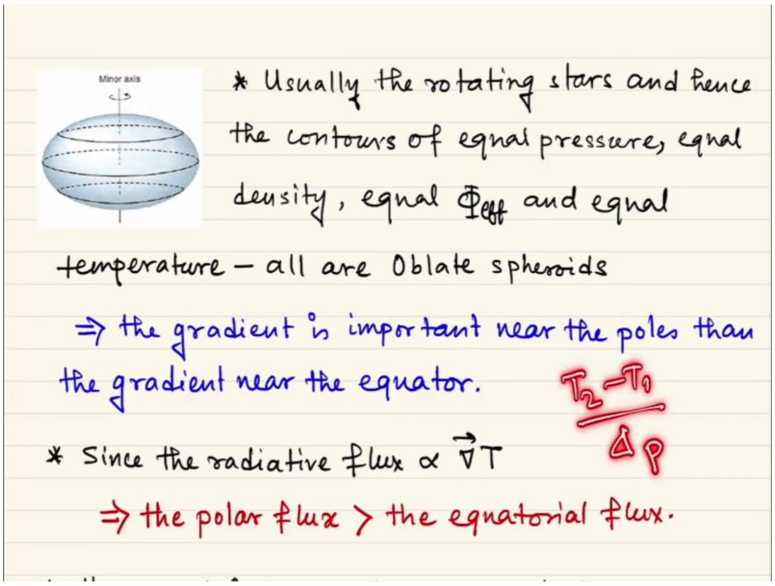

So, the temperature gradient near the poles will be $T_2 - T_1$ divided by the distance between two contours near the pole that is Δp.

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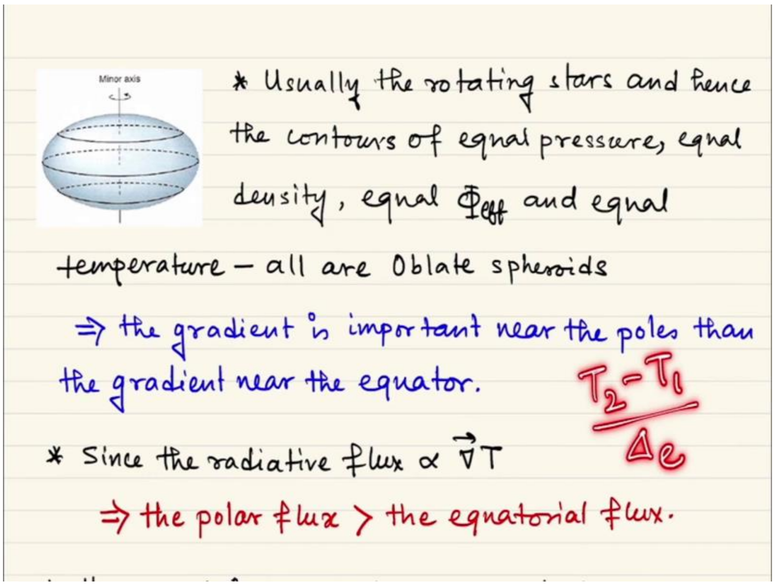

The temperature gradient near the equator will be delta equatorial. Now, as delta equatorial is greater than Δp then the temperature gradient will be greater in poles with respect to the gradient near the equator.

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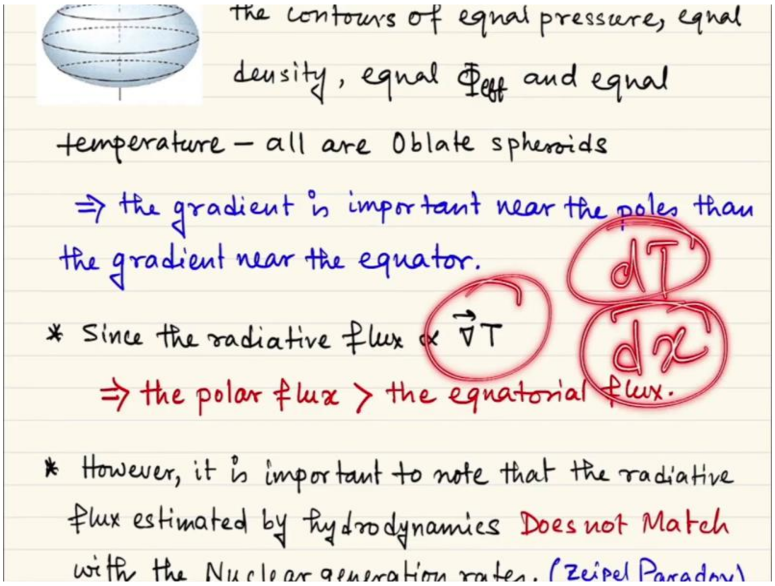

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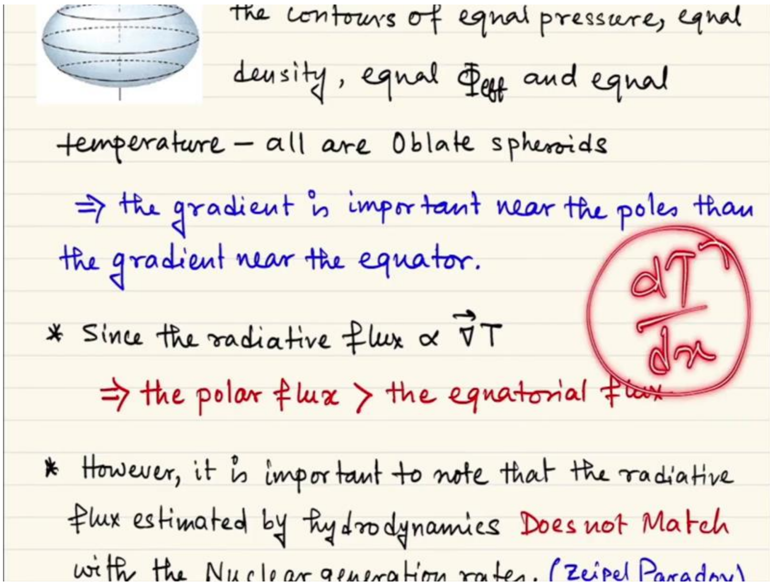

So, gradient simply means that $\frac{dT}{dx}$ type of thing. So, dT is constant, but dx is now smaller. So, I mean dx is smaller near the poles that is why $\frac{dT}{dx}$ is getting a larger value. So, the gradient is higher. Since, we know that if we are just thinking of the radiative flux, so, due to this temperature gradient some radiative flux is coming out.

Then we can easily say that the radiative flux which is coming out due to this temperature gradient, which is of hydrodynamic origin, due to the ideal gas law which is like a fluid type of thing.

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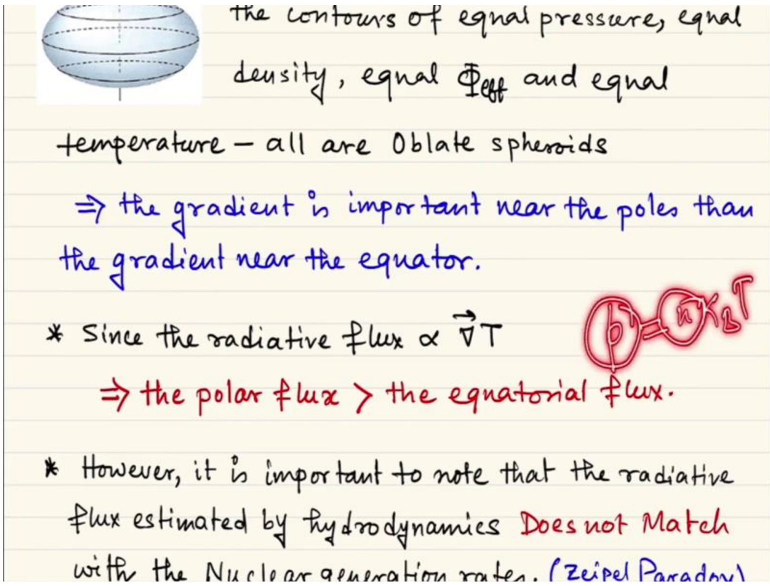

So, this is a hydrodynamic condition $p = nk_BT$. So, n is the number density, p is the pressure of the fluid, T is the temperature of the fluid, this type of thing. But, if $\vec{\nabla}T$ is greater than the corresponding radiative flux, we know from radiation physics, one can easily say that radiative flux is increases, when $\vec{\nabla}T$ increases. Not always proportional to $\vec{\nabla}T$, but this is actually increasing with $\vec{\nabla}T$.

So, as $\vec{\nabla}T$ is greater near the poles than the equator, then the polar radiation flux is greater than the equatorial flux. But the only problem we all know that the energy which is radiated should come from the normal energy source of the star, and the normal energy source of the star is nothing, but the nuclear reaction.

But this nuclear reaction gives some radiative flux estimation, which does not match with the hydrodynamic estimate of the radiative flux, and this was for the first time noticed by Zeipel and we call this as Zeipel paradox. So, how to overcome this Zeipel paradox?

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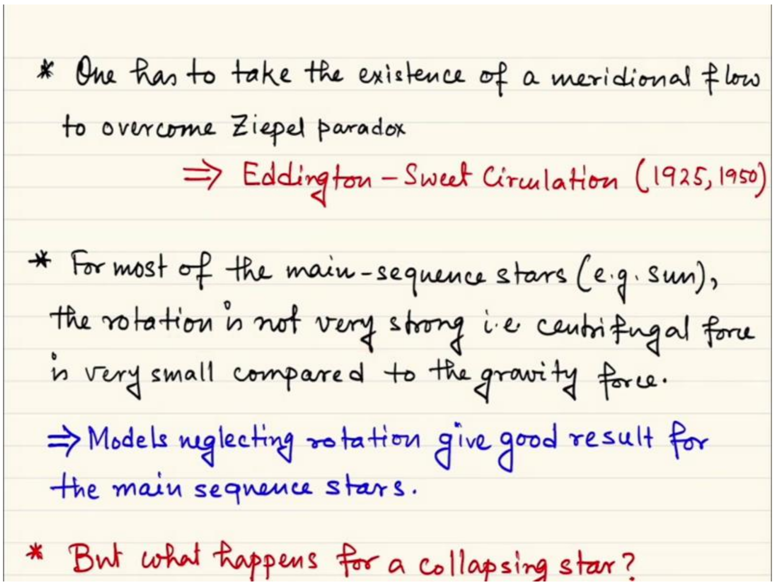

So, then one has to take the existence of a meridional flow, in the interior of this star into account, and actually, that flow changes the hydrodynamic equation, and takes part in the force balance. Finally, you will see that this will exactly give the matching with the estimated radiative flux, which should come from the nuclear reactions. So, this is known.

So, this meridional flow in the interior of the star is known as Eddington and sweet circulation, because it was for the first time discovered by Eddington and sweet did the theory in 1950, 25 years later. Now, it is true that for most of the main sequence star, for example, our sun the rotation is not very strong.

It is true that till now, we have seen that this rotation can give us something. So, for example, from rotation of a compressible fluid, we can see that the temperature, the different iso-pressure, iso-density, iso-temperature and iso-gravitational effective potential contours are all actually parallel I mean either parallel or coinciding with the oblate spheres.

Just modifying by this circulation, we have estimated the radiative flux which comes out from different parts of a star. But the question was that when actually, the effect of rotation can be important or cannot be important for a star.

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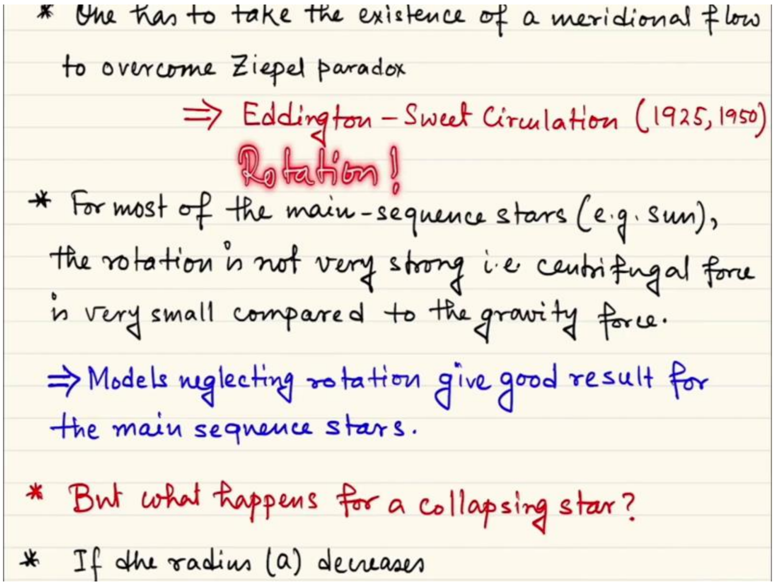

So, we have seen that in this previous part, we have already assumed that the rotational effect is there and the system has an oblate spheroidal structure, but coming back to the very fundamental question.

Now, is it true that for all stars the effect of rotation is important at all? Well, the answer is tricky. So, for most of the main sequence stars, for example, sun the rotation is actually not very strong, and when we say the rotation is not very strong, what is the meaning for that?

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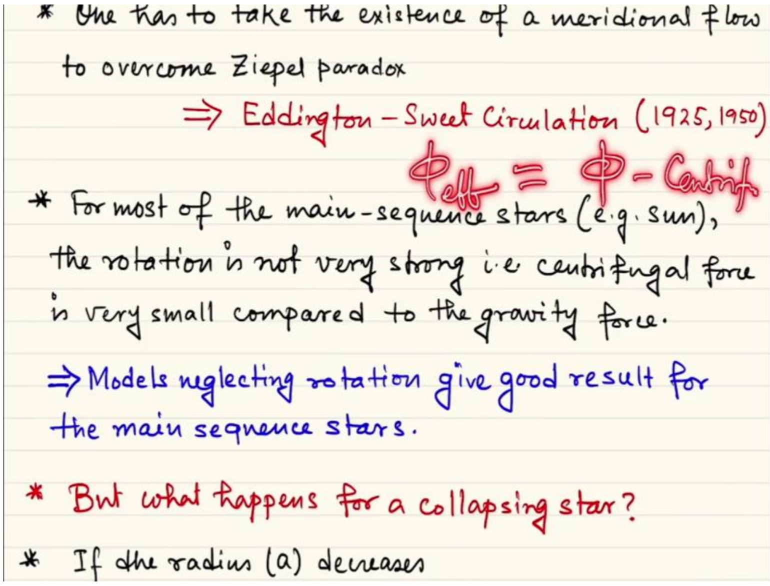

The meaning is that quantitatively it is simply equivalent to saying centrifugal force is very small compared to the gravitational force or actually, in the φ_{eff}, we have two parts one is φ another is this centrifugal part. Now, this centrifugal part is negligibly small for the main sequence star.

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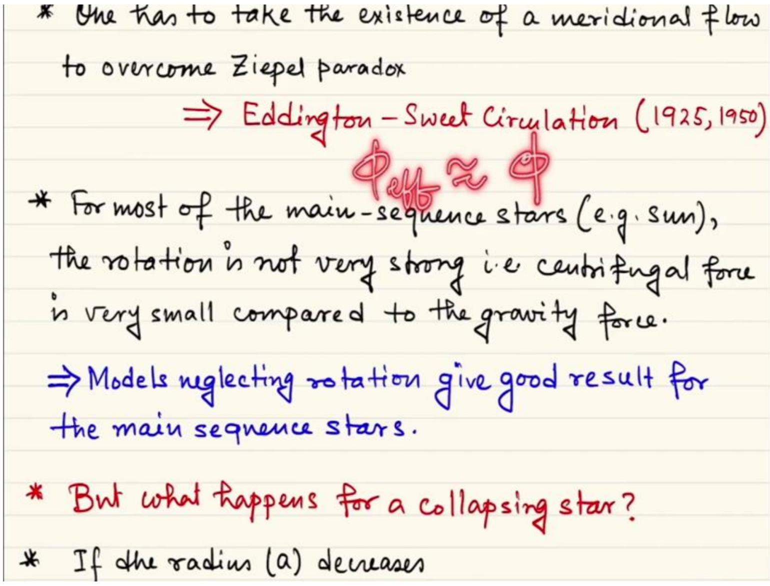

So, $\varphi_{eff}$is almost equal to φ. Now, for this type of stars actually, there are models which totally neglect the effect of rotation and we have simply reasonably good results for those stars, but it simply does not say that in astrophysics whenever you see stars you just forget about rotation.

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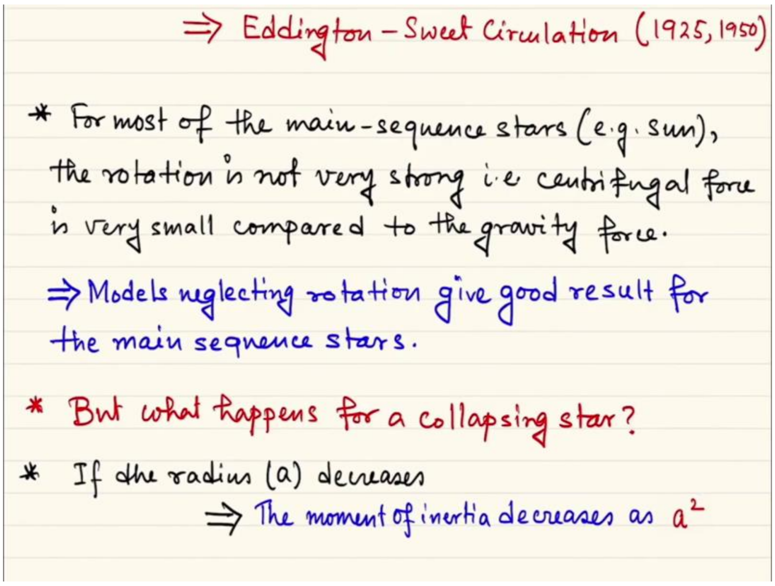

Why? Because, one question is very crucial that this was only for the main sequence stars, but what happens if we are considering a collapsing star, which is at so-called old age of its lifetime, it is not a main sequence star.

Then what happens? Then this is very tricky and very, very interesting. Let us say, for such a star which is collapsing under itself gravity, so, the radius a let us say decreases from some current value, and if radius a decreases the moment of inertia decreases as well.

Because, moment of inertia goes as a^2. So, if radius a decreases the momentum inertia also decreases as a^2.

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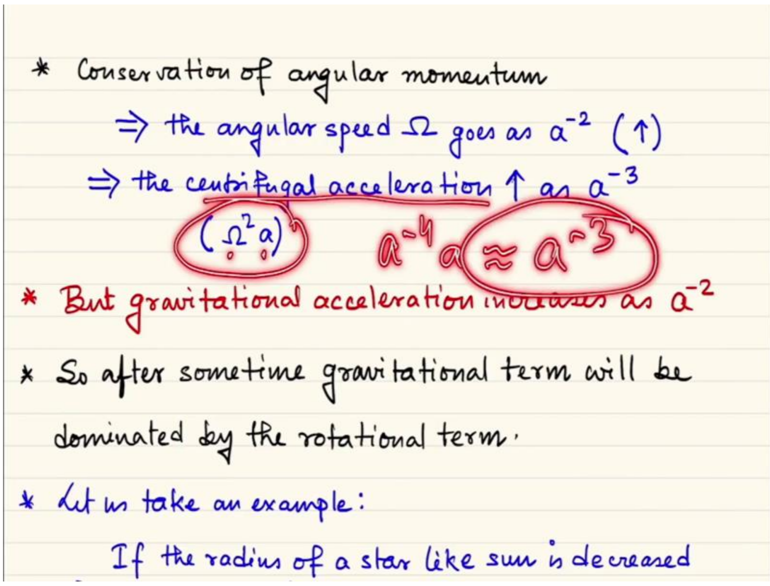

Then, if the angular momentum is conserved, because we are just thinking that there is no external deforming torque or something, so the angular momentum is conserved, the radius is decreased for some internal mechanism, for example, self-gravity. So, that is not causing any harm to the conservation of angular momentum. Then what happens? Then the angular speed simply which is Ω it goes as a^{-2}.

And if a is decreasing, Ω is actually increasing right, and then if this Ω is increasing, this follows a law of a^{-2} then the centrifugal acceleration which has a law of $\Omega^2 a$ will now follow a^{-3} correct. So, the centrifugal acceleration will also go up as a^{-3}. Now, we see that when a decreases centrifugal acceleration actually goes up at this, but gravitational acceleration increases, but it increases only as a^{-2}.

So, centrifugal acceleration will increase much more quickly. So, after some time, gravitational term will be simply dominated by the rotational term that means, the centrifugal term.

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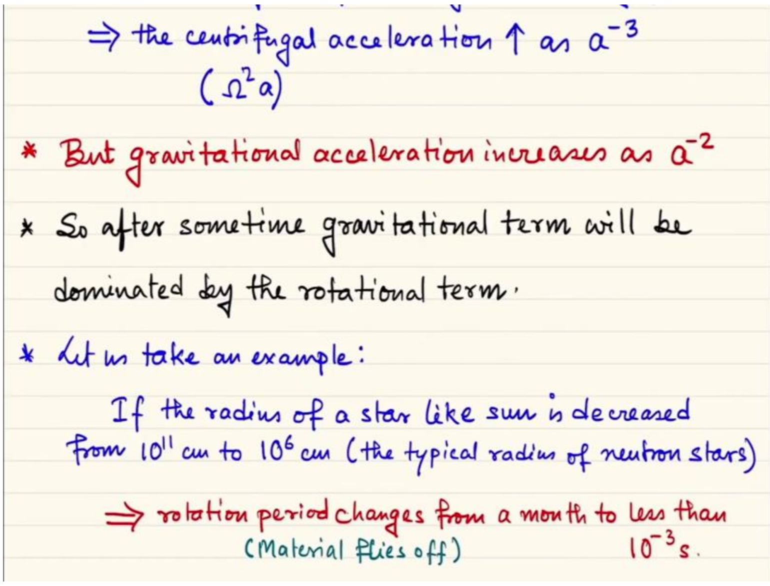

So, now, we just want to explore the whole thing just by taking simple example. So, let us say the radius of a star like sun is suddenly decreased from 10^{11} centimeter to 10^{6} centimeter, which is a characteristic radius of a neutron star. Then actually, one can calculate that the rotation period changes from a month to less than 10^{-3} second.

If that happens the centrifugal force is so strong with respect to the gravity force, then actually some matter cannot be bound by the gravitational force and some part of the material of the star actually flies off under the influence of centrifugal acceleration.

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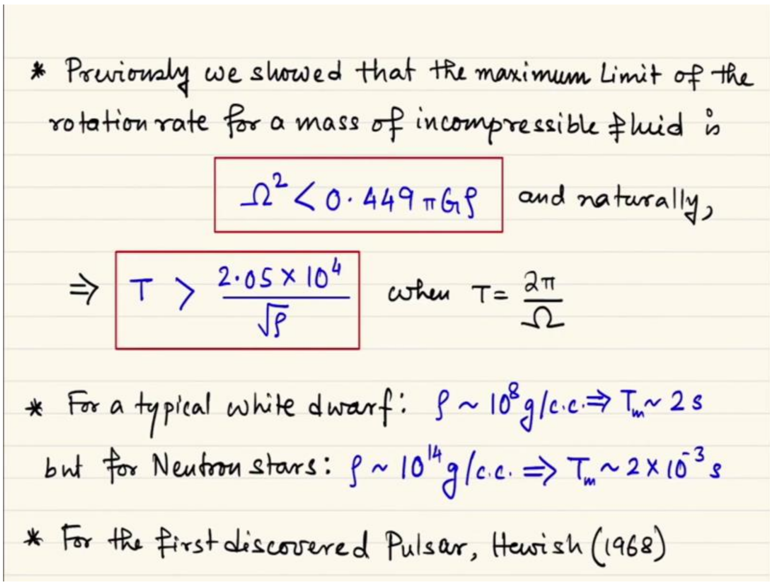

So, it is exactly something very analogous that whenever let us say, for example, you have some fluid bulb type of thing and you just rotate this fluid bulb very fast.

So, after some time, let us say some fluid will be splashing around right, that is because, the centrifugal force is much more important than the gravitational force which is trying to bind the fluid particles on the surface of the bulb, and that is why the drops of the fluid will just come around like flies off the bulb surface and it comes out.

Now, we will do some interesting thing as well. So, first we talked how should the iso-pressure, iso-density, iso-temperature contours look like, then we talked about another aspect which is very interesting that for the collapsing old stars rotation, for example, the Neutron stars rotation, actually is very important, so, we cannot neglect it like unlike a main sequence star.

Now, finally, we will see something related to that. So, when we treated the spheroidal structure, possible spheroidal structure under the interplay of self-gravity and rotation, we showed that the maximum limit of the rotation rate for a mass of incompressible fluid is given by Ω^2 which should be then less than $0.449\pi G\rho$.

So, if the rotation rate is greater than this value, this critical value then some mass will fly off. Naturally, from this one we have a lower bound for the time period, because the time period is $\frac{2\pi}{\Omega}$, and we can say that the time period should also be greater than $\frac{2.05\times10^4}{\sqrt{\rho}}$. Of course, the fluid is nearly incompressible here.

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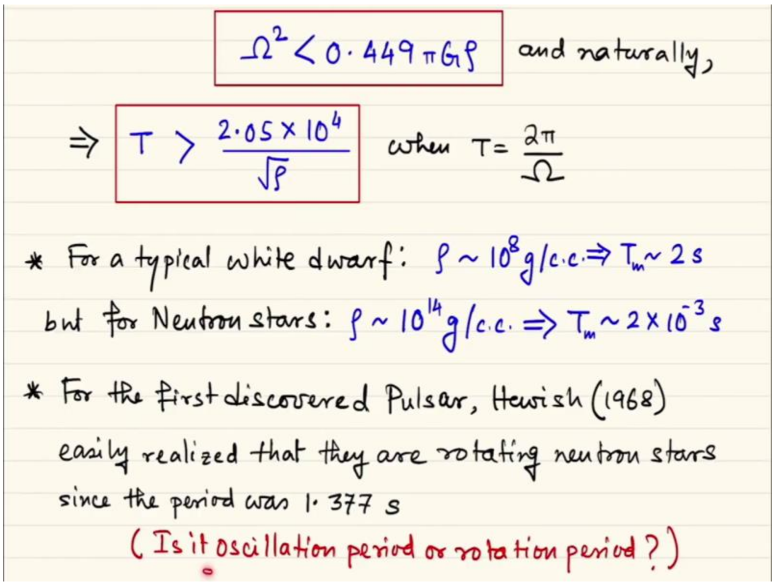

Now, we know that neither for the white dwarf nor the neutron stars, the fluid is incompressible, but we are just doing some rough theory here. So, for a typical white dwarf the matter density is almost 10^8 gram per centimeter cube, and this simply gives us a minimum time period that means, the lower interval is this one to be almost equal to 2 seconds, but for neutron stars.

Where the density is much greater 10^{14}, 6 times greater than the minimum time period permissible is 2×10^{-3} second. That is the time period, below which if the star rotates then the star will not sustain, its structure, because of the centrifugal force which is largely then dominating over the gravitational attraction force.

Now, in the year 1968, Hewish, for the first-time discovered pulsar and he saw that the period of the pulsars, the pulsars of the pulsating stars. So, period was 1.377 second, and he immediately understood that this is somehow roughly less than this. So, pulsars cannot be rotating white dwarfs, but they should be rotating neutron stars. One question that can come to your mind at this point that we have already talked about stellar oscillations and for that pulsation was also observed.

So, why pulsars were assigned not to the oscillation of the star or the stellar object, but to the rotation? That is simply because, if you just calculate the Eigen frequencies or the Eigen modes of oscillation of a white dwarf or neutron star you will simply see that the corresponding frequencies do not at all match with this observed frequency. So, that is why the only possibility was to convert that the pulsation was due to rotation.

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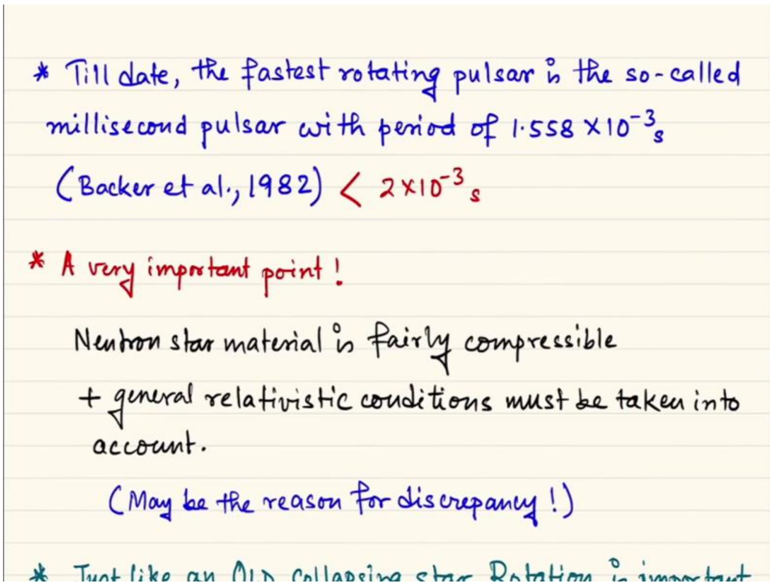

Now, till date this is very interesting, the fastest rotating pulsar is the so-called millisecond pulsar with period of 1.558×10^{-3} centimeter which was for the first time realized by Backer et al in the year 1982. Now you can tell me that what is happening here. This is even less than this one. So, neither this nor this is not exactly the cause

This is not exactly a problem because of the simple fact that these pulsars are rotating neutron stars which are very massive and very compact. So, this star material is fairly compressible first of all, and due to being very massive and very compact the general relativistic conditions that means, the curvature in the space time due to that neutron star should not be forgotten. So, these two things should be taken into account,

If you take into account, you will actually see that for white dwarf the general relativistic correction is not that important, but for neutron stars this is important, and you can actually see after inserting all this like modifications that the minimum T is well below this 1.558×10^{-3}. So, we are saved again.

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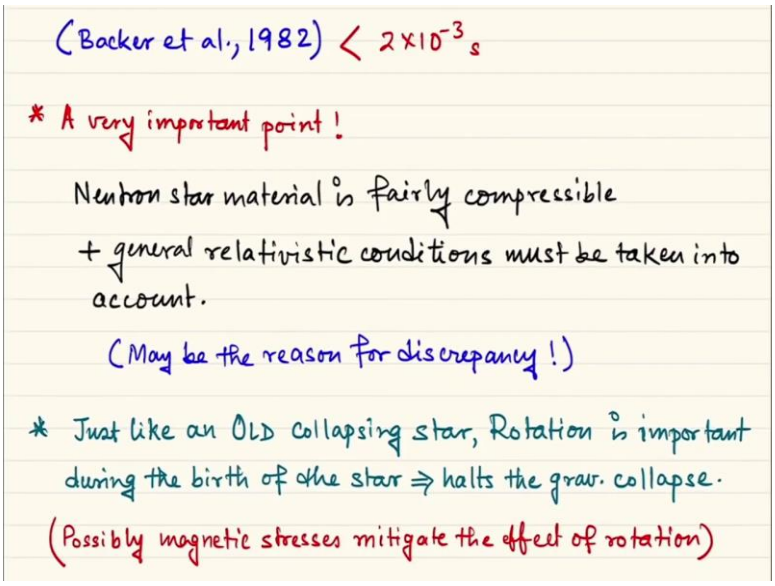

Now, you know, this is a very complex thing. So, this is one of the possibilities, of course, that the general relativistic condition and the compressibility can be the original reason for this discrepancy. But this is true that we are not 100 percent sure of that. Now, just like an old collapsing star, rotation is actually important during the birth of the star as well. This is very interesting, and what happens?

So, previously, we like discussed about the formation of a star by jeans instability and the corresponding fragmentation. So, what happens that when the under jeans instability when a star forming cloud is collapsing, thereby making the system much more prone to start formation then suddenly, if the rotation becomes important, because of the same reason we have just mentioned a few minutes ago that that the radius is suddenly decreased.

So, the increase in the centrifugal force will be dominating the increase in the gravitational force. Then, suddenly the effect of rotation simply becomes important to counterbalance and thereby halting the contraction or the collapse process.

Now, the question is, finally, we can see the stars. So, what is happening? If every time it was the reason, then we could have not seen any single star till now.

So, one very plausible theory is that the magnetic stresses, they actually somehow mitigate in some sense, the effect of the rotation, and this finally, leads to the further collapse of the star forming cloud, thereby finally giving birth to a star. So, you say that the astrophysical stories are interesting, but at every state, we have nontrivial things and we have new effects to include in order to just explain the observed phenomena and observed things.

So, this is very important in the domain of astrophysics and the research in astrophysics that we should be very open minded that means, you see till now, we have not talked about magnetic field we are doing everything hydrodynamics, but we should not be closed. So, that we just say well while doing hydrodynamics, we will not be looking at the magnetic field and the plasma aspect of the fluid, that is never working in astrophysics.

You say that some inconsistency in this hydrodynamic model must be explained in the light of the magnetic stresses. So, due to all these things, basically in the next part, I will switch to the very interesting domain called the domain of plasma. Because, this is explaining in some cases the possible phenomena, but in reality, all the fluids in astrophysical framework are plasmas. Sometimes the plasma is weakly ionized that means, there are many more neutral molecules rather than ions and electrons.

But sometimes they are quite strongly ionized. Just before ending this rotational part of this context, I would try to say one simple thing that the rotational effects are extremely complicated in case of the galaxies. In case of the galaxies, the treatment just by using hydrodynamic fluid equation is not very easy. So, then you can actually use like a collision moment equation like a collision-less system.

So, you can see in the book, this is beyond the scope of the current course, but it is also true that as I already mentioned that the ellipsoid type of structures which are already observed in case of the so-called elliptic galaxies, do not contain much angular momentum, and that is why this ellipsoid structure is really due to rotation. well, we do not know.

But on the other hand, this is true that in several cases, you can actually see the effect of the rotation. The effect of the rotation, even in several spiral galaxies, sometimes give you an estimated radiated flux, which does not match to the observed radiated flux or radiated energy flux from that galaxy, let us say from a spiral galaxy.

Most often, the main reason is thought to be the presence of so-called dark matters. So, this is all about to inspire you or rather to give you an impetus towards the interesting domains of astrophysics. So, from the next lecture, we will start discussing plasma.

Thank you very much.