Introduction to Astrophysical Fluids
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Lecture - 40
Effect of rotation on a self-gravitating mass

Hello, and welcome to another lecture of Introduction to Astrophysical Fluids. In this lecture, we continue our discussion on the effect of rotation in astrophysics. So, previously, we discussed various aspects of fluid dynamics in a rotational frame of reference. We saw that how in a rotational frame of reference, the Kelvin's vorticity theorem is modified.

Just due to Taylor Proudsman theorem, a small obstacle at the very bottom of a liquid which is very slowly rotating can actually follow the motion about the obstacle even away from the obstacle that we saw. Now, in this lecture we will do something which are much more practical for our astrophysical interest and that is nothing, but the application of this fluid dynamics in case of a self-gravitating body.

As you can easily understand, this is very much interesting and important for starts and the galaxies any self-gravitating scalar object. Now, one thing is there before starting, I am trying to mention that the analytical treatments are very much complex. So, I will not go through every single step of analytical treatments.

If necessary, I will tell you the details of the calculation. In some places, I can tell you to check that at home as usual, what I do in general in the lectures, but in some places, I will just tell you the results and if you are interested you can go through very good books by Chandra Shekhar, for example, or by Bini and Tremen. But for the scope of this course, what we are interested in are really the results of that.

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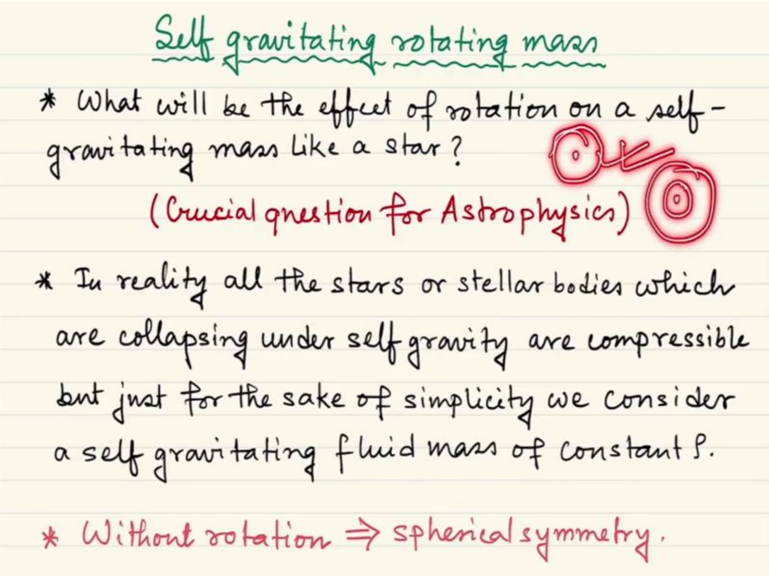

So, the self-gravitating rotating mass is something which is of the central interest of our current discussion. Now, the question is that how rotation affects the self-gravitating mass dynamics in general.

That is very important, because a star which is either in the main sequence or in the collapsing state, in both cases, it rotates. So, whether the collapsing is predominant or not, the effect of rotation can be very interesting to study to know its effect on the evolution of a star or any other self-gravitating stellar object.

Now, it is true that in reality, all the stars or stellar bodies which are collapsing under self-gravity are compressible in nature, that is true, but here for our current analytical purpose just for the sake of simplicity, we consider a self-gravitating fluid with constant density.

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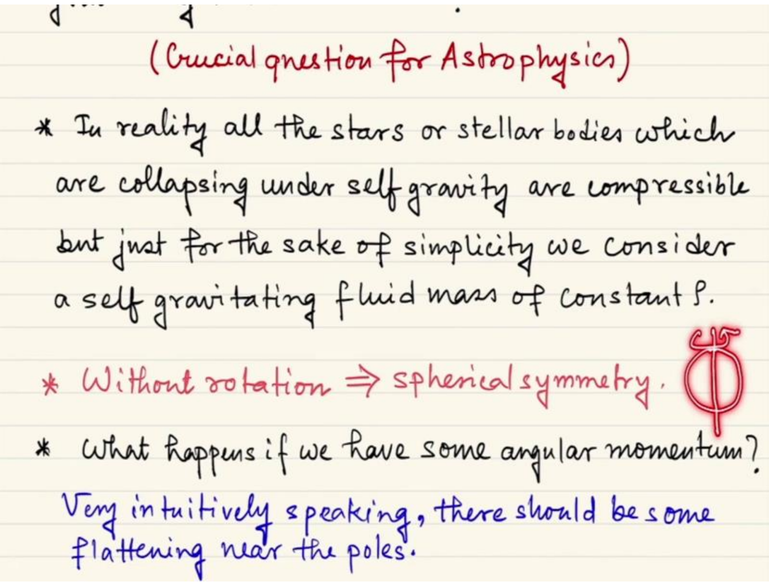

So, basically our density will be just a constant value, and we just think that the fluid is simply like a liquid. Now, it is true that if the fluid does not rotate at all then, as the fluid is only under two forces, one is the pressure gradient force and another is the gravitational force and when the gravitational force is simply counter balanced by the pressure force then we are talking about of course, hydrostatics just a very simple case.

So, in general we can think of there can be a spherical symmetry. Just even if the fluid is in motion just because the fluid is collapsing under gravitational field and the gravitational field has a radial symmetry, it is a central force, then we can think of a spherical symmetry if the fluid is not rotating, but if it rotates, then what happens to it. Now, what is the signature of being a rotating body that means that the body should have some angular momentum associated with it.

So, very intuitively speaking, from our everyday experience we can say that there should be some flattening near the poles. So, if you have seen how the potter is made there. That is also the same type of technology right. They are actually rotated with a very high speed and you will see that the central part gets flattened.

So, there should be some flattening near the pole. So, in physics whenever we talk about rotation in astrophysics, for example, mostly we will have a vertical axis of rotation.

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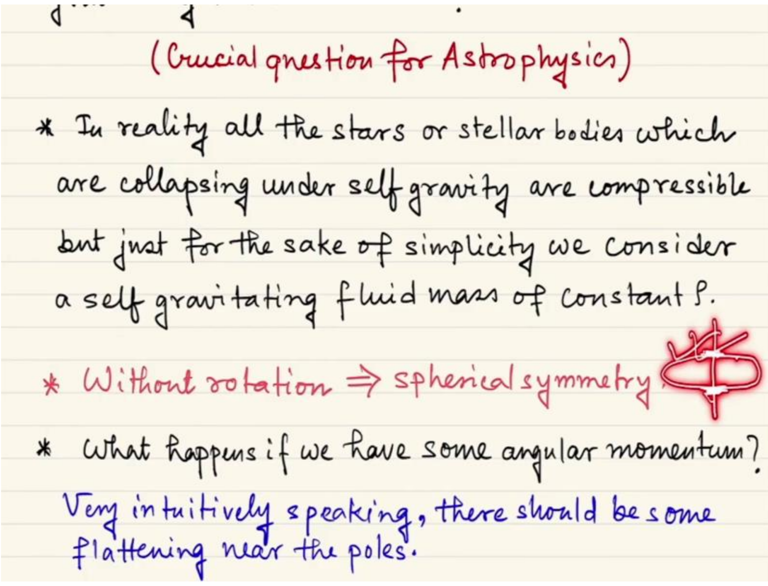

Then, what happens due to this rotation after sometimes, this the body will be flattening near the poles. So, that is the meaning of flattening near the poles. So, this body will then become oblate from sphere. We will see that.

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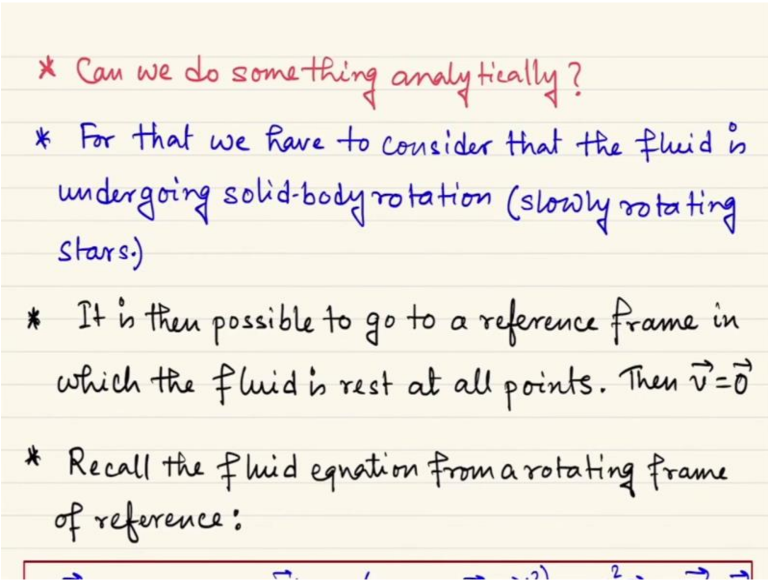

Now, of course, the thing is that can we do something analytically at this point? For that we have to consider that the fluid is not only incompressible, but also is undergoing a solid body-rotation. So, you see that incompressibility and solid body rotation these two together make our life simple and set for doing analytical things.

So, this is not a bad assumption for slowly rotating stars as we have already discussed, but for moderately or quickly rotating stars, this is not a very good solution, I mean not a reasonable assumption. So, now the question is that let us say, we are in such a situation where solid body rotation can be approximated reasonably.

Now, for those type of systems, it is actually possible to go to a reference frame as we have seen in the previous lectures, where the frame is rotating with the same angular speed as that of the fluid. So, with respect to the frame, the fluid is simply at rest at all points. So, we can simply say that with respect to the rotating frame of reference, the fluid velocity is 0, at every point.

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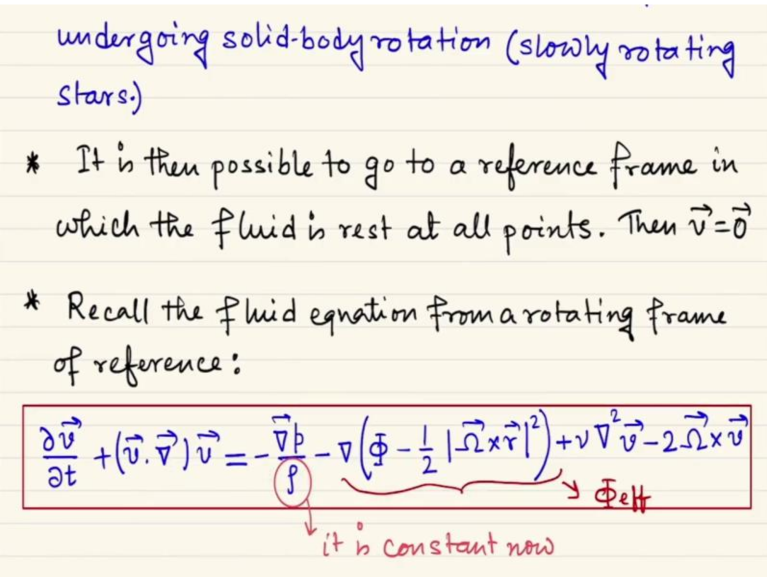

Now, recall the fluid equation from a rotating frame of reference. The total equation $\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho}$. Although here, just remember that ρ is a constant, because we are dealing with compressible fluids $-\vec{\nabla}\varphi$ which is the true gravitational potential $-\frac{1}{2}\left|\vec{\Omega}\times\vec{r}\right|^2$.

This is the part which came from the centrifugal acceleration and these two together creates φ_{eff}. So, not including the gradient, but these two is equal to φ_{eff} and plus we have the usual viscosity term and the Coriolis term. Now, when v is 0 well, we do not care about those terms which identically vanish and so, we have only these two terms $-\frac{\vec{\nabla} p}{\rho}$ and $\vec{\nabla}(\varphi - \frac{1}{2}|\vec{\Omega} \times \vec{r}|^2)$ which is surviving.

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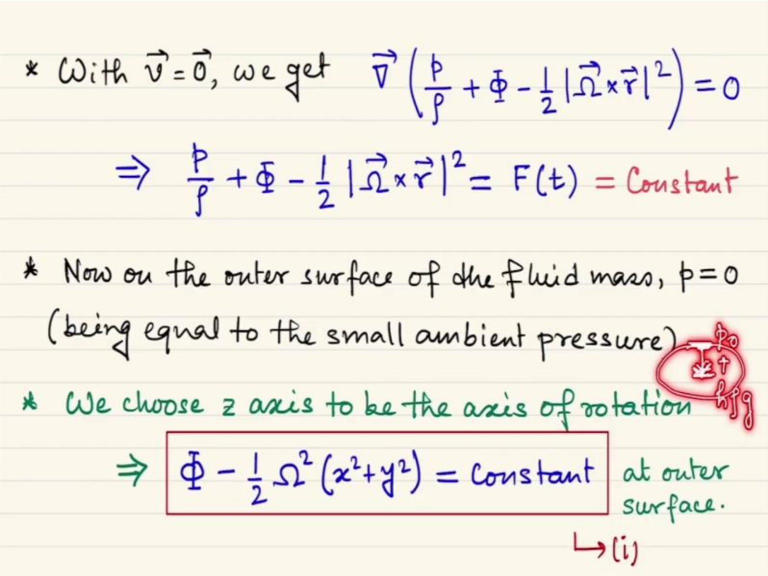

So, with v is equal to 0 is simply have $\frac{\vec{\nabla} p}{\rho}$, because ρ is now constant, plus $\vec{\nabla}(\varphi - \frac{1}{2}|\vec{\Omega} \times \vec{r}|^2)$ is equal to 0. So, the whole thing is a function of time or that can be a constant, but in general there is nothing. So, in our whole problem there is nothing which can be a source of an explicit time dependence. So, we simply say that this thing within the bracket is just a constant.

So, $\frac{p}{\rho} + \varphi - \frac{1}{2}|\vec{\Omega} \times \vec{r}|^2$ is equal to a constant. Now, this is true at any point of the fluid. Now, we are considering just the outer surface of the fluid and we say that with respect to the interior of the fluid, the outer surface which is at a pressure balance with the ambient having a very small pressure.

We can simply say that the at the outer surface, the pressure is 0. It is almost like if you have a fluid like this and you are saying that the pressure inside the fluid the pressure increases with depth. So, it is a hydrostatic problem for a liquid. So, you can call this as p_0 and here, it will be p_0 plus you all know that $h\rho g$ type of thing. So, now, you can simply understand that when p_0 is some arbitrary value we can simply say that, so, p_0 is just a reference we can set that as 0, p is 0 at the outer surface and for the outer surface only this part is constant, because p is 0.

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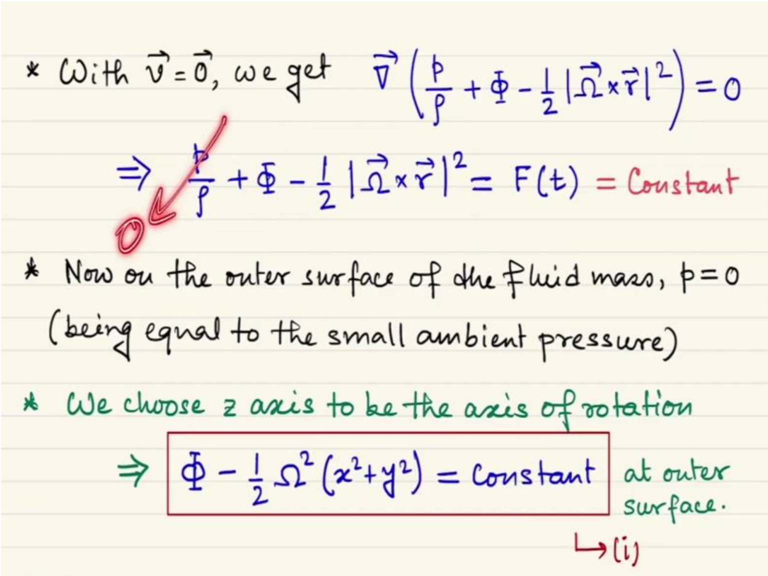

That is true only for the outer surface, and in addition, if we also assume that z-axis to be the axis of rotation.

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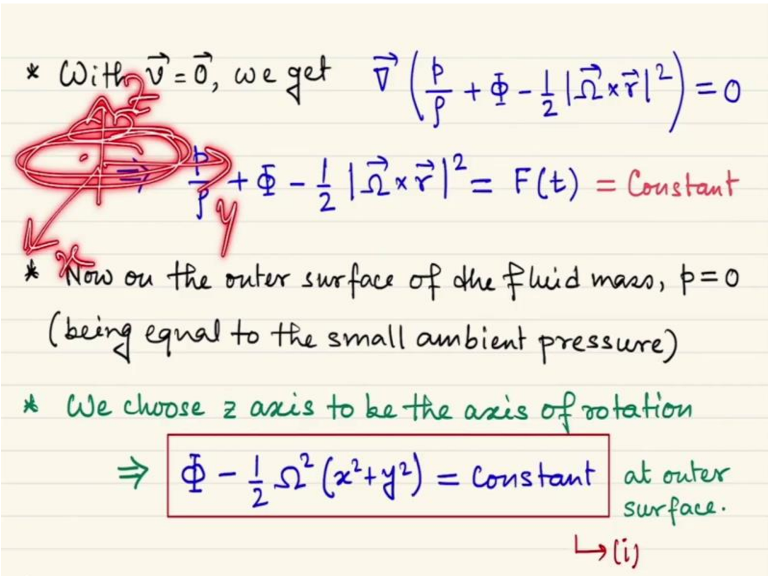

So, this is the x axis, y-axis and z-axis and we say that rotation is around, this axis then you just calculate this one. So, this is a small home task you can do ok and you will see that this is equal to constant is simply reduced to

$$\varphi - \frac{1}{2}\Omega(x^2 + y^2) = constant \qquad \text{(i)}$$

But this is true only for the outer surface where p is equal to 0.

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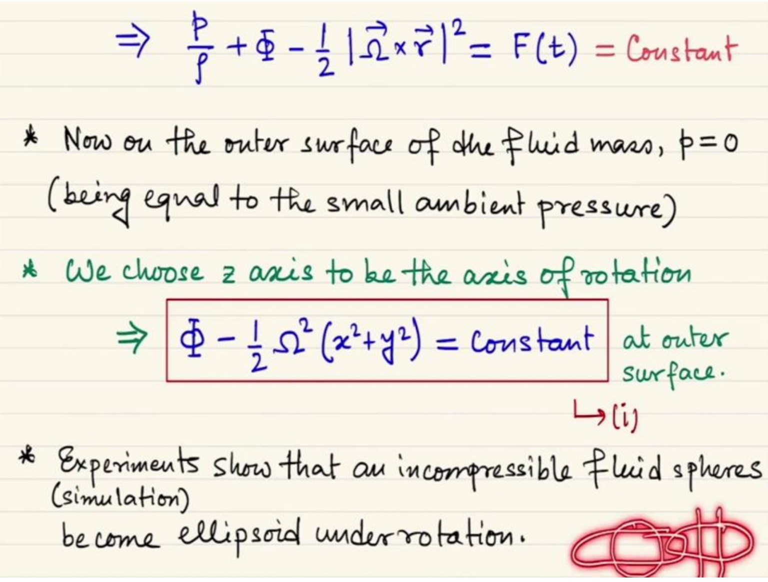

Now, experiments show that an incompressible fluid sphere becomes ellipsoid under rotation. Also, you can do that in simulations, but this is also very much intuitive. Maybe, from some as I said in a pottery farm, you can see this type of thing. So, in pottery farm, you have a soil cluster type of thing, but not this.

But you can actually see that something which is initially like this actually with time they get flattened, they get a bit ellipsoid. So, that is simply, because of the centrifugal force that is very intuitive.

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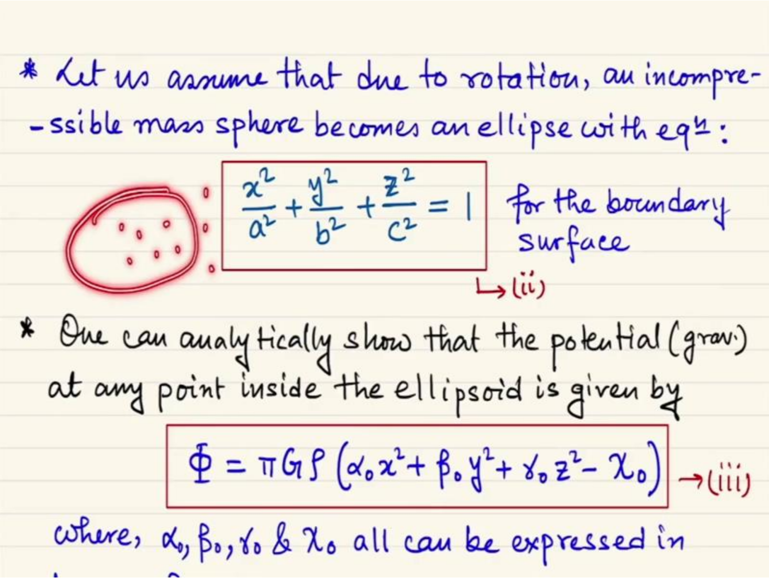

So, now, in this analytical treatment what we will try to do, we will simply assume that due to the rotation, the incompressible mass sphere actually, becomes an ellipse with equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{(ii)}$$

and that is for the boundary surface, and we will simply check that finally, where we should be self-consistent with this assumption or not.

Now, the question is that if we assume already that this should be the of the surface of the fluid under rotation, when the fluid is incompressible, then we can be interested to know the potential or the gravitational potential at any point inside the ellipsoid.

So, if we can know the gravitational potential inside the ellipsoid, then we can actually find the value of this gravitational potential at every point on the surface as well and then, mixing all this information, we will do something very interesting, but at first, we have to just check for the potential function at any arbitrary point inside and also the surface of the ellipsoid.

This is given by this expression

$$\varphi = \pi G\rho(\alpha_0 x^2 + \beta_0 y^2 + \gamma_0 z^2 - \chi_0) \qquad \text{(iii)}$$

where α_0, β_0, γ_0 and χ_0 all can be expressed in terms of a, b, c and, but there is an integration over the so-called Lagrange multiplier.

I am not going into the detail of the process. Just to tell you that only you can integrate and make the expressions out of free of the multiplier then, each of these α_0, β_0, γ_0 and χ_0 all can be expressed in terms of a, b, c.

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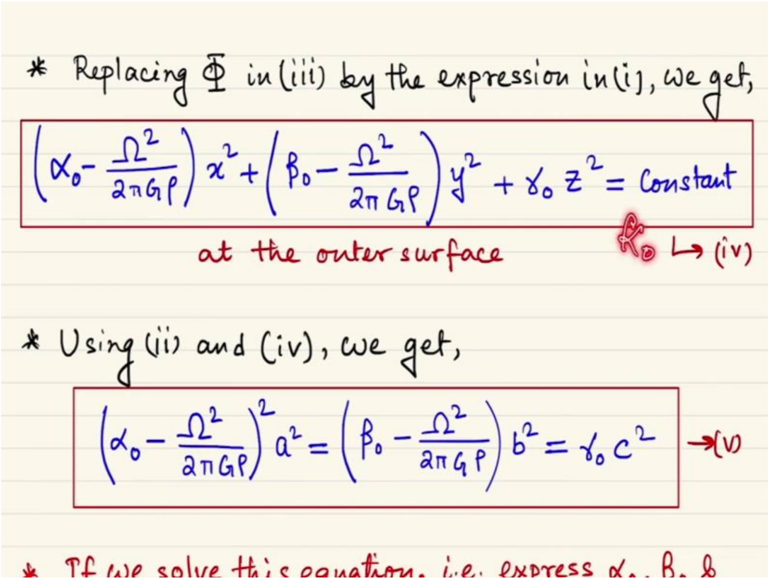

So, now the question is, whether under the given surface situation surface constraint, we can integrate that or not. So, one thing we can do. So, we now have this expression for φ and we also know that this φ in equation (iii), we can replace in equation (i).

We have this constraint equation

$$\left(\alpha_0 - \frac{\Omega^2}{2\pi G\rho}\right) x^2 + \left(\beta_0 - \frac{\Omega^2}{2\pi G\rho}\right) y^2 + \gamma_0 z^2 = constant \qquad \text{(iv)}$$

So, this is only valid for the outer surface and this is also the equation of the outer surface. Now, we have equations (ii) and (iv). So, if we just take this equation. So, we simply get the equation of the surface and not any point inside or outside, because inside or outside that should be an inequality.

The equality is only on the surface of the ellipsoid. Now, if two equations are both true for the surface of the ellipsoid, same ellipsoid, then we can simply say that their coefficients will have the same proportion.

So, if this is true, then we can simply write. This is a small home task again you can do at home that if both equations (ii) and (iv) are valid then, one can write that the coefficient of x^2

which is $\frac{1}{a^2}$ and the divided by this coefficient will be equal to or actually, we have to say, because in the right-hand side there is 1. So, we have to make this constant to be some k_0 and then, we have to divide the whole thing by k_0.

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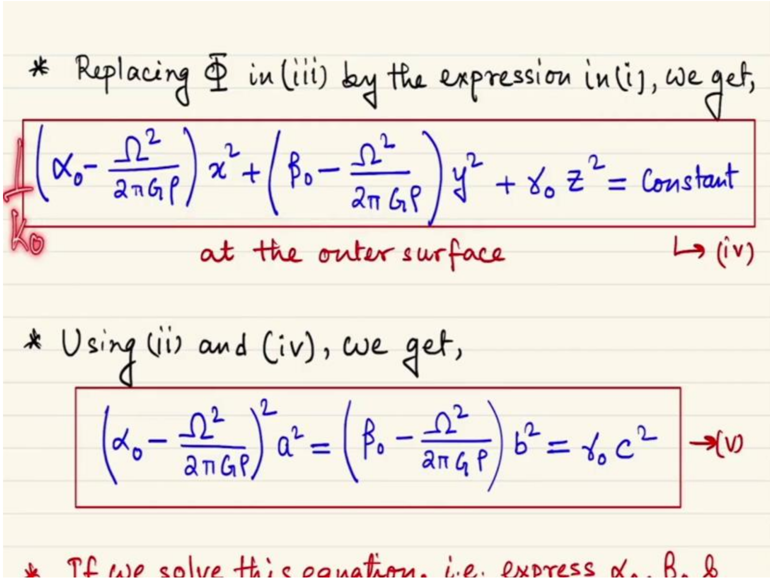

Then, we just again take the ratio. So, the ratio of the coefficient of x^2, y^2 and z^2.

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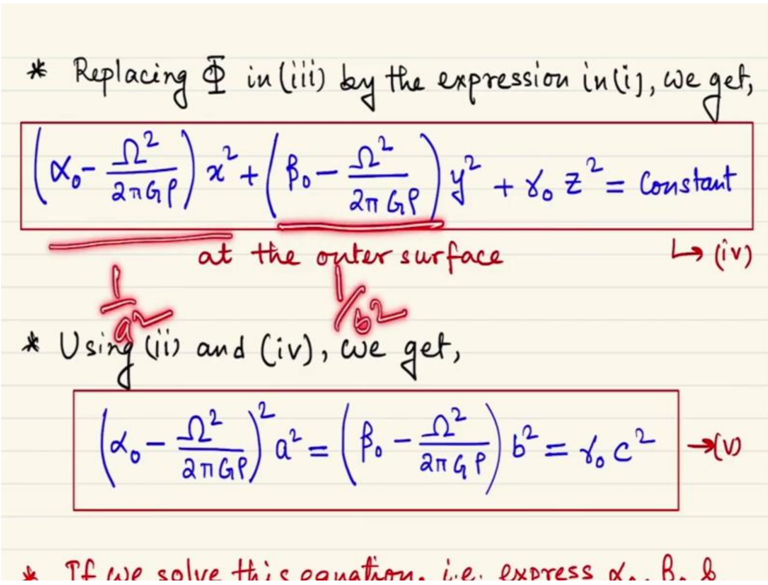

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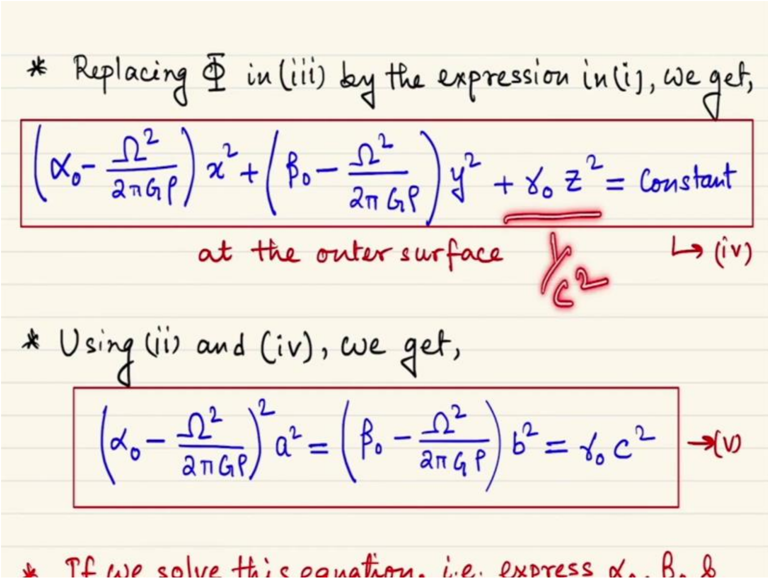

So, finally, and if you just then cancel out the whole k_0, which is the constant over here you will finally see that you have this relation

$$\left(\alpha_0 - \frac{\Omega^2}{2\pi G\rho}\right) a^2 = \left(\beta_0 - \frac{\Omega^2}{2\pi G\rho}\right) b^2 = \gamma_0 c^2 \qquad \text{(v)}$$

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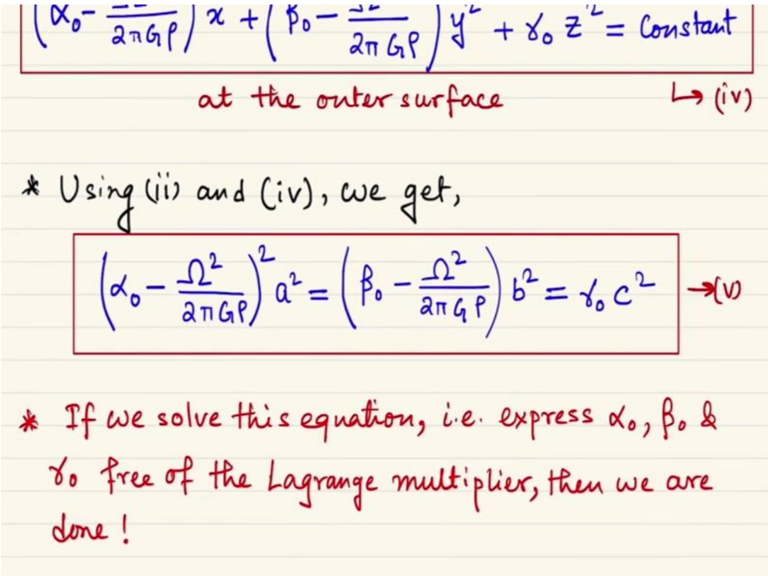

Now, if we finally, solve this equation just again by solving the integration in involving the Lagrange multipliers then, we can express α_0, β_0, γ_0 free of Lagrange multipliers and then, we are actually done. What to do for this? So, basically, one can directly search for that, but this is an extremely complicated job.

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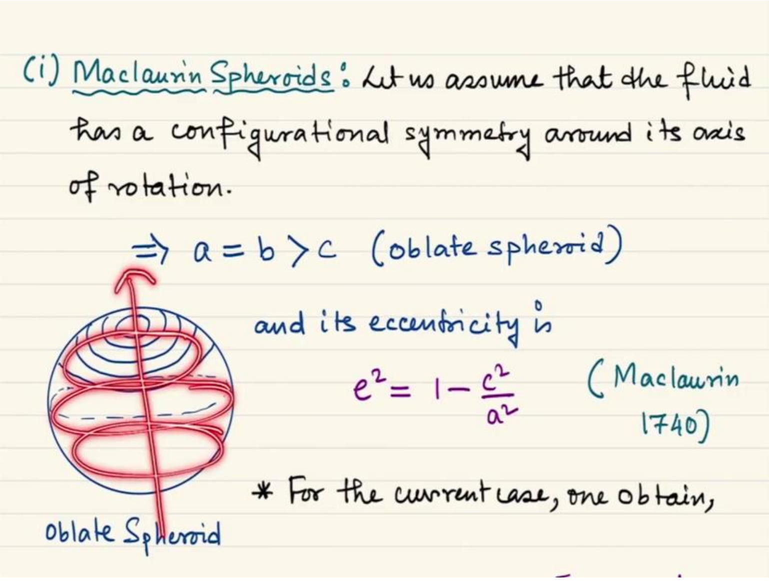

If we start with different type of geometry and, some geometries we can actually see already observationally or numerically. So, from that we can propose some specific geometry and we will see that whether solutions exist for that geometry or not. So, for example, the first one is, if we just assume that the fluid has a configurational symmetry around its axis of rotation that means, if this is the axis of rotation, then perpendicular to this, every cross section is a perfect sphere.

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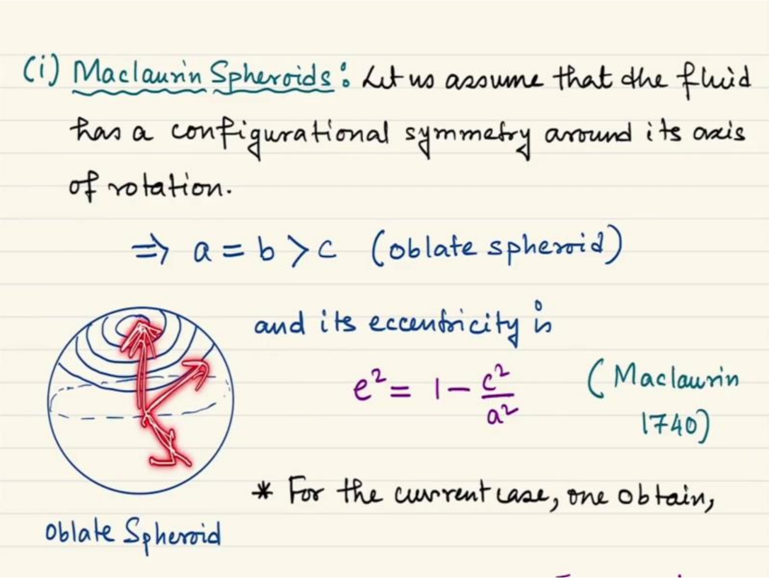

So, this body if it has three axis, then the two axis perpendicular to the axis of rotation, this one will always have the same length, because this will be a sphere. So, this type of geometrical shape is called a spheroid.

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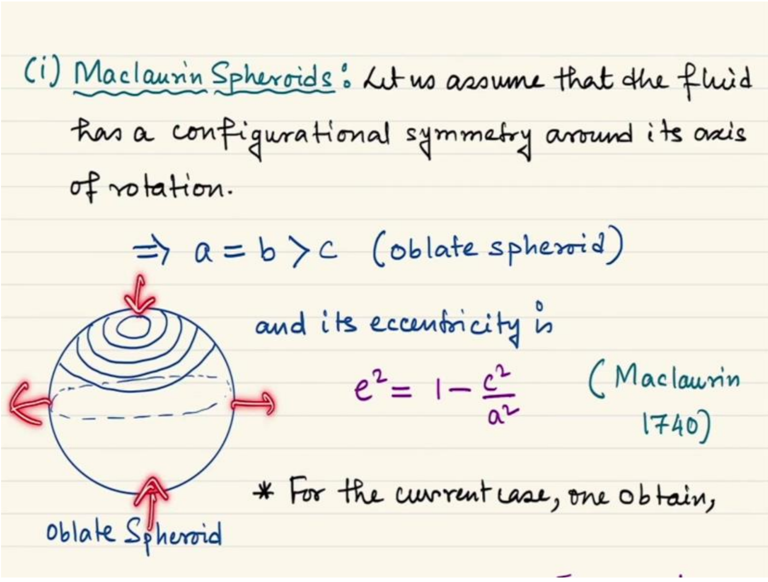

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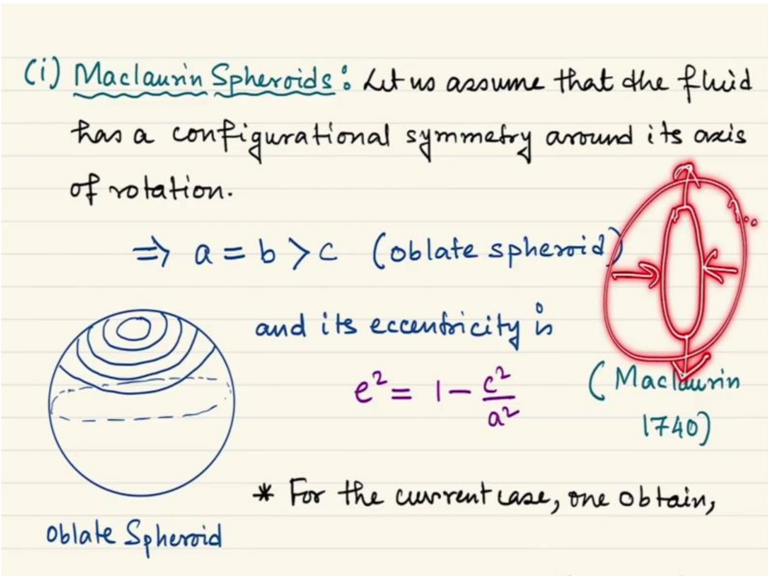

Now, spheroid can be elongated in this direction and contracted in this direction, or it can be elongated in this direction and contracted in this direction. So, if this one which is not our case is called a prolate spheroid. If this is the type of thing which is our principal interest is called an oblate spheroid. So, for oblate spheroid, we can simply say a is equal to b. So, there are not like semi-major axis or semi-minor axis.

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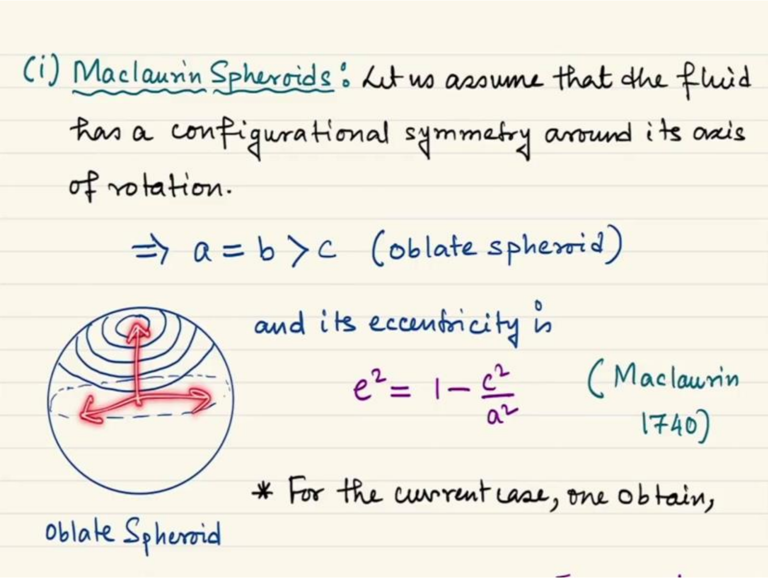

So, this is semi-major axis. So, this is called semi-minor axis in this case and the two major-axis are possible and these two major-axis are exactly equal that is a equal to b and the semi-minor axis c, which is less than both a and b.

For prolate spheroid what happens, for example, if you have a prolate spheroid like this, then, it can have, I do not know like this type of rotation.

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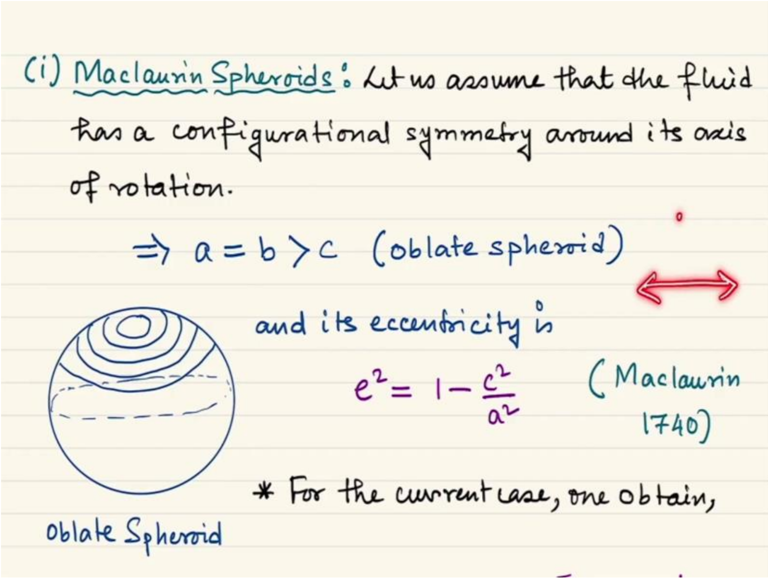

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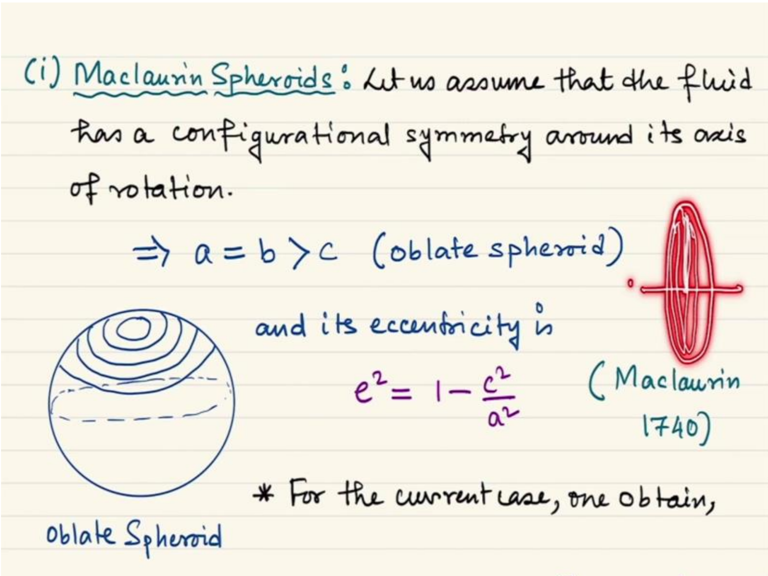

So, which is in the horizontal direction, then you have this, but then, you have circles like this. Then, this axis will be semi-major axis.

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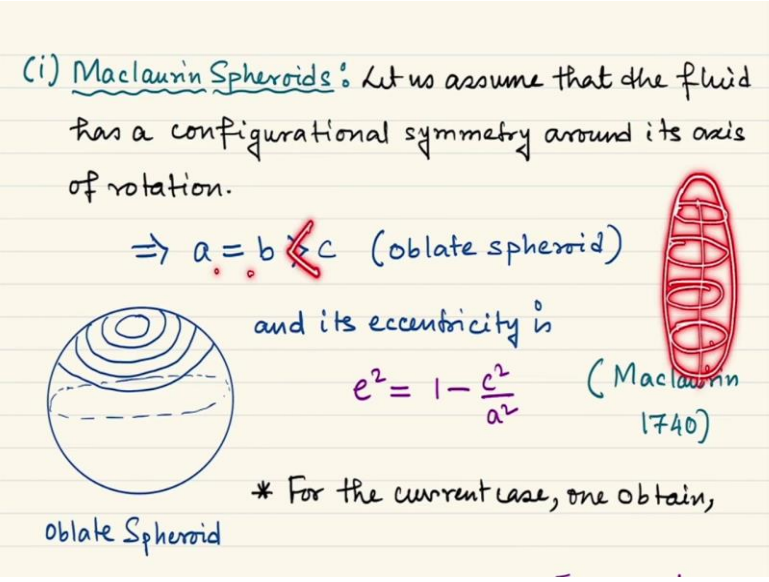

In a prolate spheroid, if you just assume that you have spheres like that, then actually, the simple condition will be a is equal to b, but that will be less than c. So, c will be the semi-major axis and which will be greater than the two semi-minor axis this type of thing. But for our case, we have this condition and the corresponding eccentricity is $e^2 = 1 - \frac{c^2}{a^2}$.

It is your home task. I have deliberately not written this. If a is not equal to b then, just find what will be the eccentricity of an ellipsoid, and if a is equal to b, then show that this reduces to this $e^2 = 1 - \frac{c^2}{b^2}$. So, I mean actually in case of this you can simply write this b^2.

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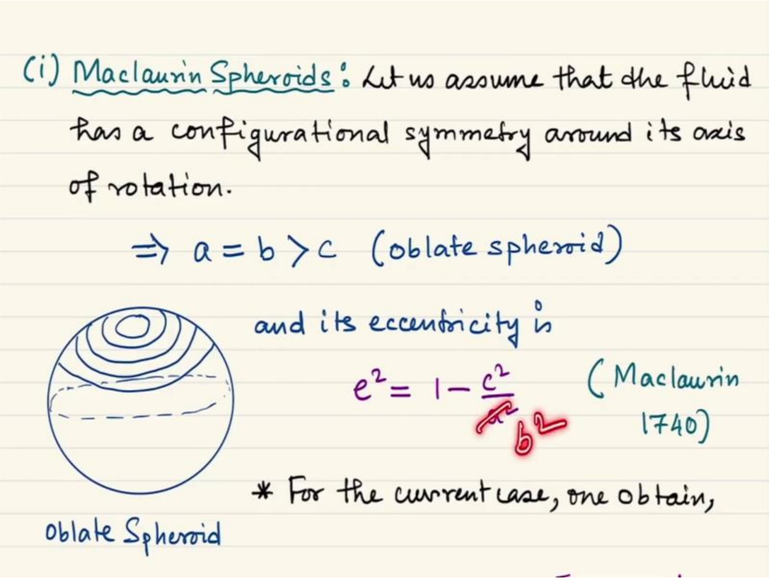

So, $1 - \frac{c^2}{b^2}$, because a is equal to b. Now, a solution for this type of spheroid was first found long time ago 1740 by Maclaurin. So, that is why this spheroid are known as Maclaurin spheroids.

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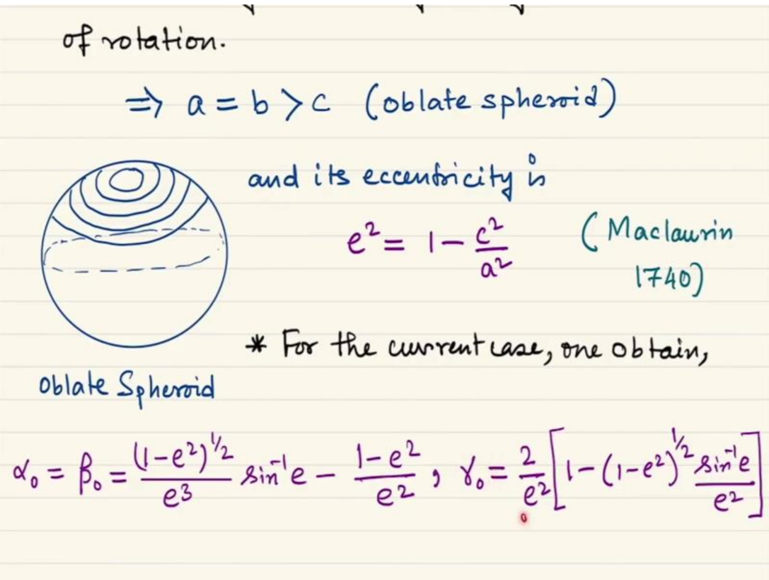

So, you see the problem is a pretty old problem. So, now, for the current case what happens that actually one can integrate and finally, one can find the values of α_0, β_0 and γ_0 free of the

Lagrange multipliers, and you can now, see that both of α_0, β_0 will be equal and will be simply equal to $\frac{(1-e^2)^{1/2}}{e^3} sin^{-1}e - \frac{1-e^2}{e^2}$.

So, e is nothing, but a function of c and a. So, basically the final message is that both α_0, β_0 they are expressible in terms of a, c here. Just for simplicity, we are just forgetting b, because b is exactly equal to a. Now, what about γ_0?

γ_0 will also be expressed in terms of e which is in terms of c and a only. So, γ_0 is also expressed being independent of the Lagrange multiplier. So, finally, we have three things. So, this is simply the indication that this is a, how to say this is a valid solution.

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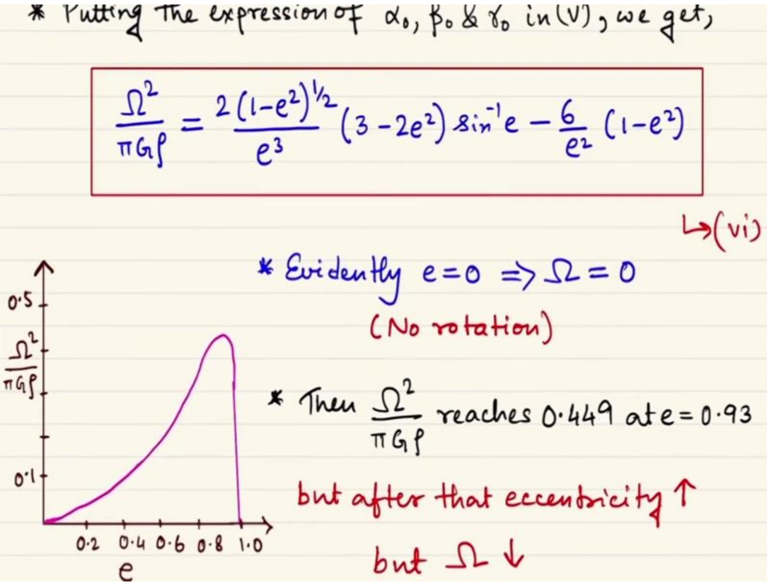

Now, if we put the expressions of α_0, β_0 and γ_0 in equation (v). What was equation (v) can you remember. If we put that finally, we have to see that we can finally, get such type of spheroids.

So, spheroid is possible, but spheroids of which eccentricity is possible that is not clear and if we write then, the α_0, β_0 and γ_0 in terms of the eccentricity then finally, we can get a simple relation comparatively simple relation between ω and e.

Of course, analytically you cannot solve that very easily, but you can do something like numerically or analytically you can try. And you can get some plots like this. So, this is at along the y-axis, you have $\frac{\Omega^2}{\pi G\rho}$ which is plotted. So, $\pi G\rho$ is nothing, but a constant. So, this is just to make these things dimension less.

So, Ω^2 is plotted as a function of e. Now, this starts from origin says that when, e is 0, Ω^2 is nothing, but 0. So, it simply says that e is 0 means c is equal to a. So, which is nothing but a perfect sphere. So, a perfect sphere basically, indicates no rotation and which is completely following our intuition.

Now, then with the rotation the eccentricity increases, but it does not increase indefinitely or limitlessly. With the increase in eccentricity, this Ω also increases, but this Ω does not increase limitlessly.

When eccentricity first increases, Ω increases up to a value where $\frac{\Omega^2}{\pi G\rho}$ reaches 0.449 and that corresponds to an eccentricity critical eccentricity of 0.93. After that if eccentricity increases, Ω actually decreases.

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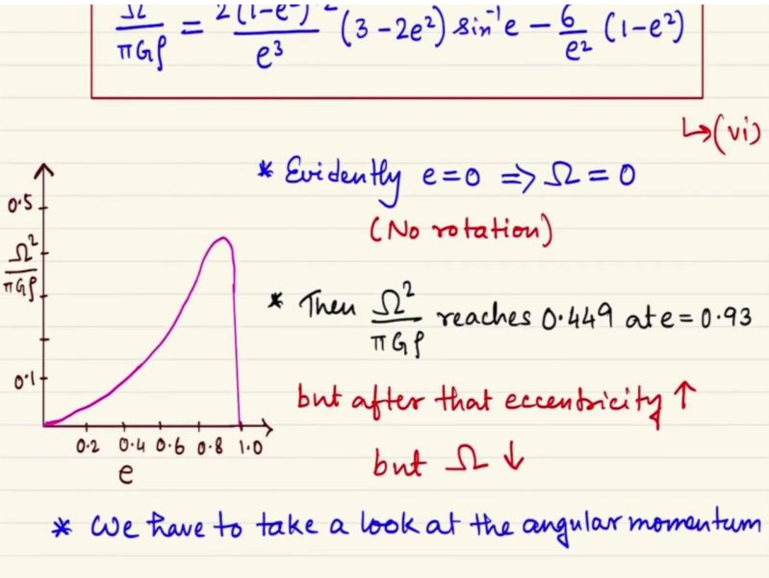

Now, why is that? There small rotation actually is causing much efficient flattening after some point. Well, for that we have to actually take a look at the angular momentum and how it changes with eccentricity, for example, or how it changes actually with Ω.

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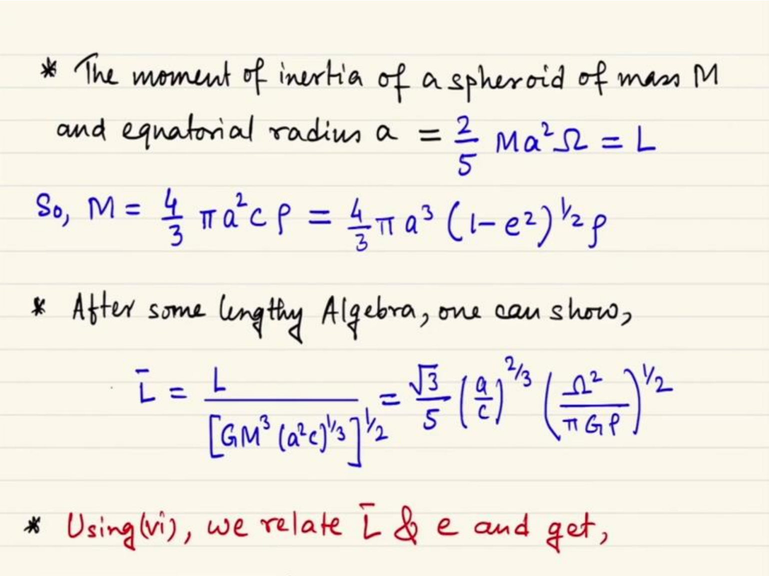

So, the moment of inertia of a spheroid of mass M and equatorial radius a is nothing, but $\frac{2}{5}Ma^2\Omega$ and which we call as L. Now, the mass of the spheroid will be nothing, but $\frac{4}{3}\pi a^2 c\rho$ this is a, b, c in general for an ellipsoid, but here it is simply a is equal to b. So, $a^2 c\rho$, and then, we just replace c in terms of a. So, we have finally, this $\frac{4}{3}\pi a^2(1-e^2)^{1/2}\rho$. Once again, in our whole analysis ρ is nothing, but a constant.

So, after some lengthy algebra, one can show this dimensionless angular momentum once again, we made it dimensionless by dividing the true angular momentum by this factor. You just check it that this has a dimension of angular momentum. If we do that, we can see that this dimensionless angular momentum is related to the angular velocity like this.

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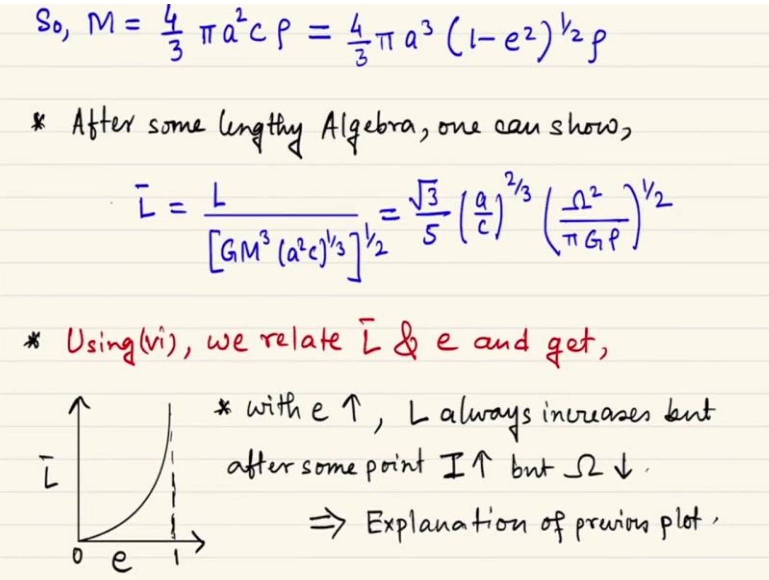

So, using equation (vi), this one

$$\frac{\Omega^2}{\pi G\rho} = 2\frac{(1-e^2)^{\frac{1}{2}}}{e^3}(3-2e^2)sin^{-1}e - 6\frac{1-e^2}{e^2} \qquad \text{(vi)}$$

We can now find a relation between L and $\bar{L}$ you know what is that $\bar{L}$ with the eccentricity e. Again, we have to solve numerically, and what we can actually see is that this is the typical behavior of $\bar{L}$ as a function of e and you will see that first e increases and finally, it saturates to some value when e reaches one. So, now, what happens, for L, this always increases.

So, with eccentricity, L is always increasing, but what happens L is increasing, but this L is increasing not enough. L is increasing in such a manner that the moment of inertia is increasing, but this Ω is no longer increasing. But the product is actually increasing that is possible, because finally, L is a product of I and Ω. So, I is increasing, Ω is not increasing that much but the product is increasing.

So, finally, it is true that although it may be paradoxical that Ω is getting down that means, that whether the effect of rotation is getting down. No, that is not true, because I is actually getting up. So, that the effect of rotation which is not actually given by Ω, but it is given by L is actually enhanced with eccentricity. So, that was the like explanation for the previous plot.

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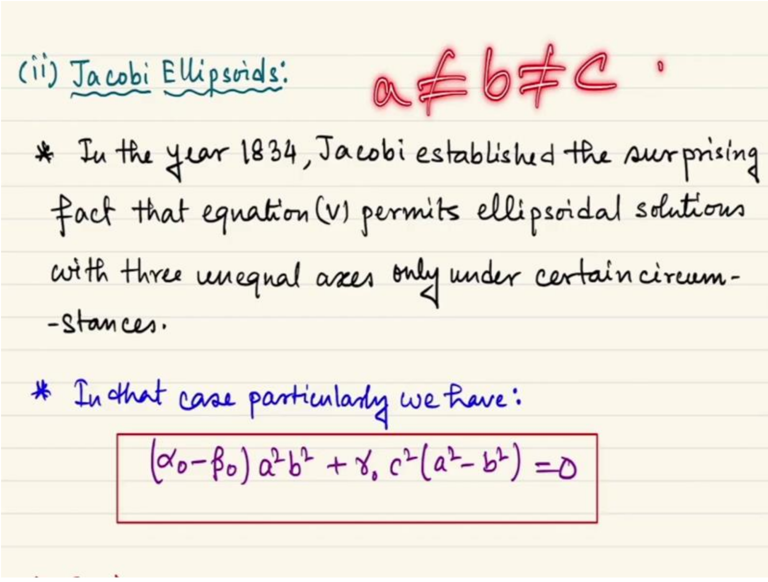

So, in this part, we saw something called the spheroid geometry. Spheroidal shape of a self-gravitating mass under rotation. Now, the 2nd possibility is Jacobi ellipsoid. So, there is another possibility where a is not equal to b not equal to c that is also possible and in the year 1834.

So, he was in 1740 and it was in 1834 almost 94 year later almost 100 year. Jacobi established the surprising fact that equation (v) also permits under certain circumstances ellipsoidal solutions as well. That means once again we can solve efficiently the integrals so that we can express all these α_0, β_0 and γ_0, irrespective of the Lagrange multipliers in terms of a, b and c and in that case finally, the constraint comes out.

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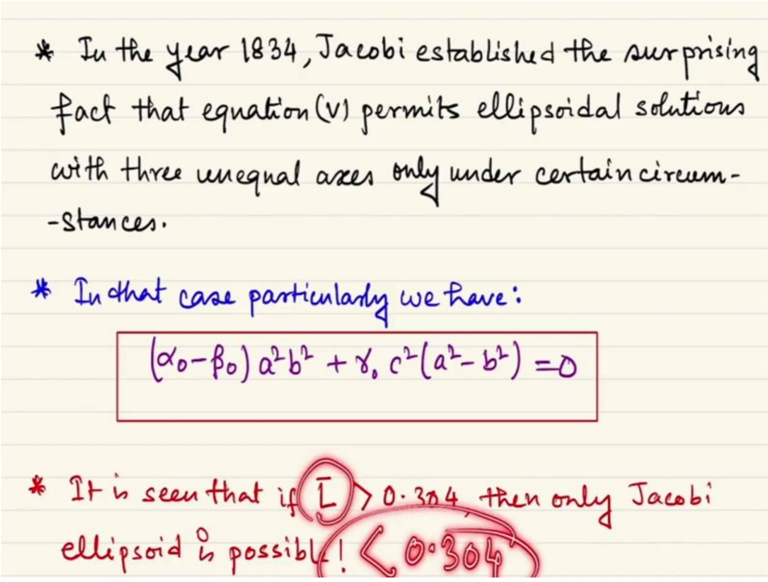

I am not going into the mathematical details. Just for your information, the constraint comes out to be $(\alpha_0 - \beta_0)a^2b^2 + \gamma_0 c^2(a^2 - b^2) = 0$. Now, so, this is exactly the constraint what we can have for a Jacobi ellipsoid.

From that just following the same type of methodology, you can check the dependence of Ω as a function of the eccentricity and also the angular momentum, and it is actually seen that by extensive analysis that when this non dimensional angular momentum $\bar{L}$ is greater than 0.304, then only Jacobi ellipsoid is possible if L is less than 0.304, the only possible geometry is the Maclaurin spheroids.

So, here, you can actually see that we have very briefly discussed the outline of two type of geometrical possibilities of the shapes of self-gravitating fluid muscles. Of course, incompressible and solid rotating uniformly fluid masses under rotation, but the true problem of course, lies in the treatment of the whole things in a compressible framework considering the differential rotations. Some of those things will be qualitatively discussed in the next lecture.

Thank you very much.