Introduction to Astrophysical Fluids
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Lecture - 04
Boltzmann equation for collisional systems I

Hello and welcome to the course of Introduction to Astrophysical Fluids. So, we just discussed that how the distribution function in the μ space can also obey the Liouville's equation and from that we also obtained so called Vlasov equation or collisionless Boltzmann equation.

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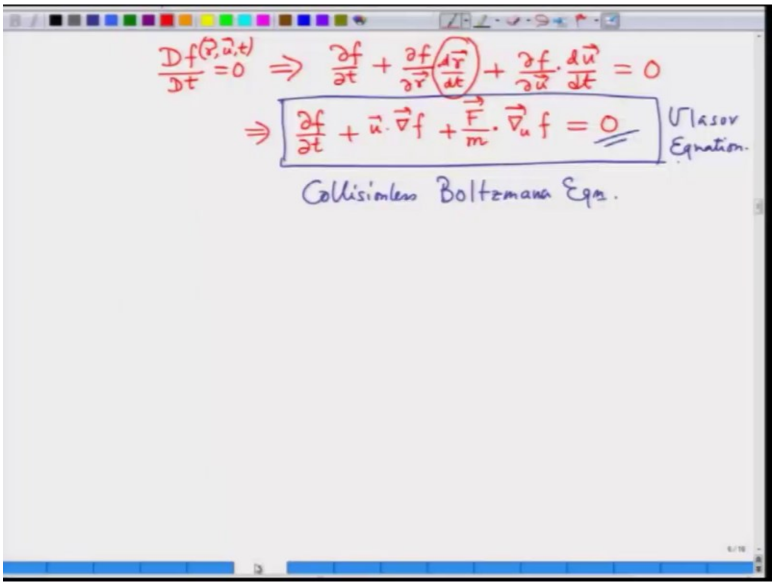

I said, if you just remember that the equation should look like this

$$\frac{\partial f(\boldsymbol{r},\boldsymbol{u},t)}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} f + \frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla}_{\boldsymbol{u}} f = 0$$

And I said that this is only possible when the system is collisionless.

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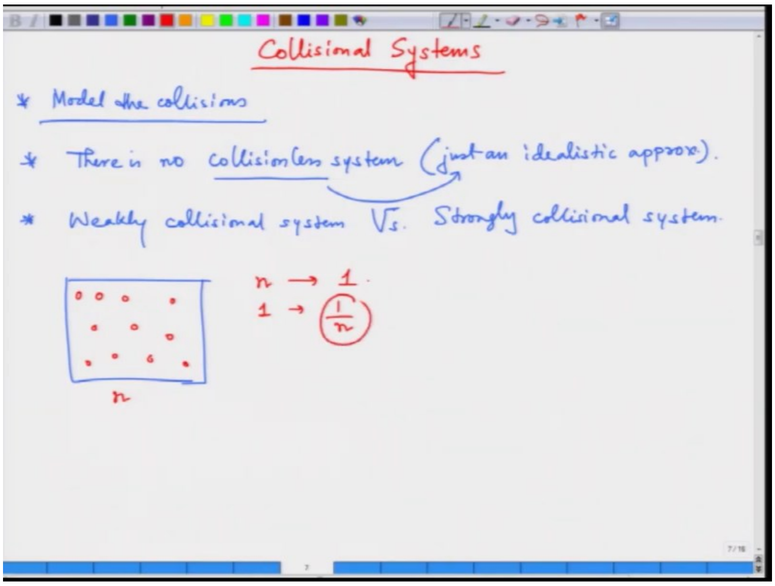

If the system is collisional, then the right-hand side will no longer be zero and how should the right-hand side be looking like? There is no unique response for that, unfortunately. But from case to case, we can try to model the collisions. Before doing that, let us first understand one conceptual thing that there is no collisionless system in nature okay. This is just an idealistic approximation. What happens in reality, we have only weakly collisional system and strongly collisional system or normal collisional system normal. Now, how to quantify that? I mean how to really define in a proper way that which system is collisional and is weakly collisional and which system is strongly collisional. Do we have a way for that?

So, in kinetic theory, maybe you have learnt the concept of mean free path. So, mean free path is nothing but the average path traversed by one particle between two successive collisions. So, from that, you can easily understand that if this mean free path basically in a given container is very large with respect to the accessible space for moving of the particles, then the system can be thought to be weakly collisional okay. That means, the most of the time the particle is moving without collision.

For example, that you have a box and the box is consisting of particles and we say that n is the number density of the box. So, n particles will be in unit volume, that is the definition of the number density. Then one particle will have $\frac{1}{n}$ volume. So, we have to understand what is the meaning of weakly collisional system and strongly collisional system and actually how can we distinguish one from the other?

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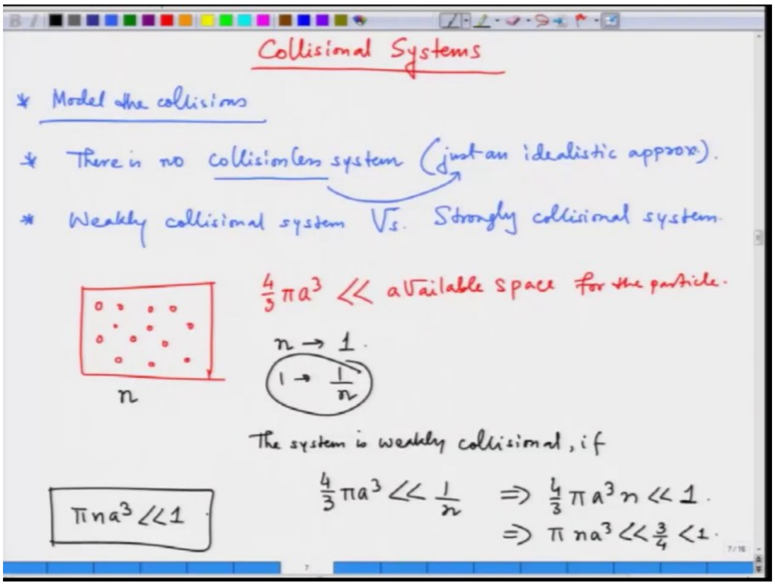

Let us take for example, a dilute gas type of system, where the gas molecules are very distance from one another. So, for such type of system, you can easily understand that the effective volume of one particle which is nothing but $\frac{4}{3}\pi a^3$ will be much less than the available space for the particle, where a is the radius of one particle.

Now, how to calculate that? There is a very elegant calculation for that. You can think that if n is the number density of the system, then n particles will have unit volume. Then, one particle will have $\frac{1}{n}$ volume.

So, if I just give you a container and the gas molecules are constrained to be in the container, then $\frac{1}{n}$ will be the volume which is available for one single particle, given n is the number density. Then, the condition for the system, the system is weakly collisional if

$$\frac{4}{3}\pi a^3 \ll \frac{1}{n} \Rightarrow \frac{4}{3}\pi a^3 n \ll 1$$

$$\Rightarrow \pi n a^3 \ll \frac{3}{4} < 1$$

$$\pi n a^3 \ll 1$$

So, if you know the molecular radius or the particle radius, sorry every time I say molecular radius because I am thinking of gas molecules, but gas particles in general and if you know the number density, then basically you can see whether the system is weakly collisional or not ok.

Now, another way of seeing whether the system is weakly collisional or not, that is the concept of mean free path. So, the mean free path basically is the average path traversed by one particle of the gas between two successive collisions right.

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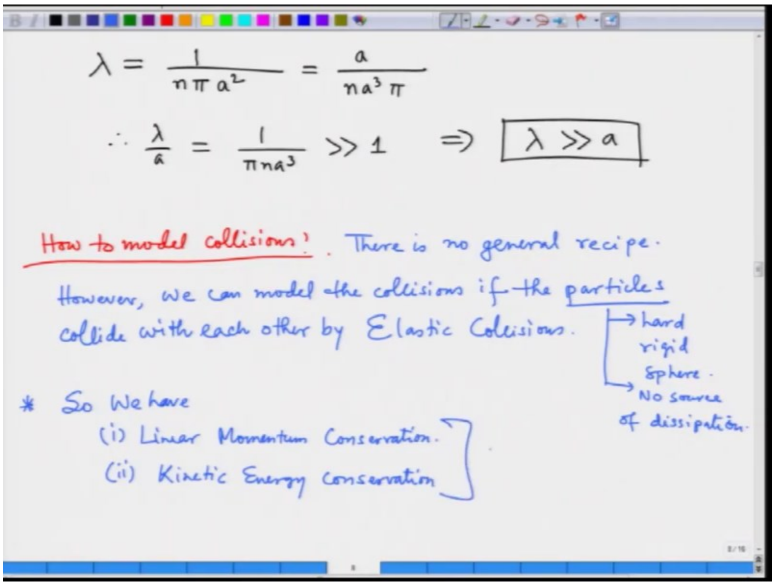

And if you remember our rough approximation gives quantitative formula for mean free path

$$\lambda = \frac{1}{n\pi a^2} = \frac{a}{na^3\pi} \Rightarrow \frac{\lambda}{a} = \frac{1}{\pi n a^3} \gg 1$$

$$\Rightarrow \lambda \gg a$$

Where, a is the particle radius. It simply says that a system is weakly collisional, if the mean free path is very-very large with respect to the molecular or the particle radius okay.

In the same manner, if the mean free path is of the order or even it is not very-very large, but it is just moderately large with respect to the molecular radius or the particle radius of the system, then the system can be thought to be collisional or moderately collisional.

So, a dense gas is an example of a collisional system. Now, the question is how to model collisions? The answer is there is no general recipe. However, we can model the collisions, if the particles collide with each other by binary elastic collisions. That means, that every single particle will be considered to be as hard rigid sphere type of thing and no source of dissipation. So, no energy will be dissipated and actually, I have to say that the kinetic energy will be conserved before and after the collision. That is the definition of elastic collision, you all know that linear momentum is conserved in any type of collision whether collision is elastic or inelastic or partially elastic; but kinetic energy is conserved only in the case of elastic collisions.

So, basically, we are taking the case where we have two conservations; one is the linear momentum conservation and other is kinetic energy conservation, before and after the collision.

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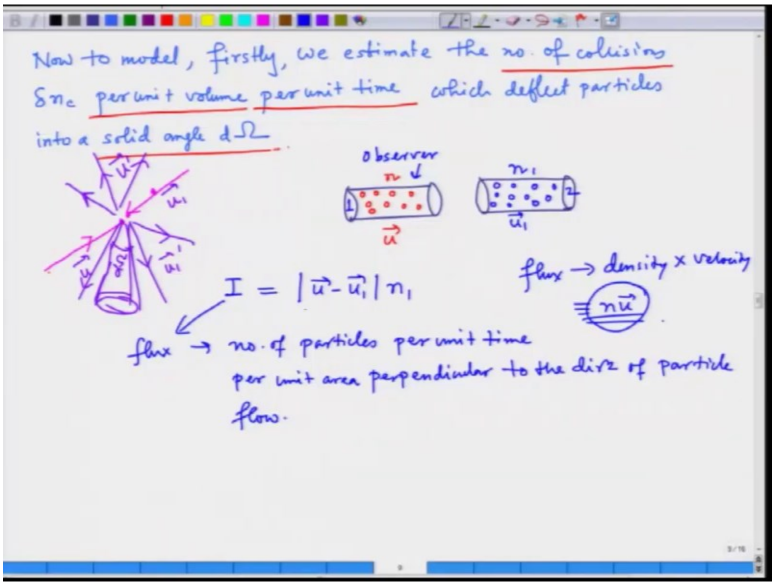

Firstly, we estimate the number of collisions, let δn_c be the number of collisions per unit volume, per unit time, these things are important; per unit volume per unit time, which deflect particles into a solid angle $d\Omega$; that is also important. Now, what is the picture for that? Let us say this particle is coming from this direction ok and this particle is coming from this direction ok; this one. (Refer Slide Time: 12:34)

Now, when they collide over here; then, after the collision, the new particles will go let us say here and here for example. So, let us consider the velocities of particles before collisions be $\boldsymbol{u_1}$ and $\boldsymbol{u}$ and after collision it became $\boldsymbol{u'_1}$ and $\boldsymbol{u'}$. Let us say, someone is counting that how many particles are coming into this direction, basically just keeping the momentum conservation intact, the two particles can go in any possible direction. So, someone can put a detector, which just calculates the particles which are deflected after the collision in a solid angle which is $d\Omega$ and we are interested in calculating.

So, δn_c will be the number of collisions per unit volume per unit time and deflects the particles into a solid angle $d\Omega$. Now, how to do that theoretically? Well, we have to start from very simplistic assumption.

Let us suppose we have two particle beams. For simplicity, we just take like a homogenous cylinder type of thing and one is just full of particles having velocity $\boldsymbol{u}$ and with number density n and another is of has a number density n_1 and velocity $\boldsymbol{u_1}$ okay.

I call this one (cylinder with velocity $\boldsymbol{u}$ and number density n) as beam 1 and this one (cylinder with velocity $\boldsymbol{u_1}$ and number density n_1) as beam 2. Now, we see that if we try to understand what is happening to an observer which is sitting on beam 1. Then to every particle of this beam, some observer is attached basically, that is the equivalence thing of saying that observer is placed on beam 1, okay. So, every particle of beam 1 will experience a flux of particle I, will be equal to simply

$$I = |\boldsymbol{u} - \boldsymbol{u_1}| n_1.$$

Why it is that? Because these particles are having or experiencing a velocity, an incoming velocity of the second beam or beam number 2 with a velocity $(\boldsymbol{u} - \boldsymbol{u_1})$ right. So, $|\boldsymbol{u} - \boldsymbol{u_1}|$ times the number density of the second beam, that will give the flux of the particles which are experienced by every particle of beam one okay.

Flux is nothing but density times velocity, $n\boldsymbol{u}$ type of thing. Another way of defining flux is nothing but number of particles per unit time per unit area perpendicular to the direction of particle flow okay. So, that is the definition of flux. So, you can easily see that I will simply be the number the flux of the particles as experienced by every particle of beam 1. So, now we understood I is the flux of particles from the incoming beam 2 as experienced by every particle of beam 1.

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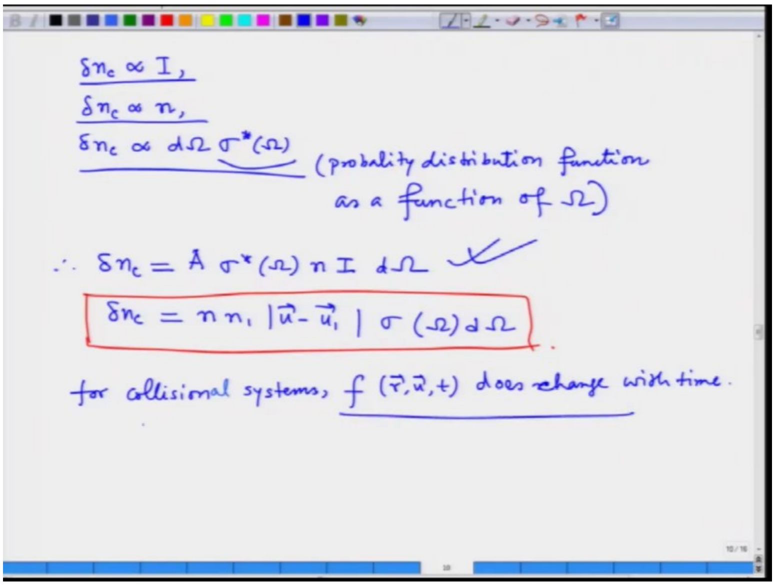

So now, we can easily say from common sense that δn_c, which is the number of collisions per unit volume per unit time and deflecting particles into a solid angle $d\Omega$ will simply be proportional to I okay i.e., $n_c \propto I$. Because if the number flux of the particles is greater, then of course, the number of collisions will also be greater, right.

Again, $\delta n_c \propto n$; that means the number density of the particles of the observer beam where the flux of the particles is coming into, if that is more and more, then again, the number of collisions is greater okay.

And finally, $\delta n_c \propto \sigma^*(\Omega)d\Omega$. So, $\sigma^*(\Omega)d\Omega$ is nothing but the probability distribution as a function of Ω. So, finally, by the law of or rule of joint variation, you can easily say this is exactly equal to

$$\delta n_c = A\sigma^*(\Omega)d\Omega nI$$

Where A is just the proportionality constant. Just remember that when I say $n_c \propto I$ is true, then n and $\sigma^*(\Omega)d\Omega$ are kept constant. When $\delta n_c \propto n$ is true, then I and $\sigma^*(\Omega)d\Omega$ is kept constant. When $\delta n_c \propto \sigma^*(\Omega)d\Omega$ is true, then the other two factors are kept constant. But when we are saying the we apply the rule of joint variation, then all the three factors will vary okay.

Then, you can easily see that δn_c is nothing but

$$\delta n_c = Ann_1|\boldsymbol{u} - \boldsymbol{u_1}|\sigma(\Omega)d\Omega$$

where I simply redefined my $\sigma(\Omega)$ as $A\sigma^*(\Omega)$ okay. That is just a nomenclature okay. Now finally, our goal is to derive the Boltzmann equation for collisional systems. For collisional systems, basically $f(\boldsymbol{u}, r, t)$ that is the distribution function does change with time. So, that is something very important. For collisionless system we do not have $\frac{df}{dt}$ right okay.

Thank you.