Introduction to Astrophysical Fluids

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Lecture - 39
Vorticity theorem in rotating frame and Taylor-Proudman theorem

Hello and welcome to another lecture session of Introduction to Astrophysical Fluids. Previously we introduced the effect of rotations in several astrophysical fluid flows. We first we distinguished between the solid body rotation and differential rotation. Then, we also showed that under certain criterion a system can sustain a differential rotation and we call that Rayleigh's criterion of stability.

And we also showed that it is very very important to study the hydrodynamics of an astrophysical system with respect to a rotational frame of reference, and that is very important for us because we are living on Earth and if we observe something, either on the surface of Earth or something exterior to the Earth, in both cases we are studying the system from a rotational frame of reference.

And of course from a non-inertial frame of reference, when we are studying something which is taking place outside Earth, the effect is a bit complicated to take in, but if let us say we are trying to study the dynamics of fluids which we can see on the surface of Earth, for example, the oceanic motions, the atmospheric motions. Now it is true that they are not in true sense astrophysical fluids, but these are important things because many planets can have atmospheres. Also maybe not normal fluids but many stars can have fluid which are actually of plasma nature, but at the end of the day the investigation of the fluid dynamics is important with respect to a rotational frame of reference, that is for sure true.

Because, whenever we are trying to see the dynamics of some plasma, I have not yet introduced plasma, but plasma can be regarded as a fluid. And if we want to see the dynamics of the plasma on the Suns surface, the plasma almost behaves like a neutral fluid, the Sun is also rotating.

So, if we assume that someone is sitting on the sun and if we know how should the equations look like with respect to a non-inertial frame of reference, then we can easily without going there, know what should be the conclusion of an observer who is sitting on the surface of the Sun.

So, this type of things of course can be applied for galactic fluids as well if someone is sitting on the galaxy for example, and it is rotating with the same. So, for example, let us say we are we belong to Earth that is true, we belong to solar system that is true, but we also belong to a galaxy, Milky way.

Now, if the galaxy is rotating somehow we are all supposed to subject to that rotation. So, due to that rotation whenever we are measuring something from here, some galactic effects can be effective.

So, if we try to study and conclude these types of things, how important can the role of non-inertial frame of reference be, that can easily be understood from this study. So, with all these things we already discussed the important aspects of astrophysical fluid dynamics, with respect to a non-inertial frame of reference, which is rotating with an angular speed $\mathbf{\Omega}$.

So, now so in this lecture what we will try to do, is to search for some interesting properties, we have investigated several properties of real fluids and in ideal fluids in previous lectures.

So, now, we will see in this lecture a very important theorem, which we discussed some weeks earlier, that is the Kelvin’s vorticity theorem or the frozen-in theorem. How could that be modified, if we are studying the whole system with respect to a rotating frame of reference.

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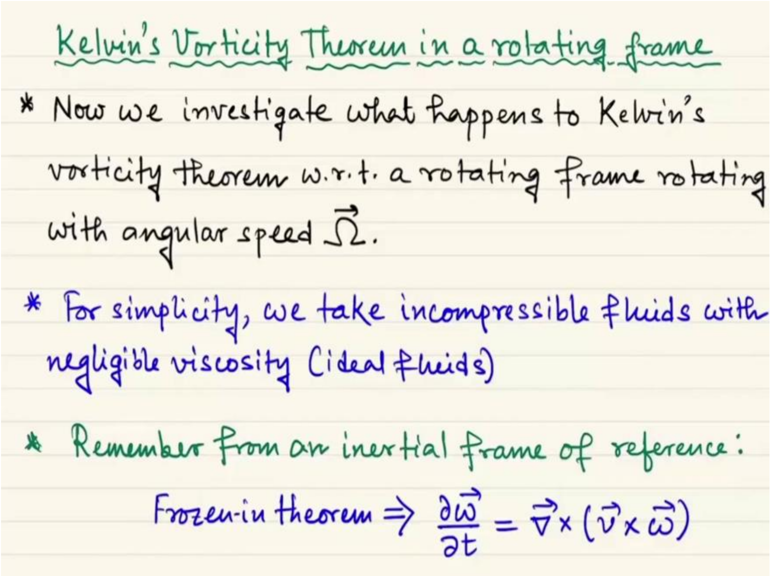

So, of course, here we will investigate systematically, what will be the generalization of Kelvin's vorticity theorem in a rotating frame! So, we investigate whether the Kelvin's vorticity theorem or the frozen-in theorem, remains the same or it has a different form if we sit on a rotating frame, which rotates with an angular speed $\mathbf{\Omega}$.

At this instant I am not saying anything on the constancy of $\mathbf{\Omega}$, $\mathbf{\Omega}$ can be anything for example, it can be variable or it can be a function of time as well.

Now, for the whole treatment we will just constrain ourself, for the case of incompressible fluids with negligible viscosity; that means incompressible and ideal fluids.

And, now you know that may be incompressible fluid is not a very reasonable assumption for galactic or interstellar medium, but at least for oceanic fluids and atmospheric fluids incompressibility is certainly a good approximation, and ideal fluids are very reasonable approximation for most of the astrophysical fluids.

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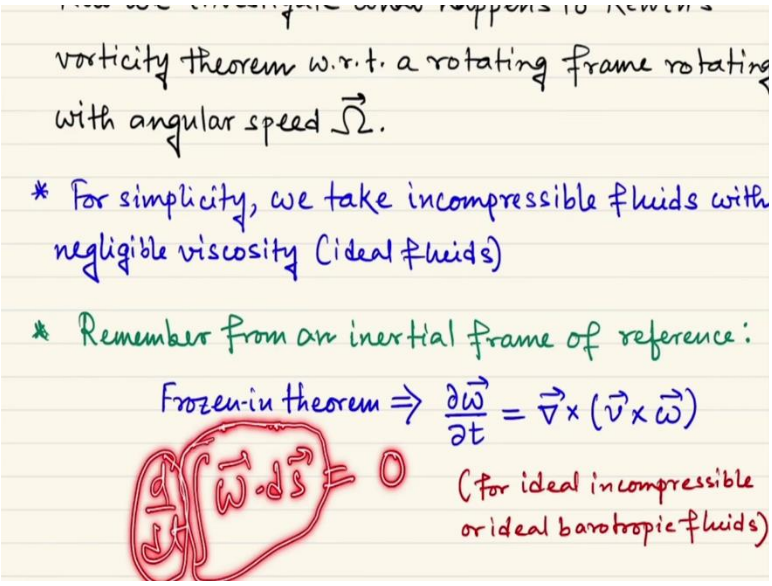

And so, if we just remember what was the form of Kelvin's vorticity theorem for an ideal and incompressible fluid. We can actually simply write that this is nothing but $\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega})$, this was the general form of the equation, which should be satisfied for a quantity whose flux to be conserved in time with a fluid element. So, that was the Frozen-in theorem.

So, frozen-in theorem simply says that if we have a vector field $\boldsymbol{\omega}$, which satisfies the type of partial differential equation above, where $\boldsymbol{v}$ is the fluid velocity. $\boldsymbol{\omega}$ can be anything, here $\boldsymbol{\omega}$ is eventually is equal to the curl of the velocity field. And, if this is true then $\frac{d}{dt}\int \boldsymbol{\omega} \cdot d\boldsymbol{S}$ will be equal to 0. That means, the flux through any surface along the trajectory of a fluid particle, that is why this $\frac{d}{dt}$ appears, will be constant.

So, that was the statement for frozen-in theorem. And, we said that the circulation is also actually something which is conserved in an ideal incompressible fluid. And, actually we also showed that this form is true for not only ideal incompressible fluid, but also for ideal barotropic fluid as well. Here, we are just we are happy to see the case for ideal incompressible fluid.

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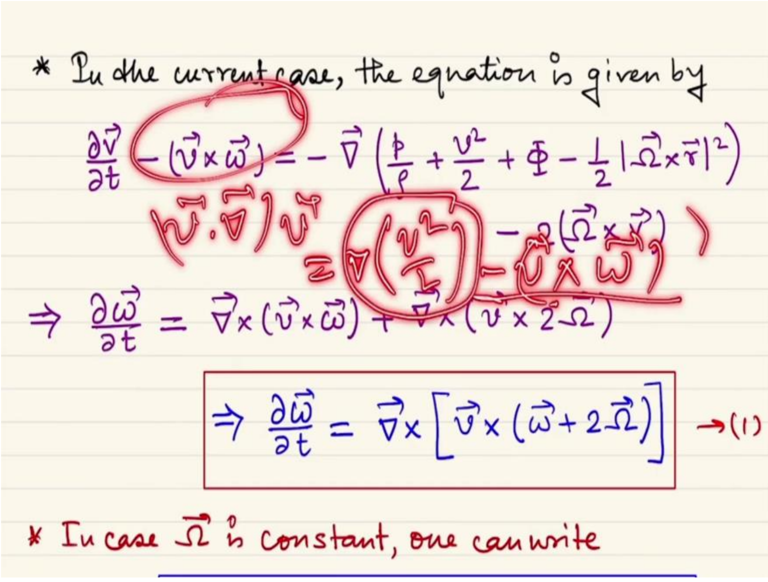

Now, here you can actually see that if you remember, just what we were discussed in the previous lecture that with respect to a non-inertial or rotating frame of reference. The equation of the momentum evolution should be written in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} - (\boldsymbol{v} \times \boldsymbol{\omega}) = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{v^2}{2} + \Phi - \frac{1}{2}|\boldsymbol{\Omega} \times \boldsymbol{r}|^2\right) - 2(\boldsymbol{\Omega} \times \boldsymbol{v}).$$

Where we have used $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{\nabla}\left(\frac{v^2}{2}\right) - (\boldsymbol{v} \times \boldsymbol{\omega})$, and since we are taking incompressible fluid, so ρ comes inside the gradient operator. And $-\nabla\Phi$ is nothing but the potential due to the body force which is conservative, for our case it can be gravity for example.

And, if you also remember, $\frac{1}{2}\boldsymbol{\nabla}(|\boldsymbol{\Omega} \times \boldsymbol{r}|^2)$ this is nothing but the centrifugal acceleration term, which is written in a gradient form. So, the final one $-2(\boldsymbol{\Omega} \times \boldsymbol{v})$ is the representative for Coriolis term.

So, Coriolis term is becoming nonzero or comes into play, when $\boldsymbol{v}$ is non-zero. What is the meaning of $\boldsymbol{v}$? $\boldsymbol{v}$ is the fluid velocity with respect to the rotational frame of reference. So, $\boldsymbol{v}$ is the relative velocity between the rotational frame of reference and the fluid flowing on it.

For example, if we just consider, an ocean is flowing, so $\boldsymbol{v}$ is nothing but the linear velocity of the ocean with respect to the Earth, because due to rotation of Earth it also has some linear velocity and $\boldsymbol{v}$ is the relative velocity, so the vector subtraction of those two.

So, if $\boldsymbol{v} = \boldsymbol{0}$, for example if something which is static on Earth surface. So, it is rotating on with Earth, but if it is static on Earth surface with respect to Earth, then the Coriolis force will be 0. But, for our case here it is nonzero and so finally, you can see that if you take the curl of the whole equation finally, the whole gradient will vanish and taking the vorticity term on the right side, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{v} \times (\boldsymbol{\omega} + 2\boldsymbol{\Omega})].$$

So, $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$ is something which is a new variable, which comes into play and which satisfies the so called equation necessary for the frozen-in theorem.

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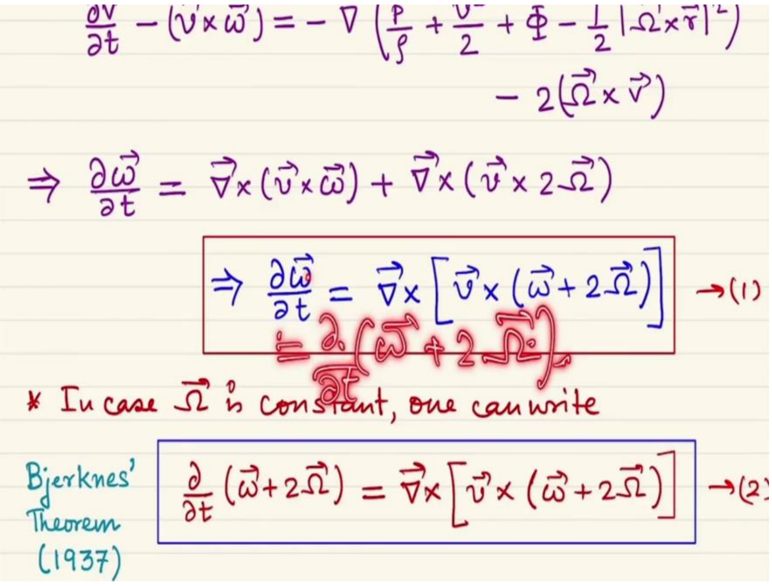

And, now we talk about the constancy of $\boldsymbol{\Omega}$. So, if $\boldsymbol{\Omega}$ is constant then one can simply write

$$\frac{\partial}{\partial t}(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) = \boldsymbol{\nabla} \times [\boldsymbol{v} \times (\boldsymbol{\omega} + 2\boldsymbol{\Omega})].$$

So, I can add some 0 on the LHS without any problem. So, the final equation looks very elegant, it is now saying that it is not $\boldsymbol{\omega}$, but $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$, which is practically satisfying the equation necessary for the frozen-in theorem. And this is known as the generalized vorticity theorem or Bjerknes' theorem, who gave and actually proved this theorem in 1937.

Now, from this equation what is the first thing you can see? if you remember this form, we also discussed that if the initial vorticity is 0, then $\frac{\partial \boldsymbol{\omega}}{\partial t} = \mathbf{0}$, so $\boldsymbol{\omega}$ remains to be 0. So, for an ideal and incompressible fluid if the initial vorticity is 0, the fluid is actually an irrotational flow as well.

Now, is it the same case for a fluid with rotation? the answer is NO. Because, if your initial vorticity is 0, you still have $2\boldsymbol{\Omega}$. So, that will give you finite non-zero $\frac{\partial \boldsymbol{\omega}}{\partial t}$. So, even when your initial vorticity is 0, just due to the rotation of the frame of reference, because you are seeing or studying the system with respect to rotating frame of reference, due to your rotation basically the system will develop some vorticity inside it.

And this is exactly what happens when you are studying some laboratory experiments on fluid dynamics on Earth. Of course, the effect of the rotation of Earth is negligible in this case, but what I am trying to say that if you make a setup, where the setup is rotating with an important $\boldsymbol{\Omega}$. Then, even if the initial vorticity is absent, with time you will see inside the fluid, the vorticities to be developing. So, that is the stark difference between the normal frozen-in theorem and this frozen-in theorem for the case of rotating fluids.

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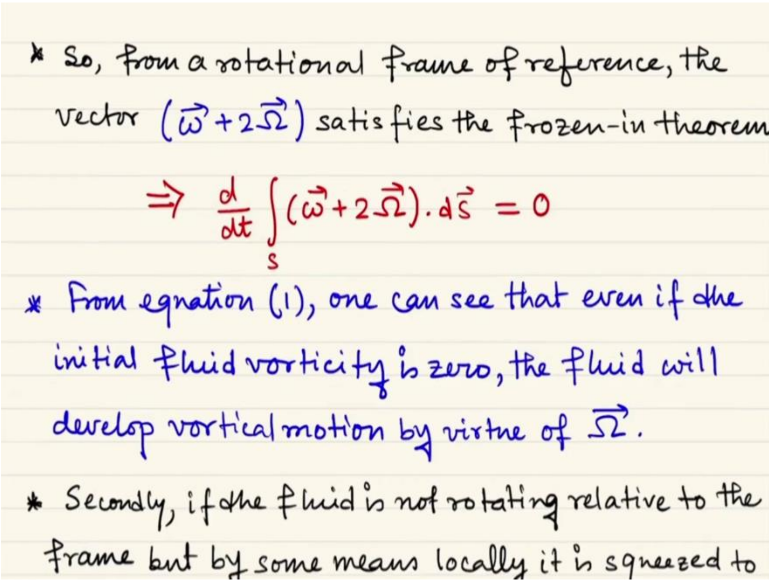

So, now you see that in a rotational frame of reference, the vector $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$ satisfies the frozen-in theorem. So, now what is the meaning of frozen-in theorem? it says $\frac{d}{dt}\int_s(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot d\boldsymbol{S} = 0$.

So, the flux of this quantity will be constant or will be invariant in time, if we are following the trajectory of a fluid element. Because, this is once again this is Lagrangian derivative.

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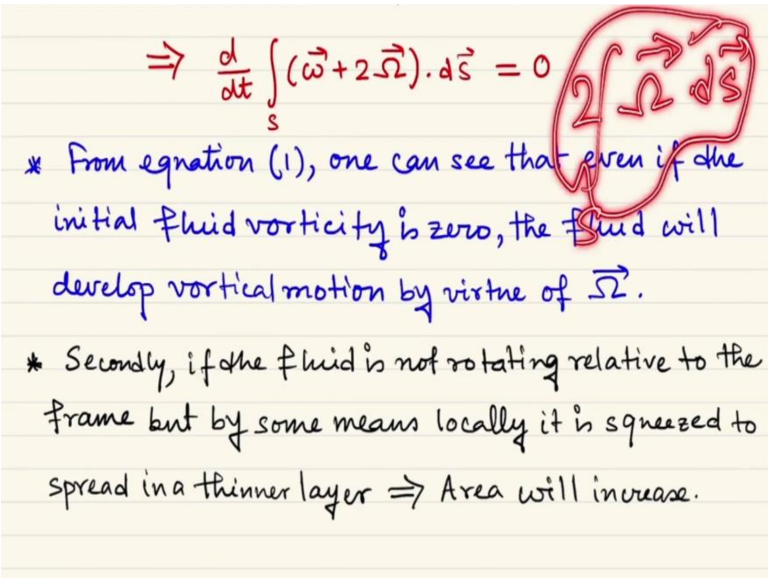

Now, from the equation $\frac{\partial \omega}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{v} \times (\boldsymbol{\omega} + 2\boldsymbol{\Omega})]$; one can see that even if the initial fluid vorticity is zero, the fluid will develop vortical motion by virtue of $\boldsymbol{\Omega}$, that we already have discussed. Now, the second point is very interesting that if the fluid is not rotating relative to the frame, but by some means locally this fluid is squeezed to spread in a thinner layer.

Then what happens? Then of course, its area will increase and if its area increases, then if the fluid does not have some considerable vorticity on its own, then we only have $2\int_s \boldsymbol{\Omega} \cdot d\boldsymbol{S}$.

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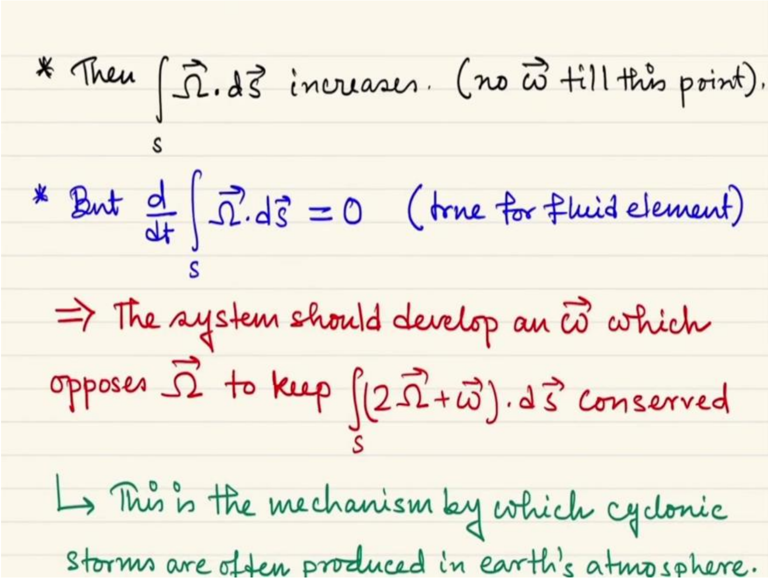

Now, if your total area increases then this flux will also increases. So, this one increase and still now there is no $\boldsymbol{\omega}$, but we know that $\frac{d}{dt}\int_s \boldsymbol{\Omega} \cdot d\boldsymbol{S} = 0$, and that is true for the fluid element. Because, that is true because at that point there is no $\boldsymbol{\omega}$.

So, this should satisfy the frozen-in theorem. And, if this is true then the system should develop some $\boldsymbol{\omega}$. So, that the $\boldsymbol{\omega}$ practically opposes $\boldsymbol{\Omega}$ to keep $\int_s (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot d\boldsymbol{S}$ conserved along the trajectory of a fluid element in time.

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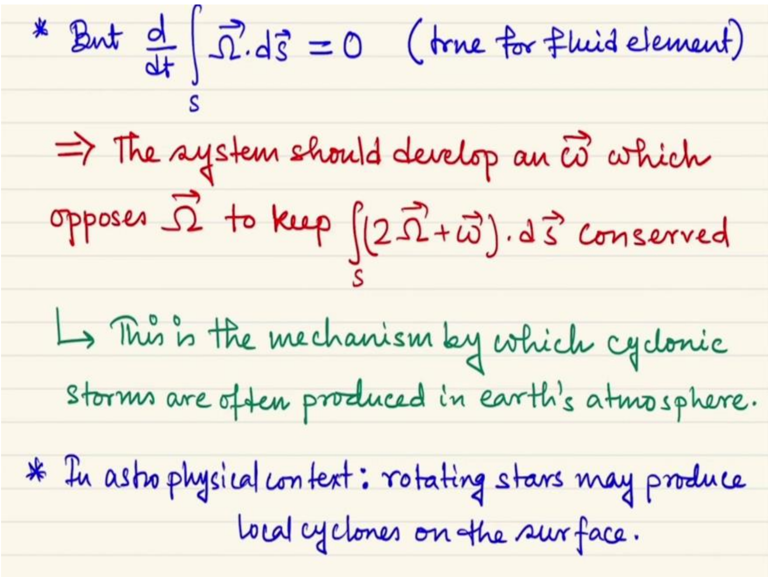

So, this is what I am trying to say, just due to this generalized Bjerknes theorem or generalized frozen-in theorem, you can actually see that the system should develop some $\boldsymbol{\omega}$ by the virtue of $\boldsymbol{\Omega}$.

So, this is exactly the mechanism by which very frequently cyclonic storms are produced in Earth's atmosphere. And, actually to be very honest this is not only the case of Earth, in planetary atmosphere, in stellar atmosphere sometimes these types of things are forming.

As very simplistic explanation of this behaviour can be done using this type of analytical tool. Now, in astrophysical context is this important? yes, because rotating stars may also produce local cyclones on their surface. So, this is also a very interesting domain of research, you can again see over internet, the papers and the ongoing research works on this topic.

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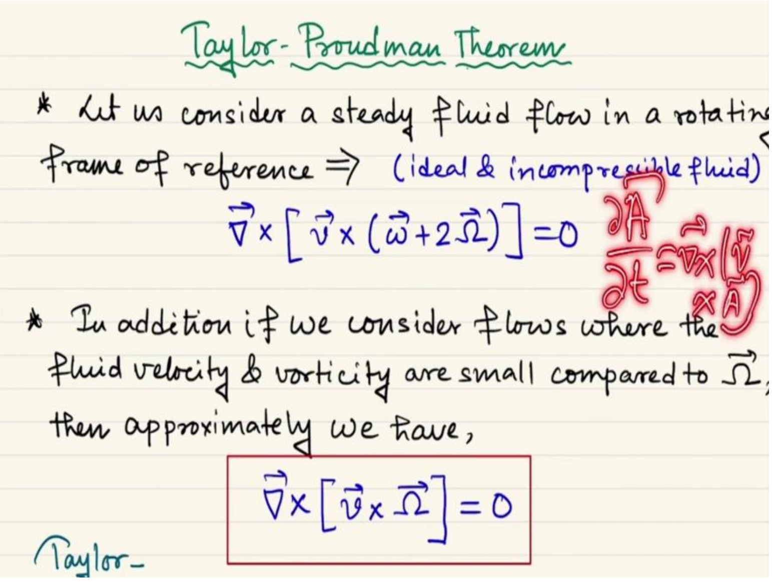

Now, after this topic in the second part of this discussion, we discuss another very important theorm. I could have divided the whole discussion in two parts, but the Taylor-Proudman theorem is almost very very much linked with the previous discussion. So, that is why I am just doing this in one single lecture.

So, we have just seen that for a rotational frame of reference, which is moving with some angular speed $\boldsymbol{\Omega}$. The, frozen-in theorem is true for the quantity, $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$, and because this quantity is satisfying something like $\frac{d\boldsymbol{A}}{dt} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{A})$, where $\boldsymbol{A}$ is the concerned vector.

Now, if we consider a steady fluid flow; that means, there is no explicit time dependence of the fluid flow, and in the rotating frame of reference with ideal and incompressible assumptions, then we will have $\frac{\partial}{\partial t}(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) = \mathbf{0}$. So, that means, the term on RHS should also vanish and that is exactly what I have written over here $\boldsymbol{\nabla} \times [\boldsymbol{v} \times (\boldsymbol{\omega} + 2\boldsymbol{\Omega})] = \mathbf{0}$.

In addition, if we now consider flows where the fluid velocity and vorticity are small compared to $\boldsymbol{\Omega}$, then you can actually neglect $\boldsymbol{\omega}$ with respect to $2\boldsymbol{\Omega}$, that is possible. If the flow is slow, not very very fast and with respect to the rotational speed of the reference frame, then you can neglect $\boldsymbol{\omega}$ and you have finally, $\boldsymbol{\nabla} \times [\boldsymbol{v} \times \boldsymbol{\Omega}] = \mathbf{0}$. $\boldsymbol{v} \times \boldsymbol{\Omega}$ is irrotational practically.

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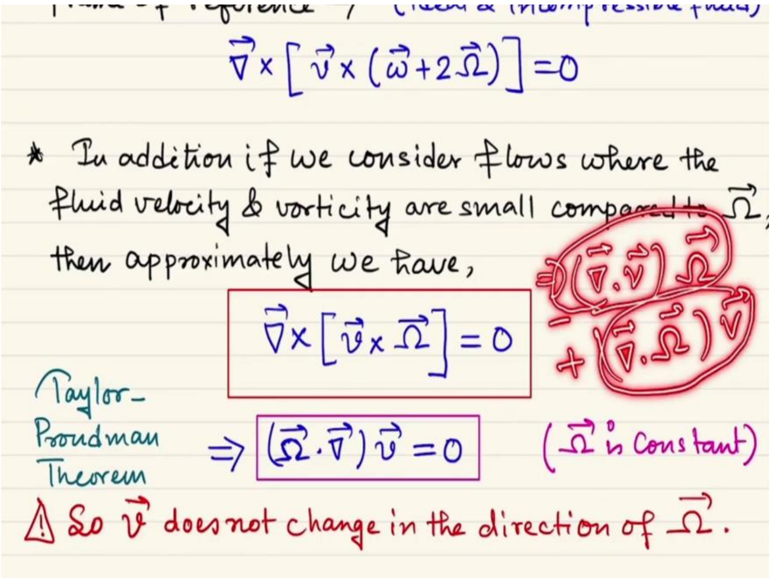

And, if you now expand them, you will have four terms. So, three terms will be vanishing if you remember, what they are, they you will have two terms like $-(\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{\Omega}$, $(\boldsymbol{\nabla} \cdot \boldsymbol{\Omega})\boldsymbol{v}$ and both of them are 0, because of incompressibility and constancy of $\boldsymbol{\Omega}$.

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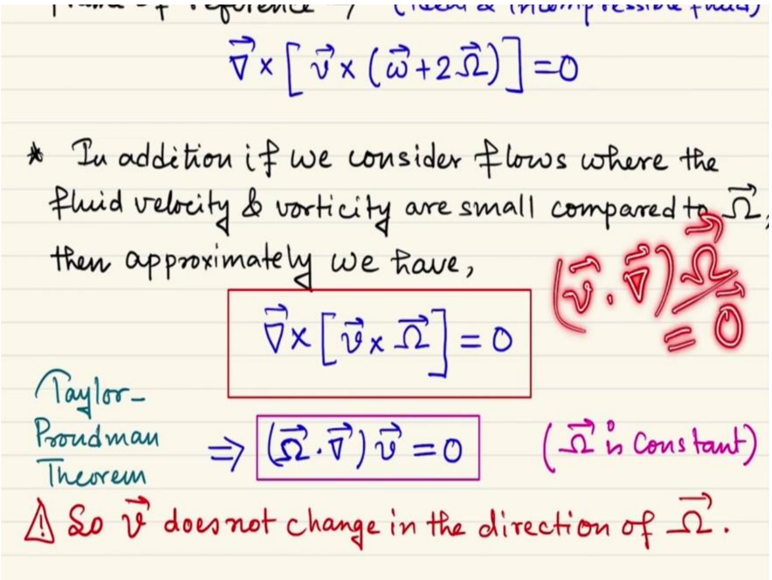

And, there will be other two terms, one will be $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\Omega}$ which will be also equal to 0, because again, $\boldsymbol{\Omega}$ is constant.

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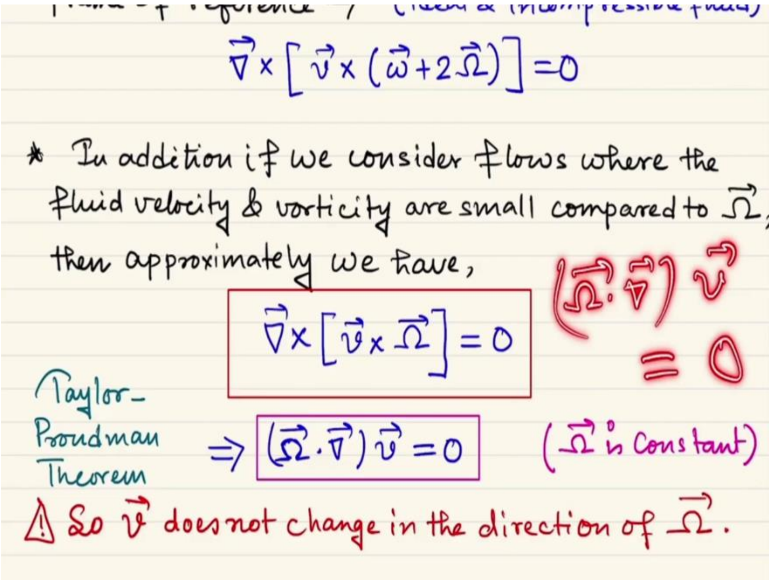

And finally, the fourth term which is rest, will be simply be $(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{0}$. So, this one is true only when we are talking of ideal incompressible, slow and steady motion of a fluid in a rotating frame of reference.

So, many things, ideal, incompressible, slow and steady. So, this is known as Taylor-Proudman theorem, Proudman gave it in the year 1916 and from the Taylors' point of view, Taylor contributed in the year 1921.

So, this theorem simply says that $\boldsymbol{v}$ does not change in the direction of $\boldsymbol{\Omega}$, that is simply the case.

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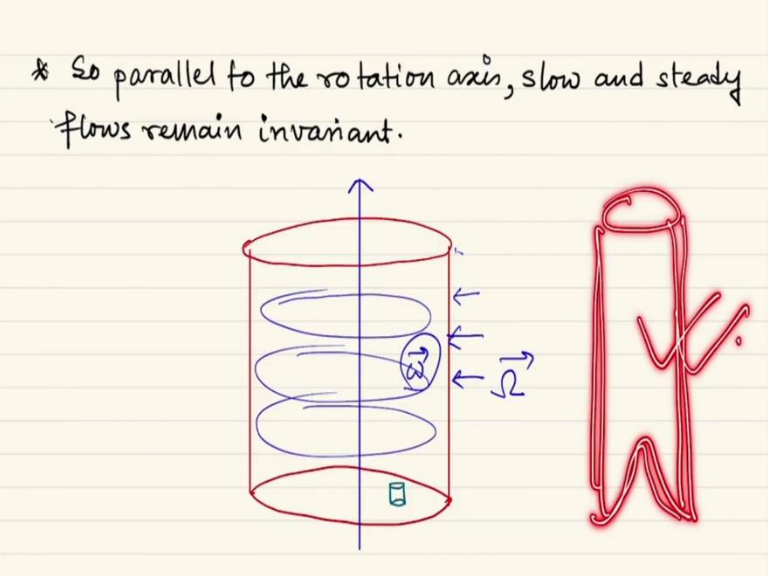

So, it simply says that parallel to the rotation axis, for slow, steady, ideal and incompressible flows, they remain invariant. And what is the effect of that for that I will tell you something here by drawing. So, let us say if you have a vessel and some fluid is moving (see above).

Now, let us say you have a small cylindrical capsule type of thing which is fixed. Now, when the fluid is steady, let us say some the fluid is first of all is steadily developing. So, the fluid is just moving like a solid body, with respect to this direction of axis of the rotation. Initially, there is no problem everything is rotating with the same angular speed. Now, let us say suddenly the angular velocity of the system or the container vessel is increased suddenly, the fluid inside, does not have let us say very strong viscosity. So, that is why the fluid will take some time, so that the solid body type of rotation again gets re-established.

But, between these two, what happens, the fluid is very very slow, so that means, the vessel which is assumed to be rotating with some angular speed **Ω**, and with respect to this **Ω**, the ***ω***, which is just developed inside the body, because of the small perturbation is very very small. And that is why we can easily use Taylor-Proudman theorem.

So, Taylor-Proudman theorem says that a parallel the axis of rotation, the fluid flow nature will be always unchanged. Now, let us say we have a obstacle inside fluid. So, the fluid will always try to skirt the obstacle, but it is expected that the fluid which will be far from the obstacle (in the vertical direction let us say), will not be considering its effect.

But it is seen, that since along this is parallel to the direction of the rotation, the skirting motion is actually totally followed even when you are at a higher layer in the fluid. So, finally, you will see there will be a column, containing much more higher layers, you can see the column which is rotating as if the obstacle is actually expanded up to a certain height.

So, this is the Taylor Proudman theorem. So, exactly the same thing the aircrafts sometimes face. Let say you have some obstacle or you have some hill or mountain type of thing and if the fluid just around this mountain or something is just rotating, then this type of rotation will also be continued, when you are at a greater height above this mountain.

So, this there will be an invisible column of influence of this obstacle. But, for that we need the global system to be rotating with a considerable high angular speed of rotation. And, here you see that this invisible column, which basically gives the effect of this obstacle is known as the Taylors column.

So, that is why this thing is very very important both for geophysical fluids and astrophysical fluids. In the next lecture, we will discuss the shape of an astrophysical body or an astrophysical fluid, under the effect of self-gravity and rotation.

Thank you very much.