Introduction to Astrophysical Fluids
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Lecture - 38
Fluid Dynamics in a Rotating Frame of Reference

Hello, in this lecture of Introduction to Astrophysical Fluids, I will mostly discuss of the general perspective of studying the Fluid Dynamics with respect to a rotational frame of reference.

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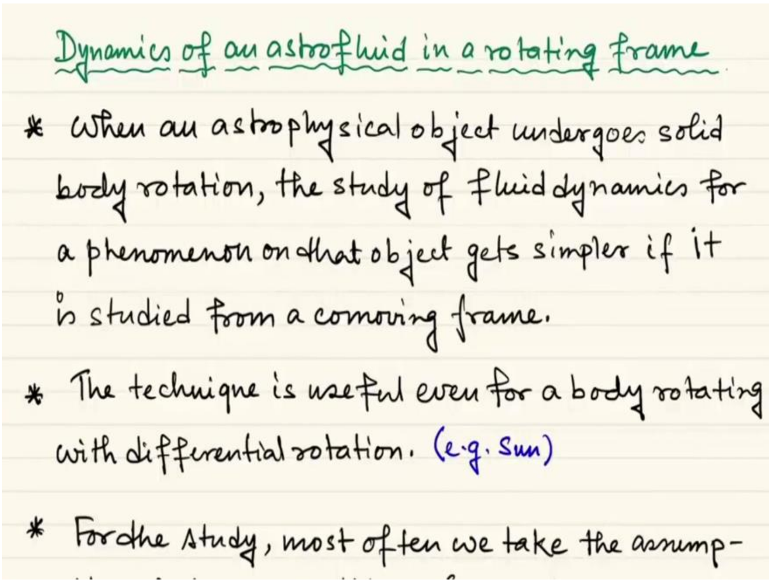

So, the title is of course, the dynamics of an astrofluid in a rotating frame, now it is true that the first question comes to our mind is that why it is at all important, to study the fluid dynamics of an astrophysical fluid in a rotating frame of reference, well the answer is very easy. For example, almost all the astrophysical objects like the planets, stars, accretion discs, galaxies, they are rotating, as we already discussed in the last lecture. Now, when an astrophysical object undergoes solid body rotation, for example the case of a slowly rotating star, as we mentioned last time right that due to the turbulent viscosity, they have always the tendency to switch from the state of differential rotation to a solid body rotation. So, for this type of case, the study of fluid dynamics for a phenomenon gets analytically much more simpler if it is studied from a co-moving frame. That means a frame which is exactly rotating with the same angular speed $\mathbf{\Omega}$.

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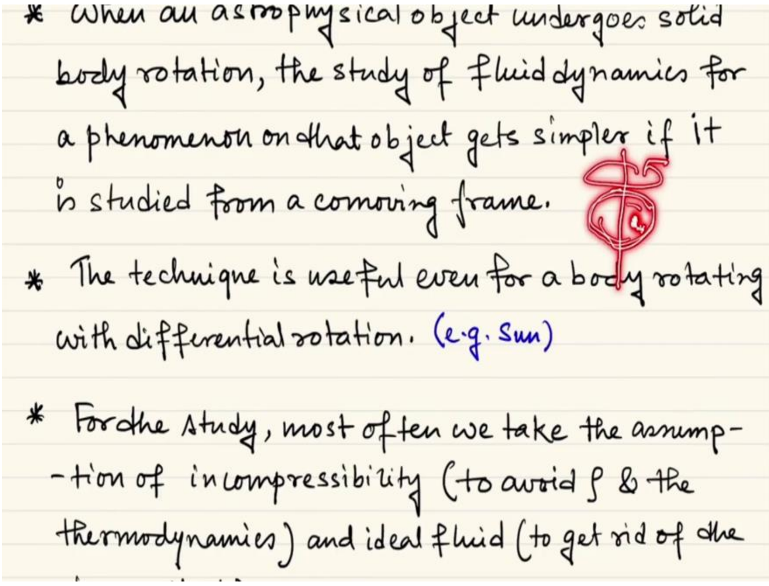

Let us say some body is now rotating along an axis, and it is performing a solid body rotation, then some object which is at rest or let us say, is stuck on this body, and if we are seeing from lab frame or some external frame of reference, then this object still has some acceleration or velocity type of thing, but with respect to a frame which is co-moving with the body, it is nothing but at rest. So, you see that considerable simplifications can be brought into a consideration, if we just talk in terms of the co-moving frames. Here, when we are talking about co-moving it is actually co-rotating frame.

Now, this technique is also useful even when the body is rotating with a differential rotation, for example Sun. It should not, rather it need not move with a constant angular speed, but if let us say the system is moving with differential rotation, then let us just choose some reference frame which is moving with the average angular velocity or angular speed of the system that actually can also simplify the problem enormously.

Now, for our study which we will discuss mostly in the next lecture, we will consider the assumption of incompressible and ideal fluid, why these two assumptions?

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body rotation, the study of fluid dynamics for a phenomenon on that object gets simpler if it is studied from a comoving frame.

* The technique is useful even for a body rotating with differential rotation. (e.g. Sun)

* For the study, most often we take the assumption of incompressibility (to avoid ρ & the thermodynamics) and ideal fluid (to get rid of the viscous effects).

So, incompressibility assumption should come because we want to get rid of ρ and the thermodynamics part. So, even if you have a barotropic closure type of thing you are still thinking of the thermodynamics. So, here this is very simplistic and of course if you want to do something more general, you are encouraged to do that, but for the scope of this course we are just using this simplest case of incompressibility.

And why ideal fluid? because, first of all this is a very reasonable approximation for an astro-fluid, and also because the viscous effects are in most of the cases, negligible. Even in case of the slowly moving stars, you can use the concept of this solid body rotation.

You know, that the effective viscosity, which causes the solid body rotation is not the viscosity which is considered in the Navier-Stokes equation, it is not the molecular viscosity, but it is the turbulent viscosity, although I have not introduced the concept yet but as an information you know this from the previous lecture.

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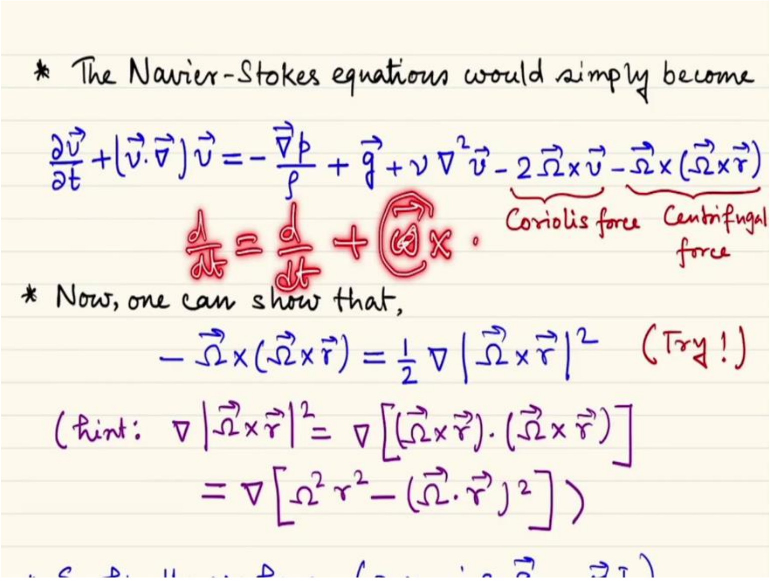

So, these two assumptions will be used for our study. So, now the Navier Stokes equation for this type of system can simply be written with respect to a non-inertial frame of reference.

Now, from your undergraduate physics course of classical mechanics you should know that $\left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{non-inertial} + \boldsymbol{\Omega}\times$. Where $\boldsymbol{\Omega}$ is the angular speed of the frame of reference.

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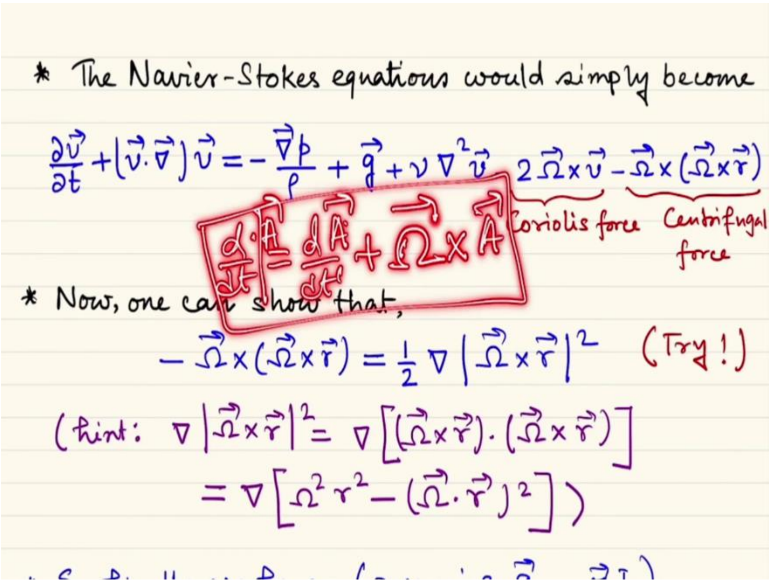

So, this equation relates the $\frac{d}{dt}$ of any vector (say $\boldsymbol{A}$) in the inertial frame to the $\frac{d}{dt}$ of the that vector in the non-inertial frame.

So, if you just use the same thing for this fluid, and then you can also write Navier-Stokes equation equivalently, just inserting the effect of this non-inertial or the rotating frame of reference

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\rho} + \boldsymbol{g} + \nu \boldsymbol{\nabla}^2 \boldsymbol{v} - 2\boldsymbol{\Omega} \times \boldsymbol{v} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

So, two additional contributions come, one is called the Coriolis force, given as $-2\boldsymbol{\Omega} \times \boldsymbol{v}$, this is actually Coriolis acceleration. If you multiply with the mass, then it will give you Coriolis force and the centrifugal acceleration $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$. And you can easily see that this whole thing is actually nothing but the Newton's law, written within in a rotating frame of reference.

Now, $-2\boldsymbol{\Omega} \times \boldsymbol{v}$ of course gets vanished when $\boldsymbol{v} = \boldsymbol{0}$. So, $\boldsymbol{v}$ is nothing but the fluid velocity with respect to the rotating frame of reference; so if something is stuck or is at rest with respect to the rotating frame of reference, then for that Coriolis force will be 0, but centrifugal force will still be there.

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$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{\nabla p}{\rho} + g + \nu \nabla^2 v \underbrace{- 2\Omega \times v}_{\text{Coriolis force}} \underbrace{- \Omega \times (\Omega \times r)}_{\text{Centrifugal force}}$$

* Now, one can show that,

$$-\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \frac{1}{2} \nabla |\vec{\Omega} \times \vec{r}|^2 \quad \text{(Try !)}$$

$$\left(\text{hint: } \nabla |\vec{\Omega} \times \vec{r}|^2 = \nabla \left[(\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r})\right] = \nabla \left[\Omega^2 r^2 - (\vec{\Omega} \cdot \vec{r})^2\right]\right)$$

* So finally we have, (assuming $\vec{g} = -\vec{\nabla} \Phi$)

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} - \nabla \underbrace{\left(\Phi - \frac{1}{2} |\vec{\Omega} \times \vec{r}|^2\right)}_{\Phi_{eff}} + \nu \nabla^2 \vec{v} - 2\vec{\Omega} \times \vec{v}}$$

Now, how to further simplify this type of thing. So, one very simple but very useful mathematical identity is that you can write $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \frac{1}{2}\boldsymbol{\nabla}|\boldsymbol{\Omega} \times \boldsymbol{r}|^2$, so you can write it as a gradient of something, and this is my request that you try this at home and you can hint the given formula or you can use any other method, but one possible thing is that you can actually write this thing as this.

Now, if you admit that, then your final evolution equation for the velocity or actually the evolution equation for the momentum should look like

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\rho} - \boldsymbol{\nabla}\left(\Phi - \frac{1}{2}\boldsymbol{\nabla}|\boldsymbol{\Omega} \times \boldsymbol{r}|^2\right) + \nu \boldsymbol{\nabla}^2 \boldsymbol{v} - 2\boldsymbol{\Omega} \times \boldsymbol{v}$$

Now, you see I said that I am assuming incompressible and inviscid fluids, but here just for this part I have not yet done any simplification, this is actually general compressible fluid with viscosity. The only thing here I have assumed is that the body forces are of conservative nature, which is true for gravity, so we can write $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$. So, if you notice carefully, the centrifugal acceleration term is inserted inside the gradient. So, now you have $\boldsymbol{\nabla}\left(\Phi - \frac{1}{2}\boldsymbol{\nabla}|\boldsymbol{\Omega} \times \boldsymbol{r}|^2\right)$, so if you call $\Phi - \frac{1}{2}\boldsymbol{\nabla}|\boldsymbol{\Omega} \times \boldsymbol{r}|^2$ as Φ_{eff}, at the end of the day your centrifugal acceleration does not do anything other than modifying the resultant gravitational potential, or your body force potential. So, if I just say that I am in such a system, where the gravitational potential is much larger than $\frac{1}{2}\boldsymbol{\nabla}|\boldsymbol{\Omega} \times \boldsymbol{r}|^2$, then already the centrifugal acceleration is neglected.

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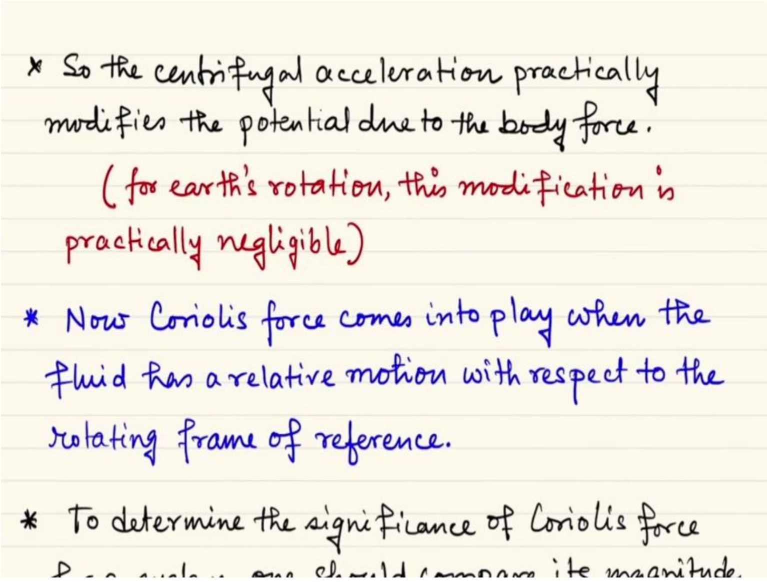

That is exactly what I said, and one should not really worry about the centrifugal acceleration and that is exactly the case for Earth’s rotation, where this modification due to the rotation is fairly negligible, because the rotation is fairly slow.

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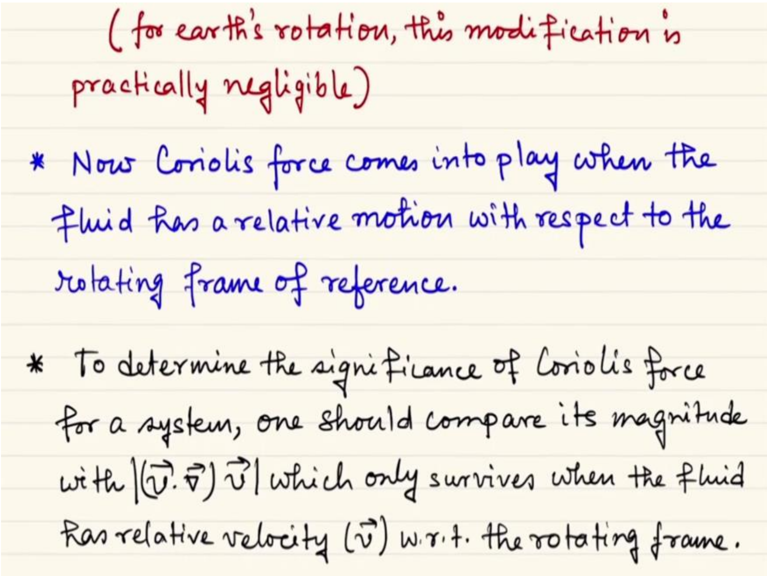

Now, for Coriolis force this is no longer true, this has a non-trivial role. But Coriolis force comes into play only when the fluid has a relative non zero velocity with respect to the rotating frame of reference.

So, in order to understand whether the Coriolis force plays a significant role in the motion or not, we have to compare that term with the non-linear advective term $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$. So, we just compare their order of magnitude, why? Because, if you see the momentum equation, when the fluid is having some nonzero relative velocity, other than $\nu\boldsymbol{\nabla}^2\boldsymbol{v}$, only $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$ and $-2\boldsymbol{\Omega} \times \boldsymbol{v}$ will be remaining. But the viscous term is unimportant for most of the astrophysical system, so those two terms should then be compared. $\frac{\partial v}{\partial t}$ is also the same case but, it can be vanishing for a steady flow even when $\boldsymbol{v}$ is non-vanishing. But, the advection and Coriolis terms remains non zero when there is a non zero $\boldsymbol{v}$. So, these two are very identical in nature, of course one is non-linear and one is linear. Because, $\boldsymbol{\Omega}$ is almost constant, at least it does not depend on $\boldsymbol{v}$, at least in our problem.

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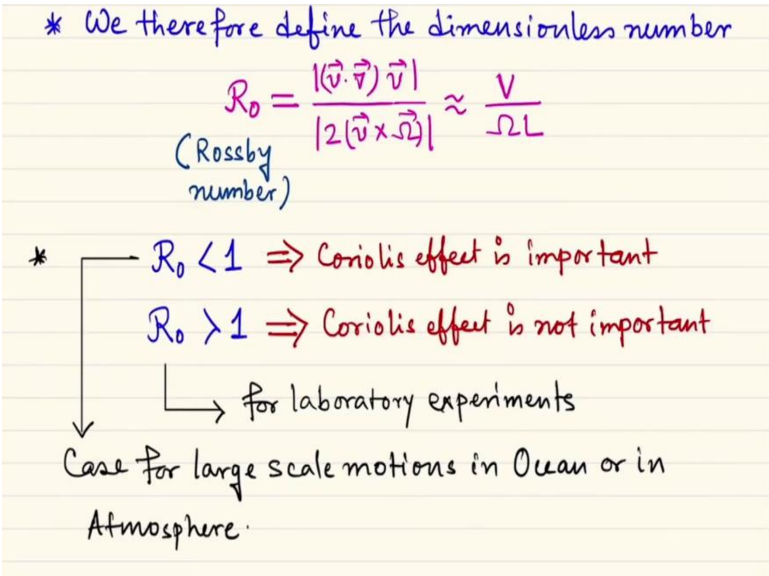

So, we can compare these two. If we compare these two finally, we define a dimensionless number, because both of these terms have identical dimensions, so if we divide the non-linear term by the rotation term finally, we will have roughly $\frac{V}{\Omega L}$. So, V is the characteristic velocity of the system, L is the characteristic length of the system, Ω is the constant rotation speed of the system. This dimensionless number is known as R_0 or Rossby number. Now, you see that in this whole treatment, until this point, I have not said anything about the constancy of $\boldsymbol{\Omega}$. But, once again the thing is that if $\boldsymbol{\Omega}$ is constant then then this type of analysis becomes much more simpler, if we place ourselves in a co-rotating frame.

So, that is why I am saying now if $\mathbf{\Omega}$ is not explicitly depending on time, then $\mathbf{\Omega}$ can be just thought to be constant and for that case Rossby number is given by $\frac{V}{\Omega L}$. But, if let us say $\mathbf{\Omega}$ is a function of time itself, then there is no problem, only then there will be an explicit time dependence of the Rossby number. So, $\mathbf{\Omega}$ need not be a constant.

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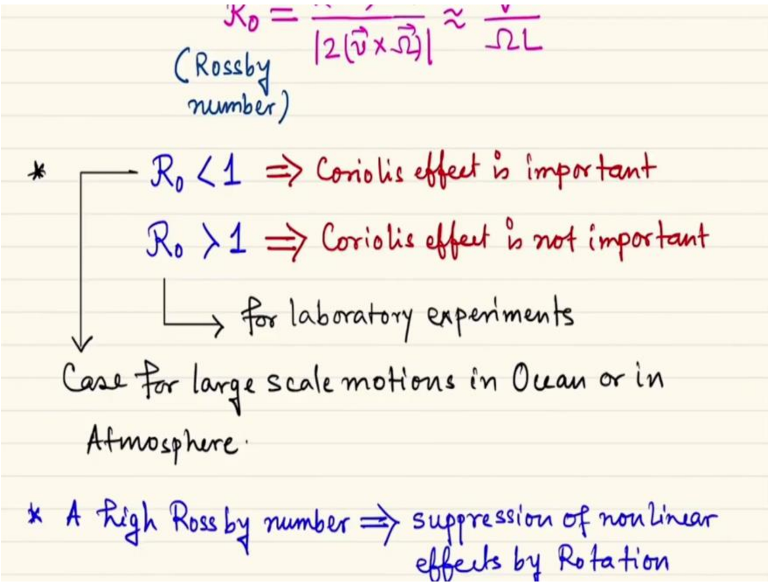

So, what is now the meaning of Rossby number! So, of course you can see that when Ross by number has a large value for example when $R_0 \gg 1$, that means that the non-linear term greatly dominates the Coriolis force term and where Rossby number is less than 1, that means, the non-linear term is less than the Coriolis term.

So, in the first case non-linearity wins and in the second case the rotational properties is winning. For example, if you are doing some experiment in your lab, now the question is that would your experiment be affected by the earth’s rotation? well you just calculate the Rossby number corresponding to the Earth’s $\mathbf{\Omega}$. If the Rossby number is very very large, then you say that I do not care about the rotation of the earth, if the Rossby number is of the order or slightly less than 1, then you have to take the effect of the earth rotation into account. And that is exactly the case where we are studying the fluid flows for the geophysical entities like atmosphere, the oceanic flow etc. So, here you can see that R_0 less than 1 simply gives us the instance, where we are talking about the large scale motions in ocean or in atmosphere.

So, once again a low Rossby number simply says that the suppression of non-linear effects by rotation (error in lecture). This piece of information is specifically important for the people, who are were interested in studying fluid turbulence under some rotation, which I have not yet introduced, but just for your information. So, let us say you have a container containing some fluid and you rotate the system. So, if the rotation is very fast or if the rotation is very slow, how does the turbulence is affected! so $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$ is somehow a measure of the strength of the turbulence, because this is the non-linearity part and you know from maybe your previous knowledge, that turbulence is nothing but a measure of the strength of non-linearity. Of course, there we talk about another dimensionless number, called Reynolds number, here we are talking about the Rossby number.

So, in the next discussion we will discuss some important properties or theorems and their eventual modifications in case of the rotating frame of reference. The first one will be the modification of the Kelvin's vorticity theorem and depending on that secondly, I will discuss a very important theorem, called the Taylor-Proudman theorem, which is very very interesting for both our laboratory experiment and for large scale geophysical and astrophysical flows.

Thank you.