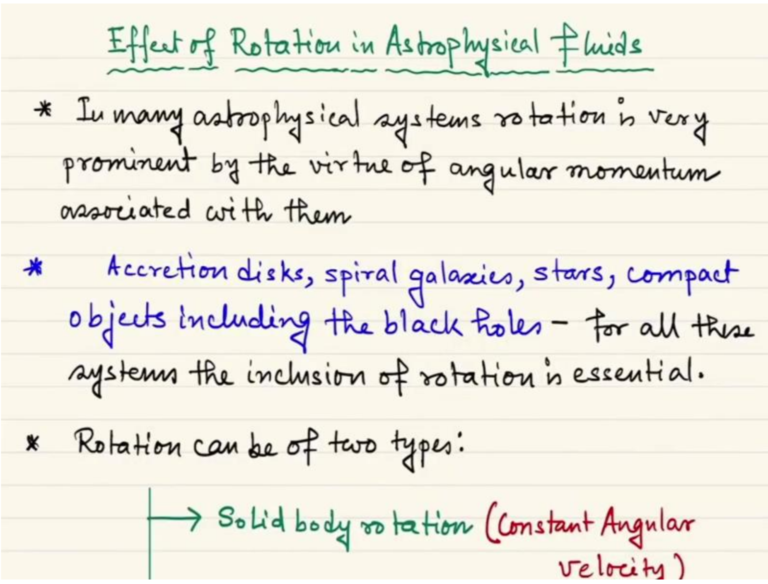
Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 37
Rotation in astrofluids and Rayleigh's criterion

Hello and welcome to another section of Introduction to Astrophysical Fluid. As promised, in this session we will discuss or we will start discussing a new topic that is the effect of rotation in Astrophysical systems.

(Refer Slide Time: 00:32)



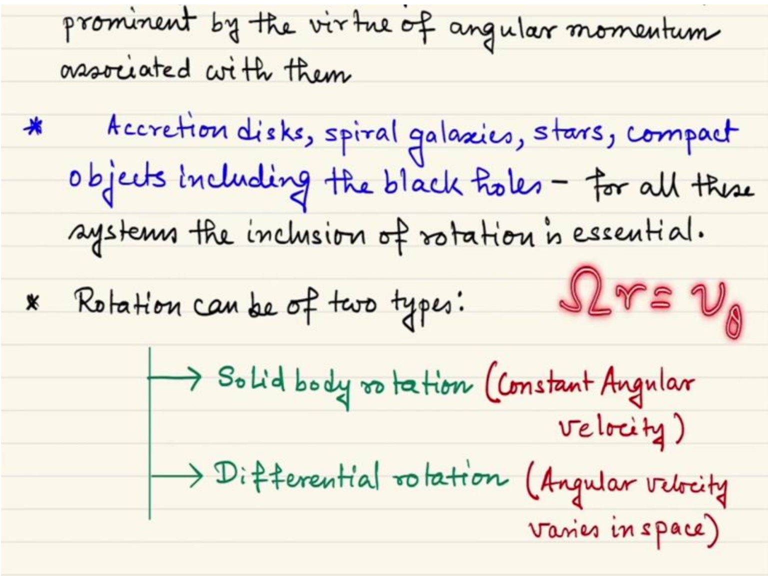
So, in many astrophysical systems, rotation is very very much important, because first of all they are the prominent effect, which can be seen by direct observations. For example, for Sun you can see really the images which can give you evidences of rotation; for distance stars or for distant objects where you cannot see directly then you have several other evidences. One evidence is that you can detect rotation by the virtue of the angular momentum associated with them, for example, for a distant accretion disc or a spiral galaxy, there are several methods by which you can calculate the angular momentum associated with the system and by using that you can actually confirm that the system is undergoing some rotation. There are techniques other than angular momentum by which you can actually also detect rotation, that is totally a different topic. So, if you are interested you can search over internet.

Now, what are the typical astrophysical systems where we can see clear evidences of rotation? First, we have already discussed the case of accretion disks, then something like that but in a larger scale like the spiral galaxies, so milky way for example. Both has shapes like a spiral disc type of thing and it rotates, there are also the problem of the rotation of the stars where you can see the almost spherical objects are rotating.

And the same thing for the compact objects; for example, the white dwarfs, the neutron stars including the black holes. The black holes have also clear evidences of rotation, so one question is that how to detect the rotation of the black hole or the spin rate of the black hole? There are very interesting works on that, you can also search for that over internet.

Now, for all the systems the inclusion of the rotation is essential. Of course, if the system is very very much massive that means like compact objects or black holes then we actually have to take the gravitational correction due to general relativity into account, but in the scope of this current course we do not need to include the generate relativistic effect; that means the space-time curvature will not be considered in this course.

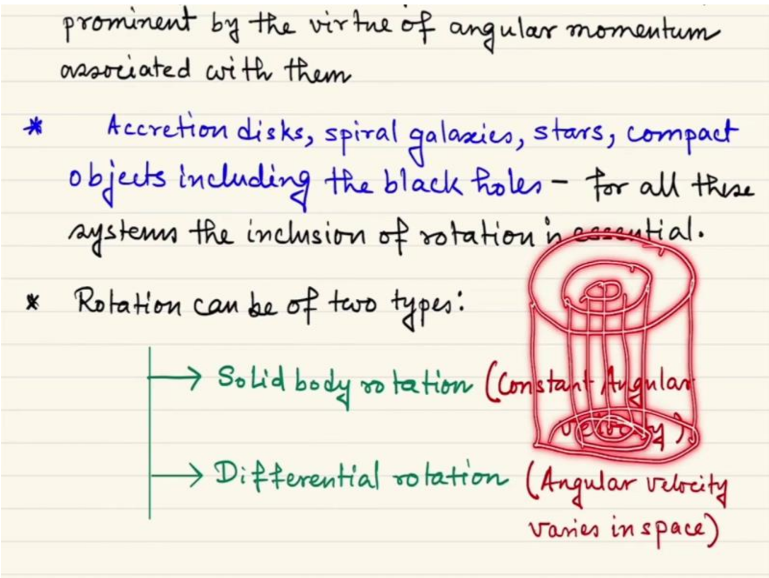
(Refer Slide Time: 03:58)



So, we will do something where simple classical rotations are needed. Now, as we already have discussed while discussing the equations of the perfect fluid and then discussing the Newtonian fluids and how the velocity gradient can have an unnecessary part other than the true viscosity part. Also we introduced two types of rotation, one was the solid body rotation, another was the differential rotation.

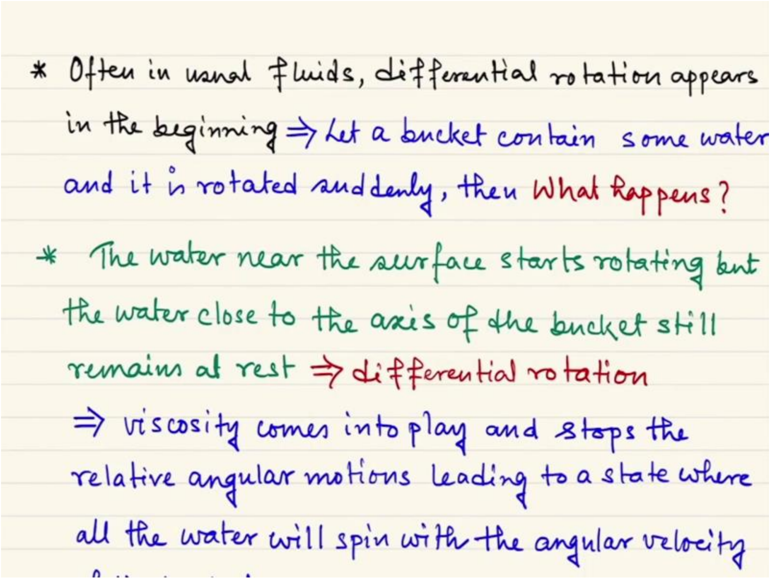
So, the solid body rotation is nothing but a rotation where the whole body is moving with the constant angular velocity. The linear velocity can be different, but the angular velocity $\mathbf{\Omega}$ should be a constant. So, remember that Ωr gives you the linear velocity in the cross radial direction (v_θ).

(Refer Slide Time: 05:40)



Now, for a solid body rotation, let us say if you are just assuming concentric shells like or concentric cylindrical shells, then they have some linear velocity gradient but they do not have any angular velocity gradient. On the other hand, for differential rotation both the angular velocity and linear velocity they vary in space and so that we have a linear velocity gradient as well as an angular velocity gradient. And when differential rotations are coming into play then what happens? the viscosity between two concentric layers become active because in rotation, the main role of viscosity is to prevent the relative rotation of one layer with respect to the other. So, if both of them are rotating in the same rate, then the viscosity effect would not work or would not be activated. But in differential rotation this is not the case, so viscosity should play a role.

(Refer Slide Time: 07:06)



Now, it is true that in usual fluids, very rarely solid body type of rotation is visible. Because there are so many things like compressibility, but if you are just taking a liquid for example, which is fairly incompressible, then differential rotation also starts to appear in the very beginning. We will see that after some time actually it actually decays, so what happens? Let us just take the concrete example of from our everyday life.

So, let us take a bucket full of water and the bucket is rotated suddenly. Just try to imagine the picture, so if you have a bucket, a cylindrical bucket, and then you just put some torque on the surface of the bucket and you want it to rotate. So, the first thing which rotates is the surface of the bucket, because you give the force on the outer surface. You give this is a shear as well, a rotational shear.

So, this gives you the rotation but the rotation starts from the surface, but the water which is very near to the axis of the bucket and far from the walls, this water is yet to start its rotation, so you can easily understand that at the very initial point the angular velocity near the wall is greater than the angular velocity near the axis of the bucket, so this actually creates an effective differential rotation.

(Refer Slide Time: 09:14)

and it is rotated suddenly, then What happens?

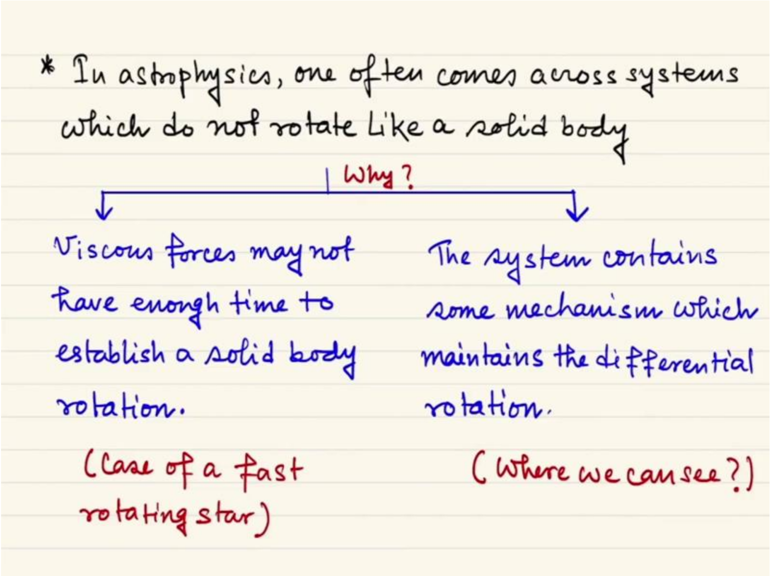
* The water near the surface starts rotating but the water close to the axis of the bucket still remains at rest ⇒ differential rotation

⇒ viscosity comes into play and stops the relative angular motions leading to a state where all the water will spin with the angular velocity of the bucket ⇒ The whole system rotates like a solid body ⇒ Solid-body rotation.
(rigid)

Then, viscosity comes into play to counter or rather to resist the relative rotation between two concentric layers, for example the first rotating layers will try to move the adjacent slow rotating layers move faster and vice versa. And finally they end up by making a compromisation, and then what happens? the relative angular motion actually stops and that leads to a state where all the water will finally spin with a constant angular velocity and that is nothing but the angular velocity of the bucket itself.

And at this point the whole system will rotate like a rigid solid body and this is actually known as the solid body or rigid body rotation. So, here we see that when viscosity comes into play, it basically transforms a differential rotation to a solid body or rigid body rotation, but it of course takes time.

(Refer Slide Time: 10:33)

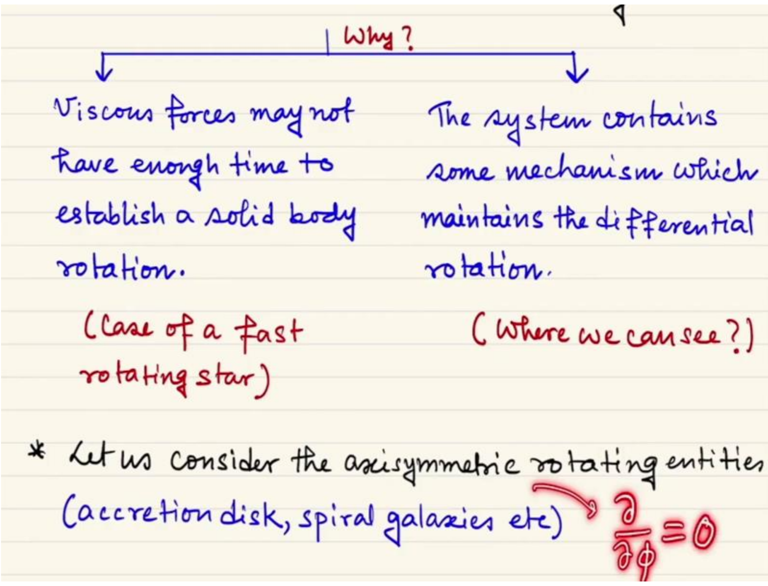


So, in astrophysics, let us say by some observation or by some indirect observation if we gather evidence that some systems do not rotate like a solid body for example, then what can be the possible explanation for that?

There can be two possibilities, one is that the viscous forces may not have enough time to develop and establish a solid body rotation; that means the system is rotating very very fast. So that is the case for a fast rotating system, which we will not discuss here in detail, but just for your information.

Another possibility is that, for the system, the viscosity has time to develop but the system contains some sustaining mechanism, which maintains the differential rotation. And now the question is that where can we see this type of thing?

(Refer Slide Time: 11:45)

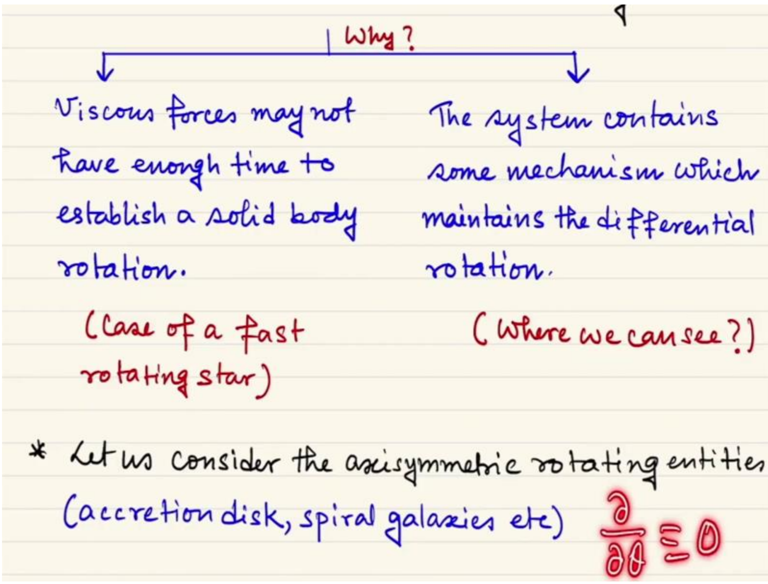


So for that, let us start by taking a very simple example which we have already studied in detail, that is the example of accretion disk and along with that we will say something very similar, that is the case of spiral galaxies. So, these are two axisymmetric rotating entities in astrophysics.

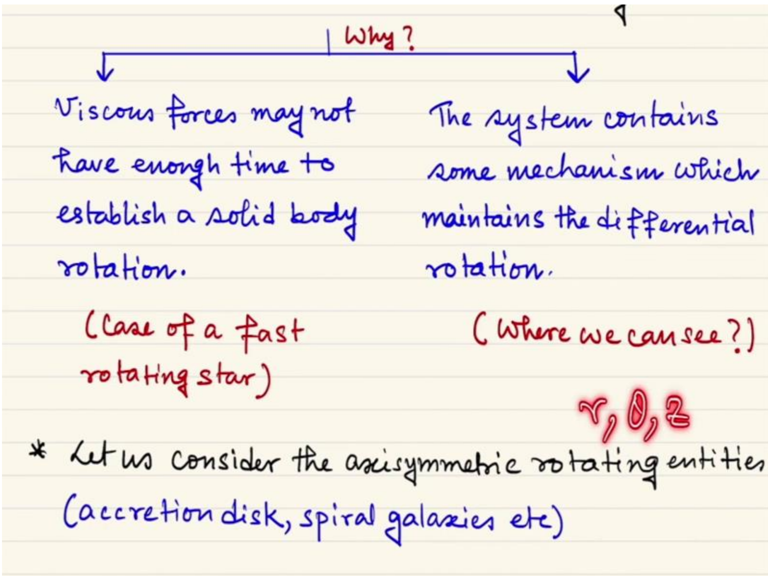
Axisymmetric means that they have a direction or axis of rotation, and the properties along the direction of the rotation is different from the properties perpendicular to it. But in the plane perpendicular to the direction of rotation it does not matter in which direction you are in. So, if you just consider the cylindrical system then basically the system changes from $\hat{z}$ direction to the (r, ϕ) plane, but in the (r, ϕ) plane it does not depend how much your θ is.

Then you can actually go from the like the inner concentric cylinders to the outer concentric cylinders or vice versa, so the system does not depend on the choice of the azimuthal angle ϕ. So, whenever we talk about axisymmetric system in cylindrical system, $\frac{\partial}{\partial \phi} = 0$.

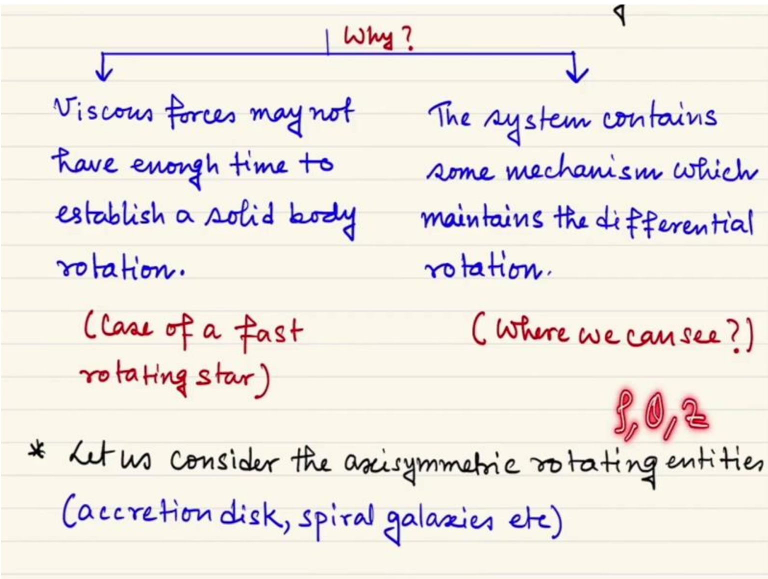
(Refer Slide Time: 13:41)



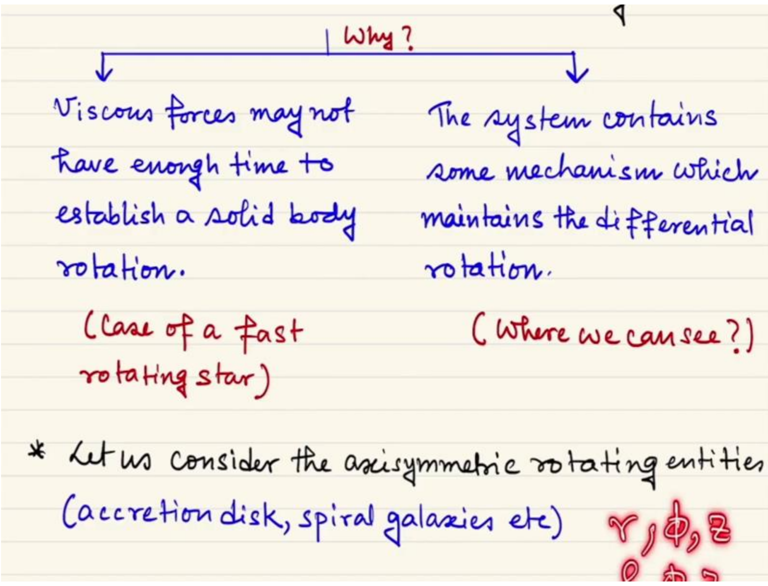
(Refer Slide Time: 13:47)



(Refer Slide Time: 13:53)

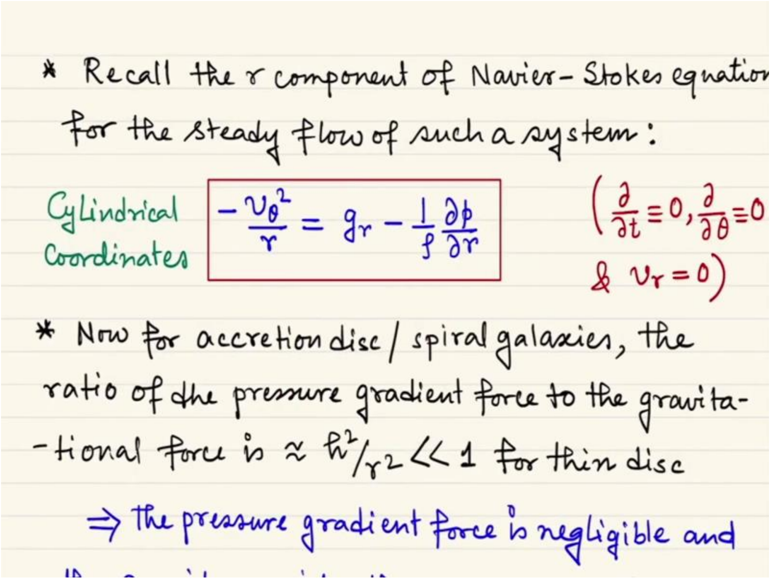


(Refer Slide Time: 13:57)



Sometimes we also say $\frac{\partial}{\partial\theta}$, it depends on the convention which we follow, either we take (r,θ,z) or we take sometimes in some literatures (ρ,θ,z) or sometimes we say $(r,\phi\,,z)$ or (ρ,ϕ,z) these type of conventions all are present.

(Refer Slide Time: 14:05)



So, now we try to analyze a bit more in detail the case of the rotation and how the rotation mechanism basically is developed in an axisymmetric rotating entity like an accretion disc. For that, we recall the r-component of the Navier-Stokes equation for the steady flow of such a system. If you remember the basic accretion disc system, we had a very small but non-zero radial velocity. The velocity was predominantly in the cross radial direction that is in the azimuthal direction and we said that the system was an axisymmetric system, so $\frac{\partial}{\partial \theta}$ or $\frac{\partial}{\partial \phi}$ would also vanish always.

And then, we also said that the system has some viscosity and that viscosity was responsible for causing the small radial inflow of mass, if you remember all these things. So, if we also remember that the main equation of dynamics for the accretion disc that came from the θ-component of the Navier-Stokes equation written in the cylindrical coordinates.

For the r and z components we had other information, for example using the z-component we obtained the condition for thin disc and using that one we obtained from the r-component of the Navier-Stokes equation, the Kepler's criterion.

So here we will actually follow the same methodology, here you will see that the r-component of the Navier-Stokes equation looks like $-\frac{v_\theta^2}{r} = g_r - \frac{1}{\rho}\frac{dp}{dr}$, because we are just considering a steady state motion of a rotation motion of an accretion disc, so that we cannot say that the system does not have enough time for the viscosity to be developed.

But remember, the steady state system is just gives $\frac{\partial}{\partial t} = 0$, but for our case, for simplicity, we also assume that v_r is very very small, so v_r can also be practically 0 and $\frac{\partial}{\partial \theta}$ is 0. If we assume all these things simply in cylindrical coordinates, the r-component of Navier-Stokes equation becomes $-\frac{v_\theta^2}{r} = g_r - \frac{1}{\rho}\frac{dp}{dr}$, g_r is nothing but the radially acting gravitational acceleration.

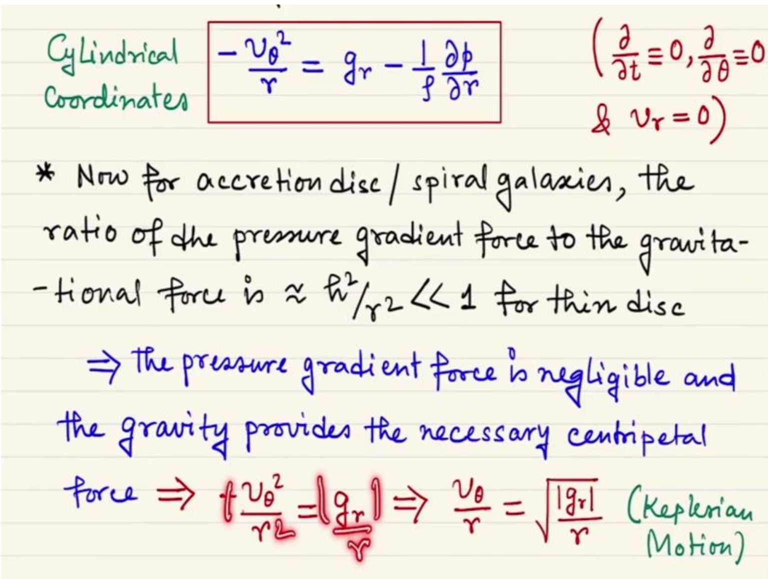
Now, you see that $-\frac{v_\theta^2}{r}$ nothing but the centrifugal force if you are in the frame of reference of the rotating system, or if you are just seeing the system from the laboratory frame of reference, then this is nothing but the centripetal force required to maintain the rotation. And g_r is the gravitational part and $-\frac{1}{\rho}\frac{dp}{dr}$ is the pressure gradient force.

So, if you remember that, for accretion disc and spiral galaxies which are thin, we can actually or in practical neglect the pressure gradient force to the gravitational force, or the net acceleration due to the pressure gradient force will be negligible with respect to the gravitational acceleration.

Because, just check your old notes where we discuss the accretion disc, the ratio of the two will be simply of the order of $\frac{h^2}{r^2}$ and for a thin disc this is negligible.

So, just to point this, it can be an interesting study to see that how this type of equation, how this ratio becomes for a non-thin disc. So, in a thin disc we are getting rid of this term and we will say that the pressure gradient force is no longer necessary for this balance equation.

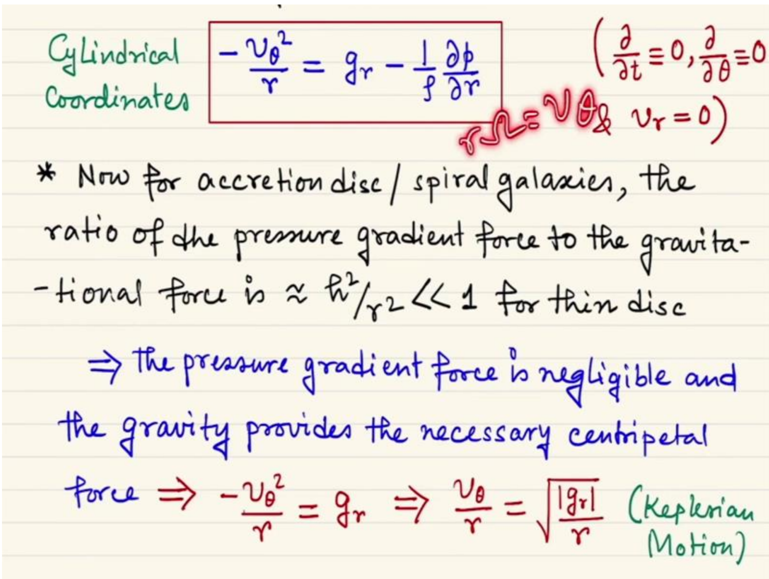
(Refer Slide Time: 19:01)



So, we will say that it is effectively the gravitational part, gravitational acceleration which provides the necessary centripetal force or centripetal acceleration. So, $-\frac{v_\theta^2}{r}$ will be exactly equal to g_r. And that then you can just say that just taking the mod that $\frac{v_\theta^2}{r^2} = \frac{|g_r|}{r}$. You know also that g_r should be radially inward directed, so that it is an attractive one, otherwise there is no meaning of rotation in this case.

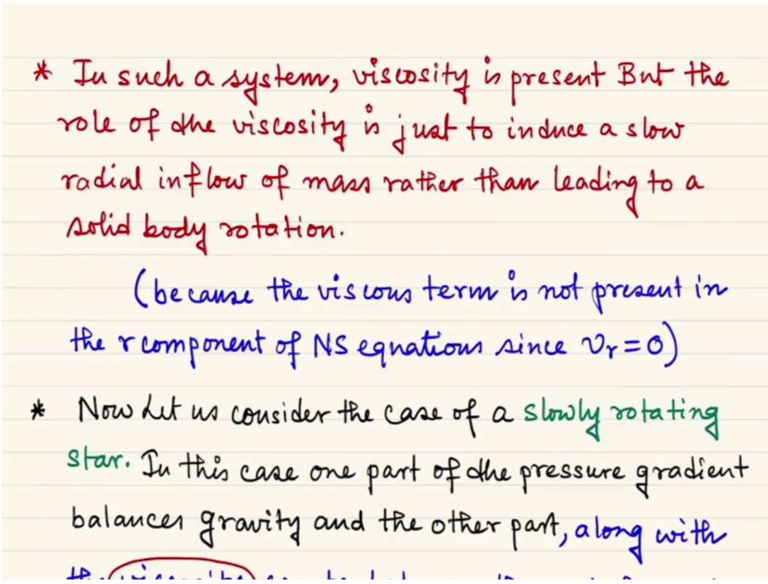
So, $\frac{v_\theta}{r}$ will simply be equal to $\sqrt{\frac{|g_r|}{r}}$. And what is this? This is nothing but our well known Keplerian motion. So, a Keplerian motion basically gives a clear evidence of a sustaining differential rotation.

(Refer Slide Time: 21:00)



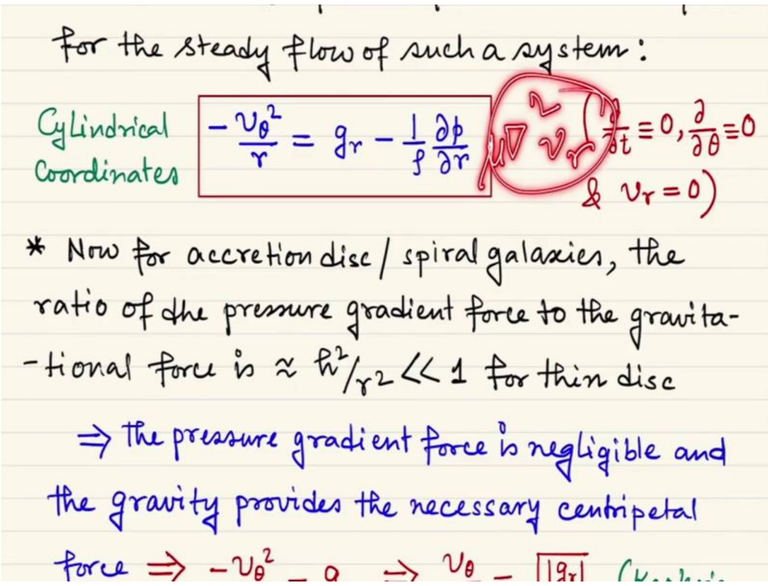
Because $\frac{v_\theta}{r}$, if you again remember, that $r\Omega = v_\theta$, so Ω is something which is depending on r, as we all know that g_r is not linearly proportional to r, so there is no chance that there will be a cancellation between r's in denominator and numerator and gives you something which is independent of r. So, we have a differential rotation.

(Refer Slide Time: 21:34)



But remember, this does not say that in this systems viscosity is not present. In general viscosity is present in such a system, but what happens that the role of the viscosity is just to induce a slow radial inflow of the mass rather than leading to a solid body rotation. So, if in other words the question is that, if you go to the equation $-\frac{v_\theta^2}{r} = g_r - \frac{1}{\rho}\frac{dp}{dr}$, you will see that there is no term from viscosity.

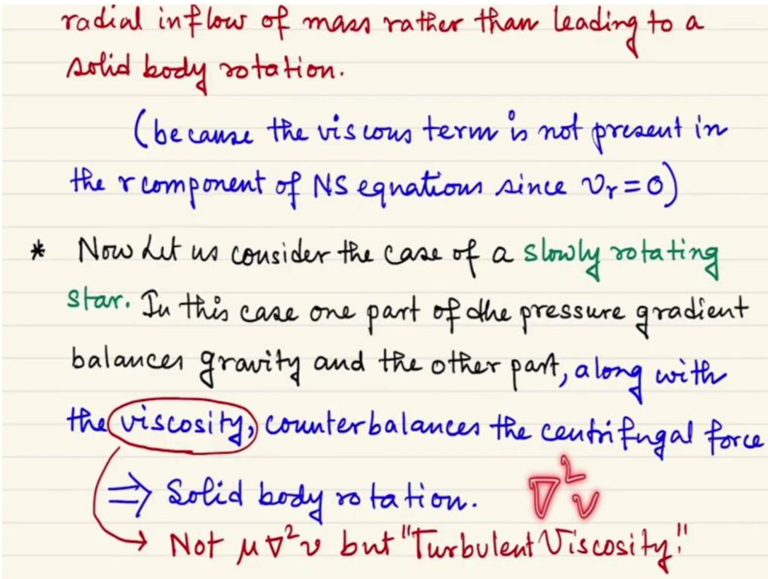
(Refer Slide Time: 22:17)



And that is because of the simple fact that v_r is so small that the viscosity term which should be $\mu\nabla^2 v_r$ will be very very small and that is exactly the case where we are talking about neglecting the viscosity for at least for this rotation purpose.

That means, the contribution of viscosity to counterbalance the net centrifugal force or to provide the necessary centripetal force, is negligible. So, viscosity is there, but its only role is to induce a slow radial inflow and that can be seen when you write the governing equation for the evolution of v_θ. So that is the θ-component of the Navier-Stokes equation, you should remember that.

(Refer Slide Time: 23:22)



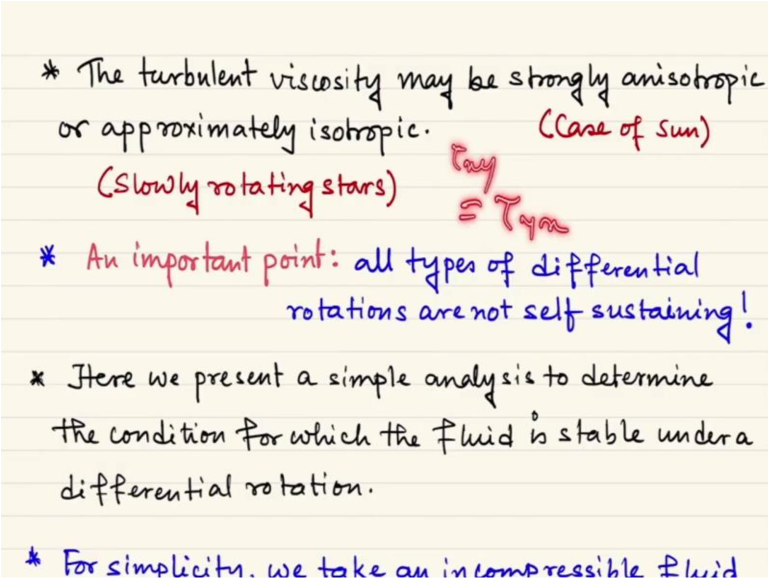
Now, that was the case for the accretion disc or spiral galaxy. Now, what happens, if we consider the case of a slowly rotating star. Now, in this case, one part of the pressure gradient balances the gravity. So, in the previous case, the pressure gradient force was negligible, centripetal acceleration was provided by the gravity, and that led to the Keplerian motion thereby sustaining a differential rotation.

Here one part of the pressure gradient, which is no longer negligible, balances gravity and the other part, along with the viscosity, counter balances the centrifugal force.

Now, the question is that, when I am saying along with the viscosity, does it is mean the viscosity term $\mu\boldsymbol{\nabla}^2 v_r$? well this is also not correct. So, actually this is not the normal molecular viscosity or Navier-Stokes equation viscosity in direct sense which is responsible here, but there is another thing, it is called turbulent viscosity which I will talk when I will introduce the general properties of fluid under turbulent conditions.

Then I will talk about the turbulent viscosity. So, turbulent viscosity is an equivalent viscosity coming from the virtue of the turbulent motion of the fluid. And this viscosity basically reduces the differential rotation and then finally leads to the system to a state of solid body rotation.

(Refer Slide Time: 25:30)



Now, turbulent viscosity just for your information is very very interesting and it is actually in general anisotropic, and when this anisotropy is very strong or non-negligible, then actually the differential rotation is not transferred or transformed into a solid body rotation. And that is exactly the case of our Sun.

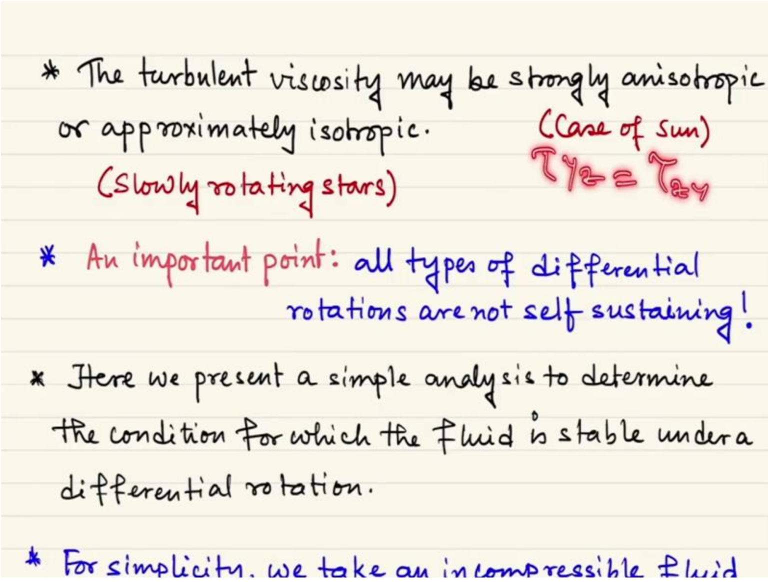
So, every time we will see something in the Sun, it will create a very important aspect for us or rather for our astrophysical knowledge, because Sun is our point of reference in some ways. So, for Sun, differential rotations are prominent. And actually, just to tell you, the signature of differential rotation including the spatial distribution of the angular velocity can actually be obtained using Helioseismology, which is the physics of the oscillation of the Sun.

So, it has radial and non-radial modes. So, using this actually people have obtained some spatial profile of the angular velocities, and it is shown that near the poles the sun moves very slowly and near the equator it rotates faster or spins faster.

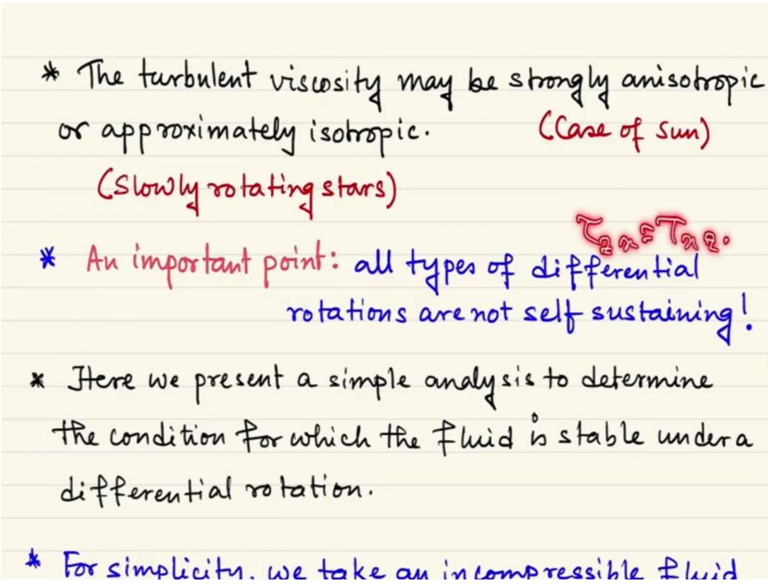
And the discrepancy in the angular velocity or the angular speed is by 10% and actually it is sometimes it is more than 10%, so this is non negligible. So, this was the case where you can exactly see clearly differential rotation, and that is not the case for the slowly rotating stars where the turbulent viscosity is approximately isotropic.

That means, if the turbulent viscosity let us say is represented by some matrix, by some second rank tensor, then the matrix will be symmetric. So, something like τ_{xy} will be equal to τ_{yx}, for example.

(Refer Slide Time: 28:28)

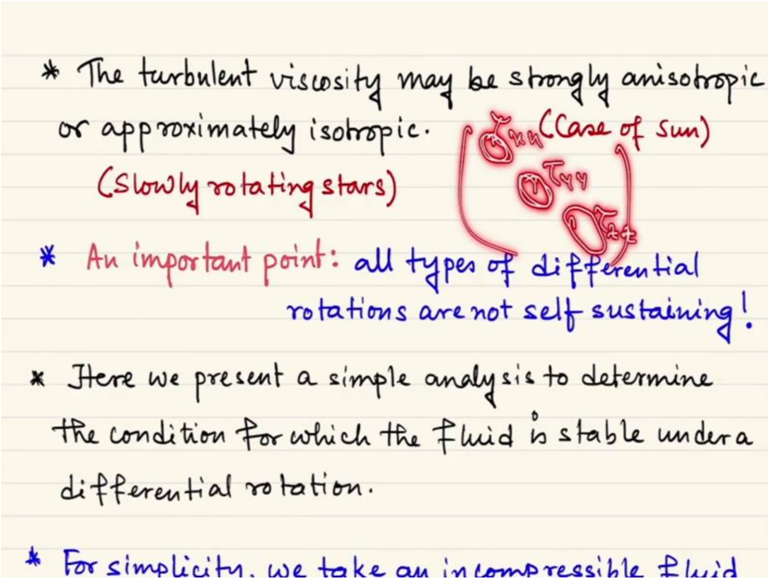


(Refer Slide Time: 28:34)



And not only that then $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$. Finally, you can see that the off-diagonal terms and the diagonal terms will be almost equal.

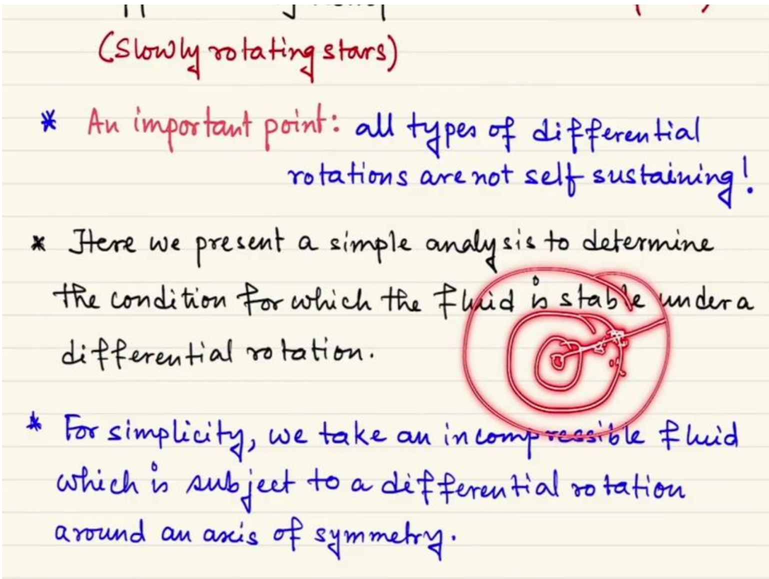
(Refer Slide Time: 28:58)



That means and the diagonal terms will also be very very close. That means $\tau_{xx}, \tau_{yy}, \tau_{zz}$, will also be very close and only then you have this turbulent viscosity nearly isotropic. So, we will come to that point. So, you know from your knowledge that the viscosity comes from the off-diagonal part of the pressure tensor.

So, what about turbulent viscosity? That I will discuss later. So just for your information if this is nearly isotropic, which is the case for slowly rotating star systems, then any differential rotation actually decays with time thereby leading to the final state of a solid body rotation. But this differential rotation can be sustained if the turbulent viscosity itself is strongly anisotropic and that is the case for the Sun.

(Refer Slide Time: 30:10)



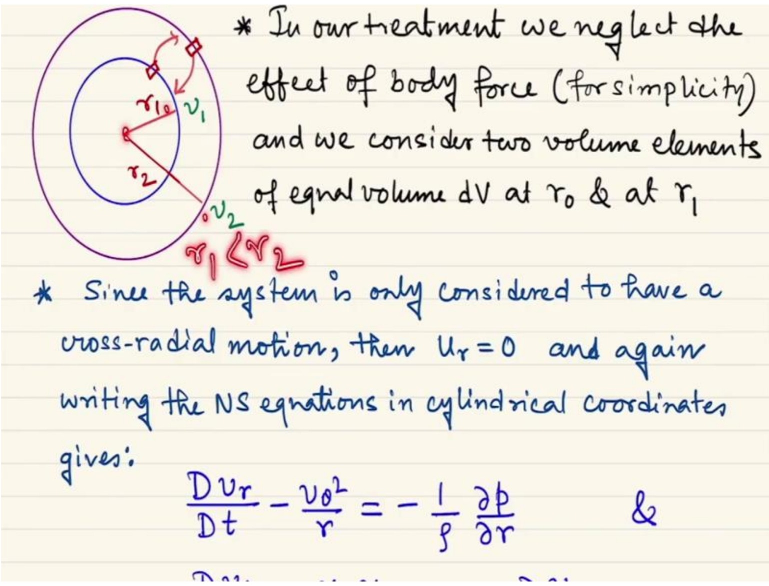
Now, there is a very very important point; all types of differential rotations are not self-sustaining, and what is the reason for that? Well, it depends on the dynamics and the structure of the system. So, a general analysis or general answer for this question is not very easy to obtain.

However, by doing some very simple analysis we can actually determine some condition for which a fluid is stable under a differential rotation. That means, a fluid is already moving with a spatial gradient of angular velocity. Now, if we perturb the system, would the system try to get back its original configuration with the differential rotation?

Let us say, the perturbation is done in such a way that the differential rotation state is made to tend towards a solid body rotation. Now, would that be a stable state? That means, would the system try to bounce back to its initial state or it would further move away!

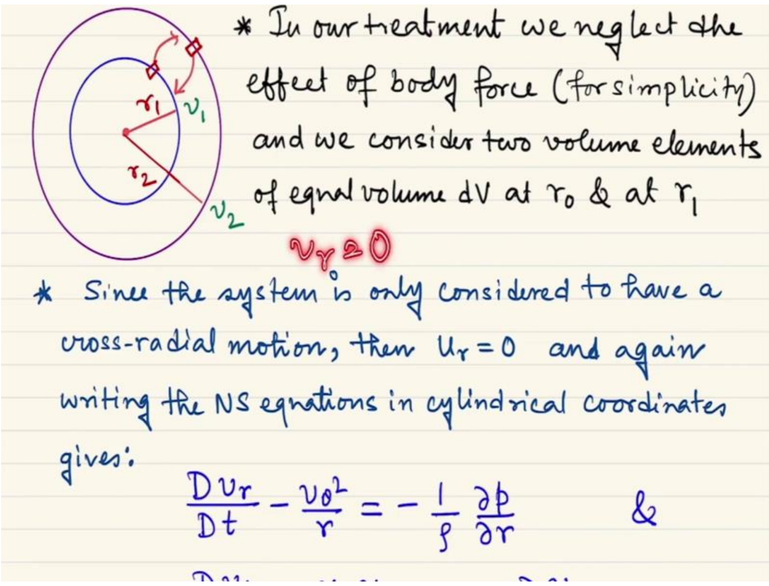
So, if it would try to get back its original position or original state then the fluid is called stable under a differential rotation. That means, the system prefers to be in a state of differential rotation. For simplicity, here we take an incompressible fluid which is subject to a differential rotation around an axis of symmetry.

(Refer Slide Time: 32:09)

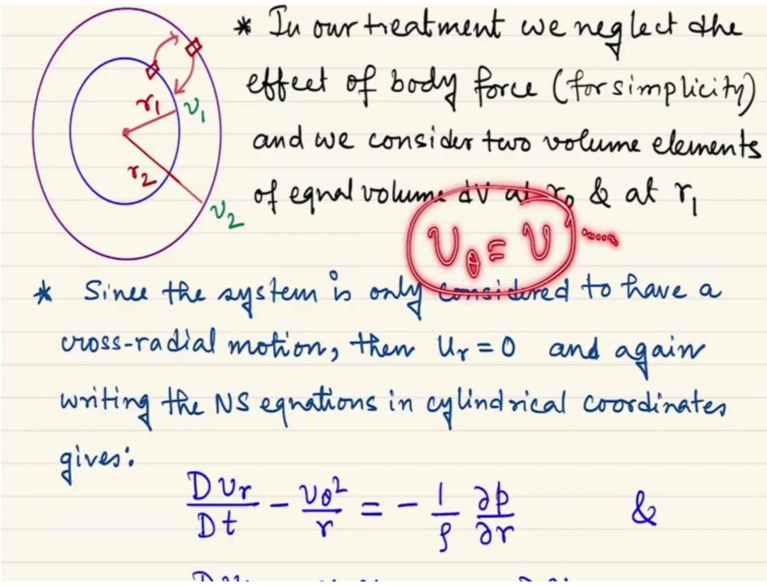


So, now we are just taking two concentric rings; one is at r_1, another is at r_2, of course $r_1 < r_2$. And we consider the fluid which is anywhere in this axisymmetric case and the fluid is only moving in the $\hat{\theta}$ direction.

(Refer Slide Time: 32:46)



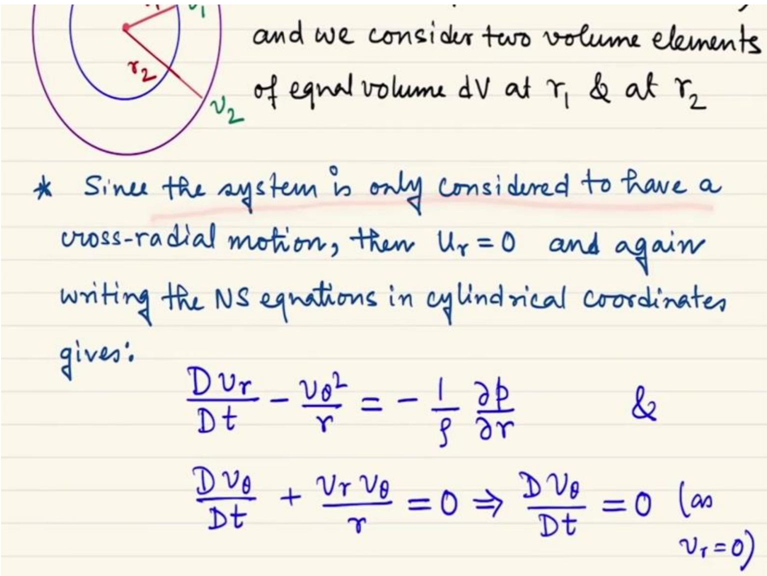
(Refer Slide Time: 32:51)



So, it has only azimuthal motion, so $v_r = 0$, practically. This is the case for accretion disc as well and v_θ is the whole velocity. So, you can see that outer ring is moving with linear velocity v_2 and inner ring is moving with linear velocity v_1.

Now, in our treatment, here we neglect the effect of body force, that is the gravity for the sake of simplicity. And one interesting study can be done to see what the effect the gravity gives. So anyways, we now consider two volume elements of equal volume, dV, at r_1 and r_2.

(Refer Slide Time: 33:59)



Now, since the system is only considered to have a cross radial motion or azimuthal motion, then $v_r = 0$, and again writing then Navier-Stokes equation in cylindrical coordinates because of the axisymmetry we have finally, $\frac{Dv_r}{Dt}$. So, $\frac{D}{Dt}$ is nothing but the material derivative; that means, we are now concentrating on a fluid element or fluid particle.

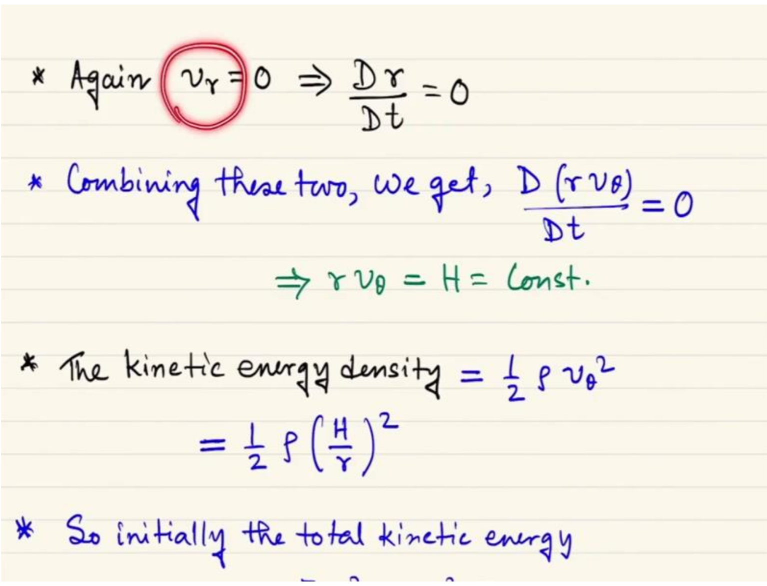
So, $\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$. So, the acceleration of the fluid element is due to two things; one is the centripetal acceleration and one is the pressure gradient force.

So, now the question is that the, how should it look like for our case? So, of course, you can see that for our case, because v_r is 0, and you will say that the total centripetal acceleration is provided by the pressure gradient force. That is the conclusion from this component.

What is the conclusion coming from the other component? We simply say that for the θ-component or the cross radial component, we have $\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} = 0$, all the other terms are 0 either because they are depending on θ or they have other negligible things.

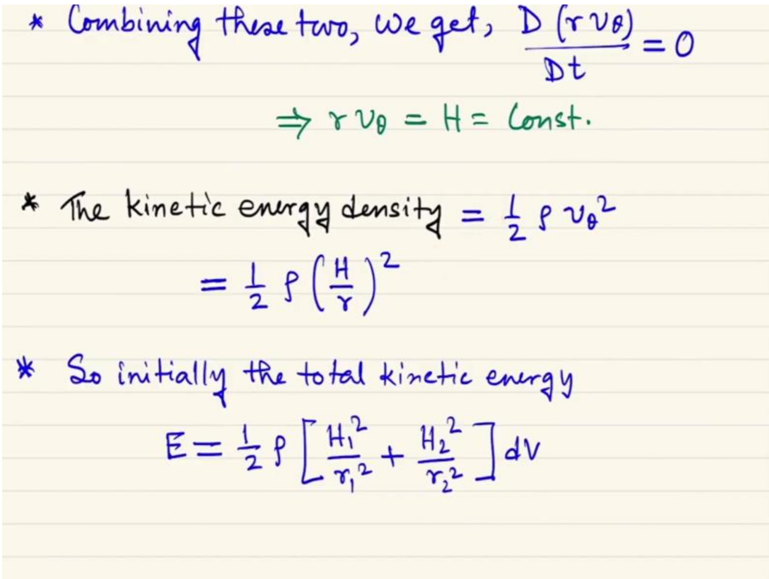
So, now $v_r = 0$ makes thing again simpler, which simply gives us $\frac{Dv_\theta}{Dt} = 0$. So, that means, the linear velocity of the fluid element does not change with time. Is this clear? So, the velocity, the linear velocity with which one fluid element is moving is constant.

(Refer Slide Time: 36:32)



Again v_r is equal to 0, what is the meaning of that? That means, $\frac{Dr}{Dt}$ is equal to 0. This is the simple meaning of this. And if we combine the two, then we can write that $\frac{D(rv_\theta)}{Dt} = 0$. So, for a particle not only v_θ, not only r, but r times v_θ is also constant in time, and this is nothing but the mass density of the angular momentum, angular momentum per unit mass. We call this H, and this is constant along the trajectory of a fluid element. So, this is a Lagrangian invariant, because we are now talking in terms of the Lagrangian derivatives.

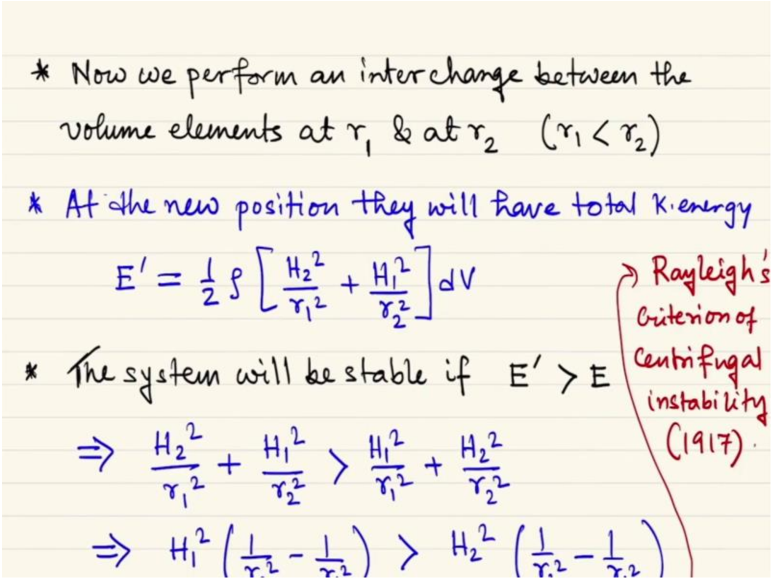
(Refer Slide Time: 37:27)



Now, what is the kinetic energy density at any position? that is the half times the density times the velocity square and the velocity means only v_θ. So it is nothing but equal to $\frac{1}{2}\rho\left(\frac{H}{r}\right)^2$. Because v_θ is nothing but $\frac{H}{r}$.

So, before doing any change we calculate the total kinetic energy of those two fluid elements, and they will be nothing but $\frac{1}{2}\rho\left[\frac{H_1^2}{r_1^2} + \frac{H_2^2}{r_2^2}\right] dV$, simple. They do not have any other force like body force that is neglected over here. So, the only energy is the kinetic energy.

(Refer Slide Time: 38:50)

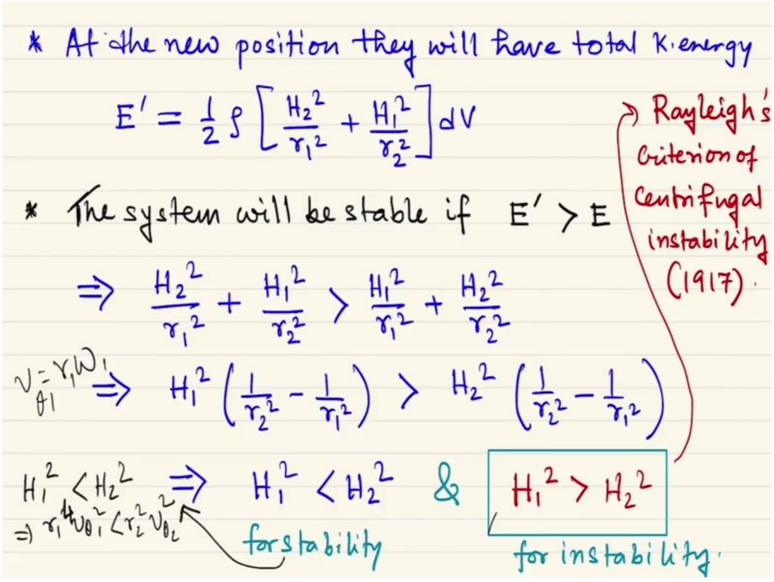


So, now you can simply say that we make an interchange of those two fluid elements, which was primarily at $r_1$and r_2, where $r_1 < r_2$ once again, if in case you forget.

Then what happens? at the new position they will have the total kinetic energy $E' = \frac{1}{2}\rho\left[\frac{H_1^2}{r_2^2} + \frac{H_2^2}{r_1^2}\right]dV$. Now look, these particles they are just picked and they are put in a new position. So they will come with their instantaneous angular momentum density, which they had before the perturbation.

So, we will see that if they are just picked and they are just put in the new positions, would they suit the new position or not, that is the question. So, if we simply pick them from their original positions and swap between them, in terms of the position, but they are always associated with their original angular momentum. Then, the question is that would that system be stable. How would the system respond?

(Refer Slide Time: 40:26)



Of course, the system will be stable, one can easily understand, if $E' > E$. Because, if the systems final energy is less than the initial energy the system would always try to go to the final state and then it will try to move further away from the initial state.

But, only if the new state has an energy, which is greater than that of the original state then the system would try to bounce back to its original state and that is the stability. So, for stability here, this can be confusing sometimes, that here when we talk about the stability that means the stability with respect to the differential rotation.

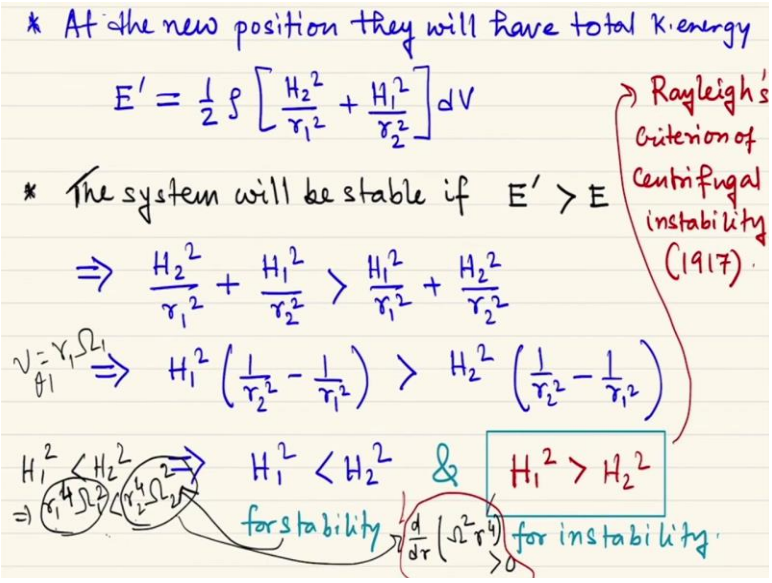
So, for our system the stability will be attained if the energy in the final state would be greater than the energy in the initial state. Because we want the system to get back to its initial state and that should be the state with less energy. And if it is true, then you can write that this is simply equivalent to writing $\frac{H_1^2}{r_2^2} + \frac{H_2^2}{r_1^2} > \frac{H_1^2}{r_1^2} + \frac{H_2^2}{r_2^2}$.

And then I take this all the terms with H_1^2 in one side and all the terms with H_2^2 in on the other side, we have $H_1^2\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) > H_2^2\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)$. Now, since $r_2 > r_1$, then $\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)$ is a negative quantity, but H_1^2/H_2^2 is positive as they are squares. So, cancelling the common negative quantity from both sides, we have $H_1^2 < H_2^2$ and that is the condition for stability.

Whereas, the opposite condition $H_1^2 > H_2^2$ will be the case for instability. So, if you write in terms of for example, Ω and θ, so what was H? If you remember, H was nothing but

rv_θ, so for stability that simply says, $r_1^2 v_{\theta 1}^2 < r_2^2 v_{\theta 2}^2$. Now, what is v_θ? It is nothing but $r\Omega$.

(Refer Slide Time: 44:34)



So, that would simply imply $r_1^4\Omega_1^2 < r_2^4\Omega_2^2$. So, at differential way, if we just change now those two volume elements between two concentric layers, which are infinitesimally close to each other than an equivalent way of writing this, because this is true for any layers $\frac{d}{dr}(\Omega^2 r^4) > 0$.

So, this is known as the Rayleigh's criterion for stability. So, sometimes we say this is Rayleigh's criterion of centrifugal stability and the opposite condition is known as the Rayleigh's criterion for centrifugal instability.

And Rayleigh in the year 1917, for the first time he derived this one. So, you see that here in this lecture by a very simplistic condition or a simplistic analysis process, we have not done much formal things, just using very simple axisymmetric cases, steady condition, incompressibility, using all these assumptions, finally, what we concluded?

That not every differential rotation is auto sustained or self sustained. So, if the Rayleigh's stability criterion is satisfied then once a differential rotation is established in a system, it will be there, it will not be decayed. Now, if the differential rotation is not sustained, then what is the fate of that? then this is the criteria for instability, and that means, the differential rotation is not sustained, that means the fluid does not prefer to be in a state where different layers are moving with different angular velocities. And then what is the fate, what is the destiny of that system? To be in the state of solid body rotation.

Of course, Sun is not like such a system, because in Sun we have sustained differential rotations. In the next lecture, we will discuss a bit more about these rotations and also we will try to understand how such a system can be studied from a non-inertial frame of reference.

Thank you very much.