Introduction to Astrophysical Fluids
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Lecture - 36
Oscillation of stars (Contd.)

Hello and welcome to another lecture session of Introduction to Astrophysical Fluids.

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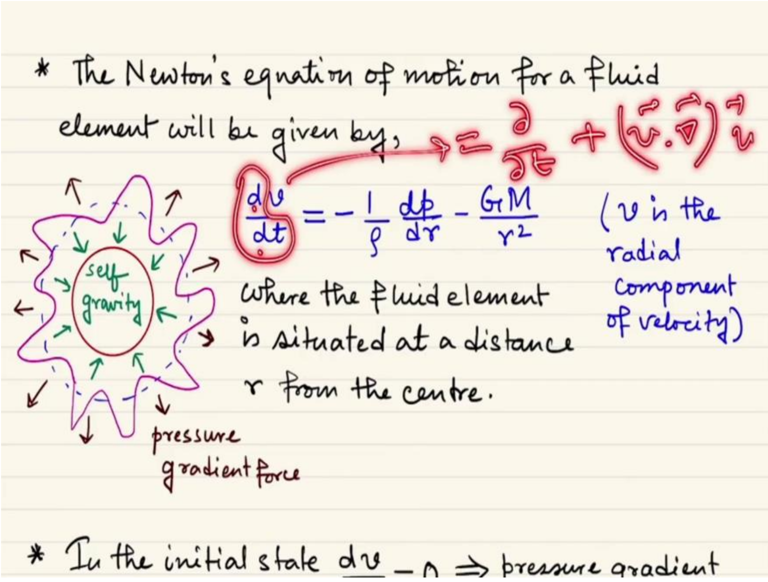

So, here you can finally see some picture of a star which is oscillating. So, let us say as a initial condition, initial state at rest (green line). And then after perturbation, the perturbed figure is given (pink line) [see the figure above].

And you can actually see that more or less, if you are at a distance r from the center, more or less you are having the same nature of the perturbation, irrespective of the position on the star.

So, this is a rough image of course, we know that the pure radial oscillation or pure radial perturbation is an idealization in this case. So, in general, we can just think that this type of figures are there, roughly considering radial symmetry. And then we just consider the equations of motion or the Newton's equation, for a fluid element, which can be considered as a particle here, and we will write the force equation for that.

So, the particle is situated let us say at a distance r from the center. So, if the particles velocity is $\boldsymbol{v}$, then $\frac{d\boldsymbol{v}}{dt}$ is nothing but, if you remember in terms of the Eulerian derivatives that is equivalent to $\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$.

So, $\frac{dv}{dt}$ is equal to $-\frac{1}{\rho}\frac{dp}{dr}$. Why this is $\frac{dp}{dr}$? This is in general gradient of p, but here once again we are only considering the system in the radially symmetric direction, so only $\frac{d}{dr}$ will survive, $\frac{d}{d\theta}, \frac{d}{d\phi}$ will not survive.

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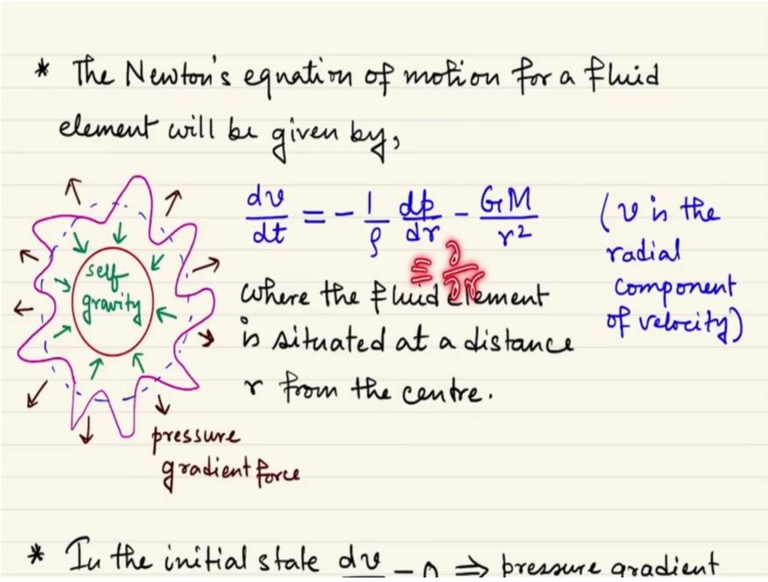

So, $\frac{\partial}{\partial r}$ basically now has become $\frac{d}{dr}$. So, this is something is given. And $-\frac{GM}{r^2}$. So, $-\frac{1}{\rho}\frac{dp}{dr}$ is the pressure gradient force, and $-\frac{GM}{r^2}$ is the gravitational force due to self-gravity. So, $\frac{dv}{dt} = -\frac{1}{\rho}\frac{dp}{dr} + \frac{GM}{r^2}$ is the equation for a fluid element which is situated at a distance r from the center, and M is the mass of all the star which is situated from 0 to r, the whole mass. So, basically the fluid element experiences the gravity field created by the whole mass which is inside the radius r, that is the meaning of M.

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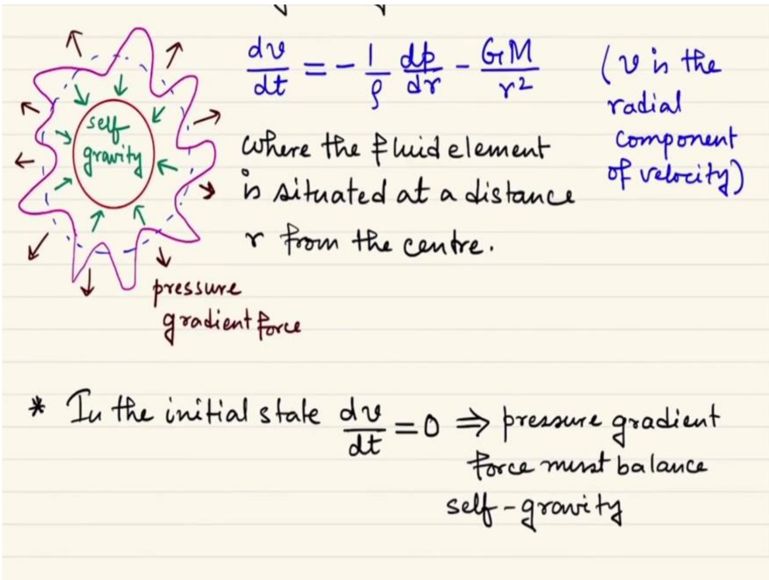

So, what happens at the initial state? So, our initial state is assumed to be not only a steady state, but as a static state. So, we just say $\boldsymbol{v} = \mathbf{0}$ and $\frac{dv}{dt}$ is also $\mathbf{0}$. So, the pressure gradient force will simply be exactly balanced by the self-gravity.

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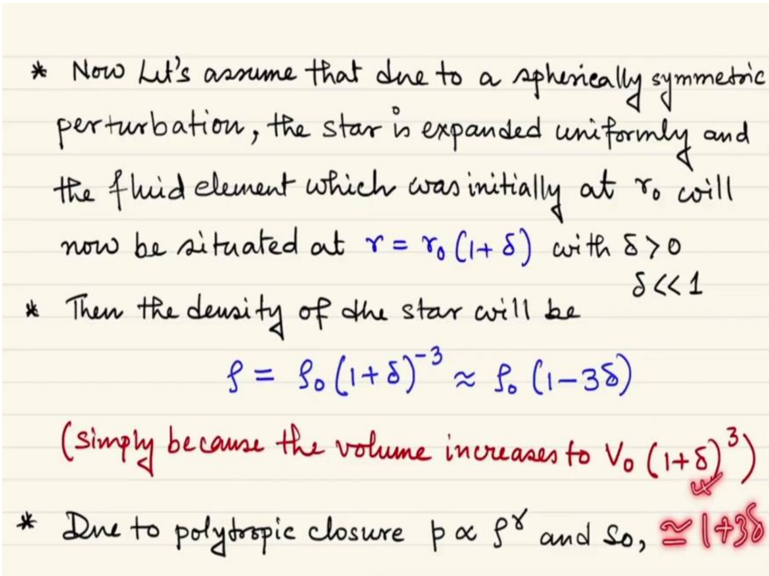

Now, let us assume that due to a spherically symmetric perturbation, the star is expanded uniformly in all directions. And the fluid element which was initially at r_0 will now be situated at some r. So $\boldsymbol{r}, \boldsymbol{v}$ are the generic variables, so the equation is true for initial state, for current state and any later state. So, we are now calling all the initial variables at as r_0, ρ_0 and p_0.

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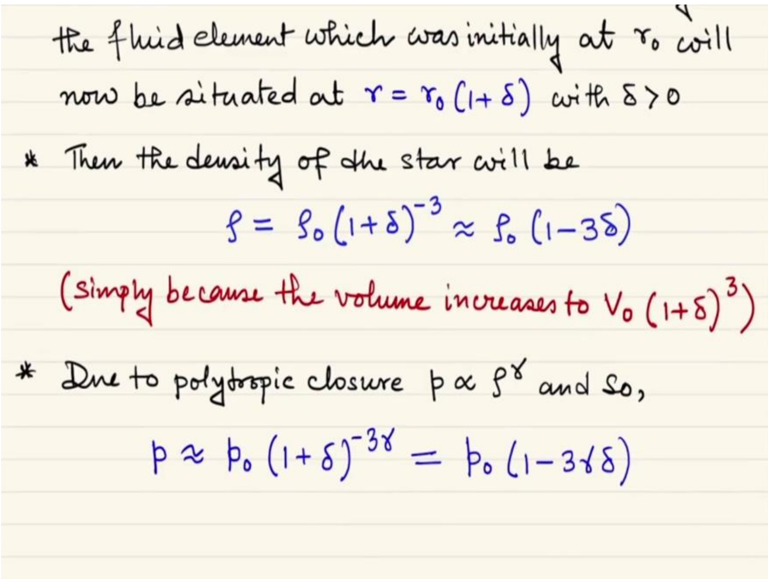

That means ρ_0 is the density of fluid element which was at r_0, so this is the initial density and p_0 will be the corresponding initial pressure.

So, now what we will do? We will simply perturb the position of the particle from r_0 to some value r, where $r = r_0(1 + \delta)$, and $\delta > 0$ but it is very very less than 1, a very small perturbation.

And if this is true, then the density of the star can be written as $\rho = \rho_0(1 + \delta)^{-3}$, that is simply because the volume changes as $V = V_0(1 + \delta)^3$, because the volume will increase as the radius cube, which increasing as $(1 + \delta)$ and so the volume increase will be proportional to cube of that. And if δ is very very small, it will be simply $(1 + \delta)^3 \approx (1 + 3\delta)$.

And for the density it is actually reducing because density is nothing but mass by volume, and the mass is unchanged for the system. So, at some point I said something a bit confusing that this ρ_0 is the initial density of the whole fluid. This is not the density of the fluid particle of course, this is the density at the point r_0.

Now, this density is now changed to $\rho = \rho_0(1+\delta)^{-3}$. Of course, this is not a representative density, this is the just the variable. So, initial variable becomes the final variable after perturbation. But they are actually changing from one point to the other, so that is something to not to forget. And the volume is increased because we have dilated the star a bit by dilating the radius.

Now, we have assumed polytropic closure. So, p will be, if ρ is decreased by a factor $(1+\delta)^{-3}$, p will be decreased by $(1+\delta)^{-3\gamma}$. So, the final pressure is equals to $p \approx p_0(1-3\gamma\delta)$ and that is true for every point of the fluid.

This is just relating initial set of variables and later set of variables, like current set of variables.

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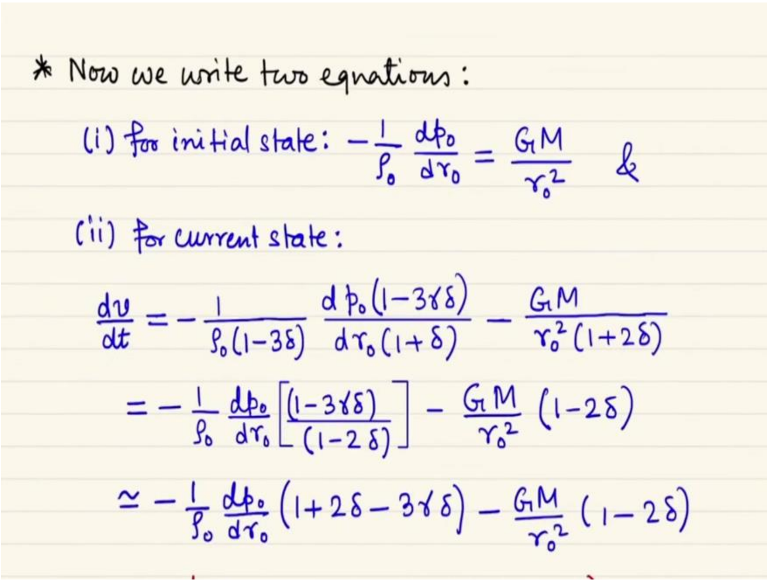

If this is clear, then you can easily say that for initial state I can just write the balance equation of the pressure gradient force and the self gravity force using the initial coordinates $-\frac{1}{\rho_0}\frac{dp}{dr_0} = \frac{GM}{r_0^2}$. And for current state, I write the equations with the current variables $\frac{dv}{dt} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM}{r^2}$, so $\frac{dv}{dt}$ is no longer equal to 0.

But ρ is nothing but $\rho_0(1-3\delta)$, p is nothing but $p_0(1-3\gamma\delta)$, r is nothing but $r_0(1+\delta)$. And r^2 will be nothing $r_0^2(1+\delta)^2$ and which is almost equal to $r_0^2(1+2\delta)$, once again just keeping the terms up to first order.

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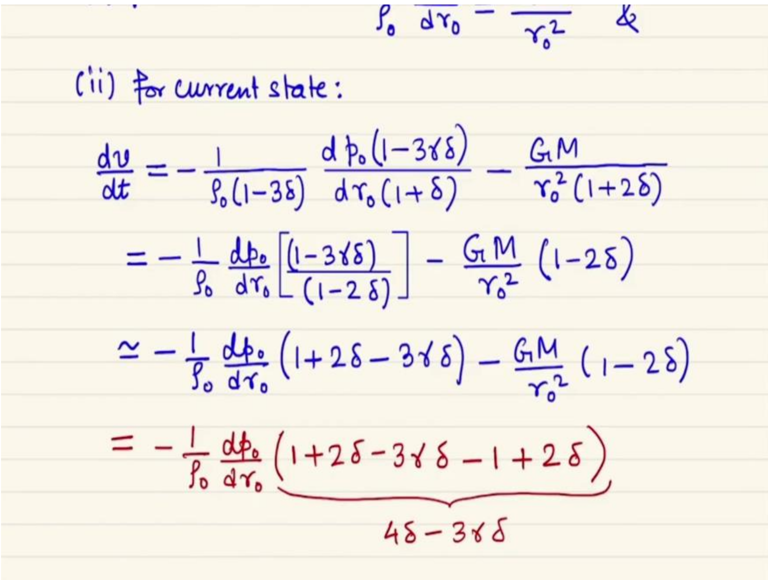

Now, if we do that, finally, we can say that this is nothing but

$$\frac{dv}{dt} = -\frac{1}{\rho_0(1-3\delta)}\frac{dp_0(1-3\gamma\delta)}{dr_0(1+\delta)} - \frac{GM}{r_0^2(1+2\delta)}$$

$$= -\frac{1}{\rho_0}\frac{dp_0}{dr_0}\left[\frac{1-3\gamma\delta}{1-2\delta}\right] - \frac{GM}{r_0^2}(1-2\delta)$$

$$\approx -\frac{1}{\rho_0}\frac{dp_0}{dr_0}(1+2\delta-3\gamma\delta) - \frac{GM}{r_0^2}(1-2\delta)$$

$$= -\frac{1}{\rho_0}\frac{dp_0}{dr_0}(1+2\delta-3\gamma\delta-1+2\delta) = -\frac{1}{\rho_0}\frac{dp_0}{dr_0}(4\delta-3\gamma\delta)$$

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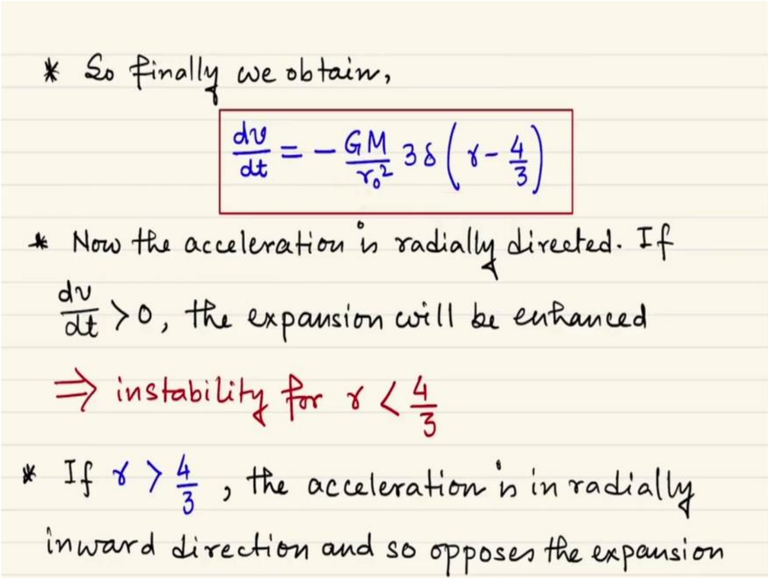

So, finally, what we obtain? So, finally, we obtain that $\frac{dv}{dt} = -\frac{GM}{r_0^2} 3\delta \left(\gamma - \frac{4}{3}\right)$. So, this is the final equation actually. This is the final equation one should write for the acceleration of a fluid particle situated at a distance r from the center.

Now, the acceleration can be negative, can be positive. Now, what is the implications of that? We were trying to dilate the same thing by putting some outward perturbation to the radius. So, we try to increase the radius, now if the acceleration, so of course, you know like these two forces where in the gravity and the pressure gradient force, the acceleration is also radially directed.

And if the acceleration is radially directed, then if this is positive that will simply indicate that the system will be dilating. And if this is negative, then this will indicate that the system will try to get back to its original position. So, it will counteract, the act of dilation. And when $\frac{dv}{dt} > 0$, simply because $\frac{GM}{r_0^2} 3\delta$ is positive, so the only possibility that this becomes greater than 0 only when $\gamma < \frac{4}{3}$.

So, if $\gamma < \frac{4}{3}$, then the system, if dilated a bit it will continue expanding thereby supporting or enhancing the dilatation and that is why we will have instability.

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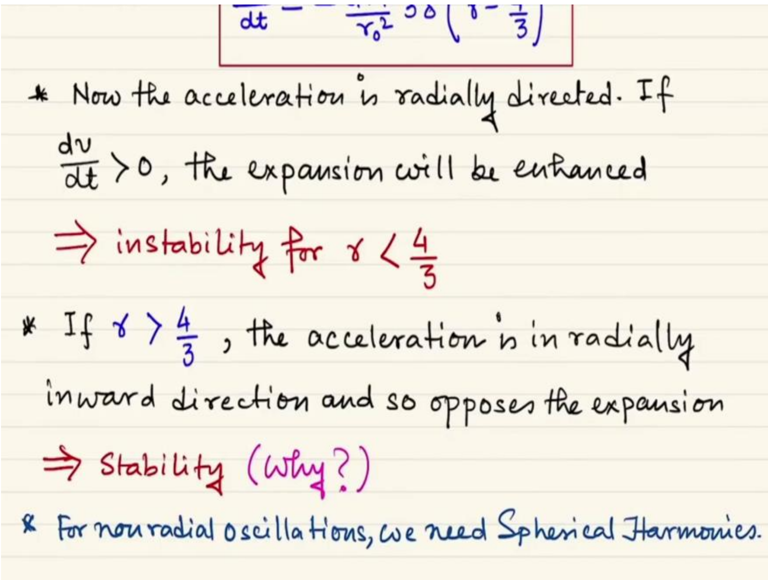

However, if you have $\gamma > \frac{4}{3}$, then the acceleration is actually in radial direction, but in inward direction, so it opposes the expansion, and then it leads to the stability.

Now, here at this point just think a bit. We understand the result is interesting, the result is simply obtained. So, we have enough reason to be happy. But before being happy just think what is the physical reason, because finally we are doing some astrophysics. So, some physical reasons should be thought, why it is that when γ exceeds critical value let us say $\frac{4}{3}$, finally gives a rise to the stability, otherwise it is unstable.

Of course, a very simple and very apparent explanation is that, whenever you are trying to perturb the system let us say by increasing the density for example, then what happens? Then you are just increasing the density at one point and so the self-gravity force will be important at that point. But if your density is increasing due to your polytropic closure, your pressure will be increasing and that will actually increase the hydrostatic pressure to balance the effect of the self gravity.

Now, if your γ is higher and higher, higher than some critical value, then some increase in density will make the increase in pressure so important that it will be enough or it will be sufficient to counteract or counter balance the self-gravitational effect or collapse type of thing efficiently. Whereas, if γ is less than $\frac{4}{3}$ value, then what happens?

Then the pressure perturbation is not sufficient to counterbalance the gravitational effect. And the same thing can be actually thought if we decrease the density. So, for example, here we are just decreasing the density. So, the self gravity force basically decreasing. But with the decrease in density how the decrease in pressure will act because if pressure wins again, so here we are trying to curb the gravitational force. So, if the pressure is still very much enhanced, then it will actually expand.

So, we can have actually two types of instability. One is instability by expansion, one is instability by collapse. Both should be counter balanced by their corresponding enemies. So, here you will see that we try to dilate the system. And then this dilatation is actually counter balanced by the reduction in the hydrostatic pressure gradient force. So, this is exactly the same mechanism by which one can actually think of stability and instability in this framework.

Now, when stability is there, then every time you are increasing or decreasing the density, so increasing and decreasing density means increasing or decreasing the gravitational effect. So, your pressure should be efficiently increased or decreased respectively, so that you can nullify or counter balance as soon as possible the effect caused by the perturbation, so that is the very apparent and straightforward discussion.

Now, it is true that there is an interesting thing about $\frac{4}{3}$, so $\frac{4}{3}$ has a very much interesting importance in the context of Chandrasekhar limit. So, you search and you let me know that whether you got it or not.

So, you know like for a normal classical gas, what is the adiabatic index? If it is consisting of monatomic gases, it is $\frac{5}{3}$. Can we have some gas for which the adiabatic index is $\frac{4}{3}$? Check ok. So, these are very interesting things and also I said that there is a relation with Chandrasekhar limit.

Coming back to our question that here we have not done proper linearization, proper dispersion relation. We have just taken a fluid element and we have tried to analyze its motion, whether it is under dilatation, whether it is trying to accelerate outward or inward, very simple. Of course, this type of simple analysis is good for symmetric cases. And actually if you do the formal calculations, you can actually see that the stability criteria which we get here is exactly the same that $\gamma > \frac{4}{3}$. Now, one can actually do a proper analysis and from that one can also find out the stability criteria.

Now what happens? So, here in this case, we have done very simplified radial oscillations. But what happens if we consider non-radial oscillations, can we have something better, something much more formal, then should we do something much more formal? Well, then the solution is no longer as simple as this.

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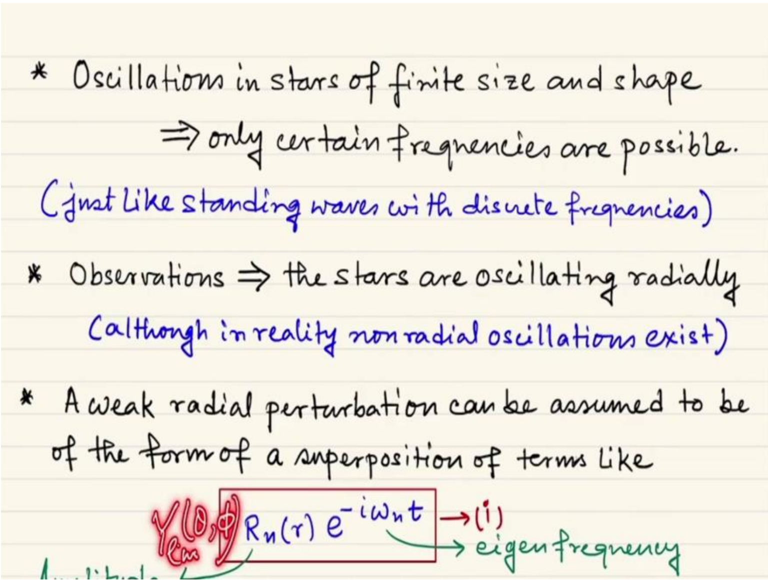

Then here you will have also something called Y_{lm}, there will be another two index which will be the function of θ and ϕ and they are known as spherical harmonics. If in case you do not know, search for it what they are. So, in case of non-radial oscillations, we have to use the spherical harmonics and we have to perform a formal analysis process.

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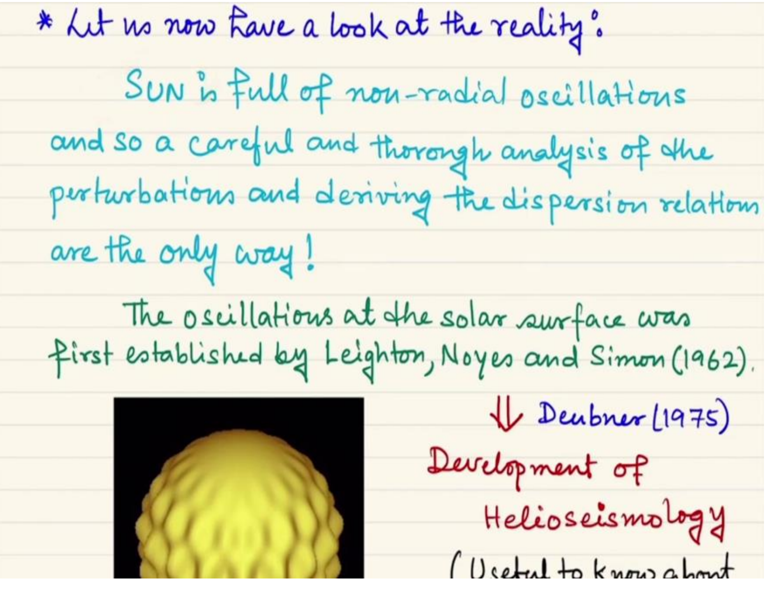

Finally, the question is that if the radial oscillation is so simple, why should we at least at all think about non-radial oscillations? because let us now have a look at the reality. And what is the reality? Our neighbor and the central star of our solar system, Sun. How can we forget that one? And like Cepheid variables Sun is also an oscillating star, but Sun is full of non-radial oscillations. And solar oscillation is a subject of very very interesting and ongoing research.

So, many people are working currently on this subject of solar oscillations. And this is known as the subject of Helioseismology. In general, when we are talking about the subject of the oscillation of stars, we talk about the Asteroseismology and just for the Sun this is known as Helioseismology. So, since Sun is full of non-radial oscillations, a careful and thorough analysis of the perturbation and deriving the dispersion relations is the only way, we cannot avoid that.

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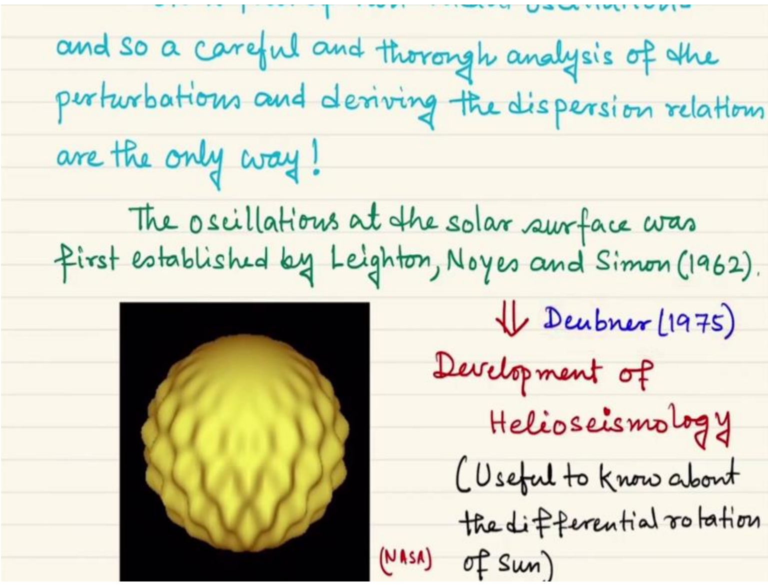

And here in this picture you can actually see that this is one of the schematic picture (see above) and this picture is actually taken from internet, so courtesy to NASA. So, this picture gives you a clear idea about the oscillation of at the Sun's surface. So, the oscillation of the Sun surface was actually first established by 3 people, Leighton, Noyes and Simon in the year 1962, so this is much more recent.

And finally, the people actually thought that maybe some oscillations were there at the surface, maybe they are local oscillations, and they are not global. But how to understand whether this is just a local oscillation or local perturbation, or this is not really something like related to the global mechanics or global dynamics of the star, we have to actually then find the corresponding eigen frequencies.

If the eigen frequencies of those oscillations match with the eigen frequency of the global star system including its geometry and the boundary conditions, then this is an indication that the oscillations which we see at the surface level they are actually the oscillation of the whole body. So, we discussed much earlier two type of forces, body forces and surface forces, so I always love to make analogies in this philosophical part, so here this is also like body oscillation and surface oscillation. So, surface oscillations if this is totally uncoupled from body, then this is not of interest, this does not belong to this current analysis. But if this is a part of the bulk oscillation, then this is something which we are now studying.

We are now studying the body oscillation thing. So, for the first time, if you see that Deubner in the year 1975, he actually showed that this is actually related to the proper oscillation of the Sun and this is the like related to the eigen frequencies of the Suns bulk.

And then the proper development of this Helioseismology was caused or was brought into the picture. And now it is of course a question that why this is useful? Because Helioseismology other than the oscillation mechanism and the luminosity and the temperature variations of the star of the Sun periodically, there is another very interesting and fundamental utility of the Helioseismology, that is, this helps us know about the differential rotation of the Sun. And actually we all know, Sun does not rotate like a solid body. So, every single concentric spherical shells or layers they are rotating with different angular velocity.

And by using proper analysis of Helioseismology, one can actually know what is the variation of this, what is the radial dependence of the angular velocities or in any case the dependence of the angular velocities in various parts inside the body of Sun. So, that was all for this of course if you are further interested you can check all the books and the internet, there are very good books about this.

So, one book which is very intensively discussing the Sun's oscillation or other stars oscillation, but mostly non-radial part that is the book by Unno Osaki. So, you can see the books name is known as Non-radial Oscillations in Stars. So, just have a look, you also have on internet so many things.

So, this is a very very interesting topic where you can actually find your topic of research as well. So, that was all about the waves, oscillations and instability, linear instability part for our course. So, in the next discussion we will start a new topic that is the effect of rotation in astrophysical fluid dynamics.

Thank you very much.