Introduction to Astrophysical Fluids
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Lecture - 34
Waves and instabilities in a two-fluid interface II

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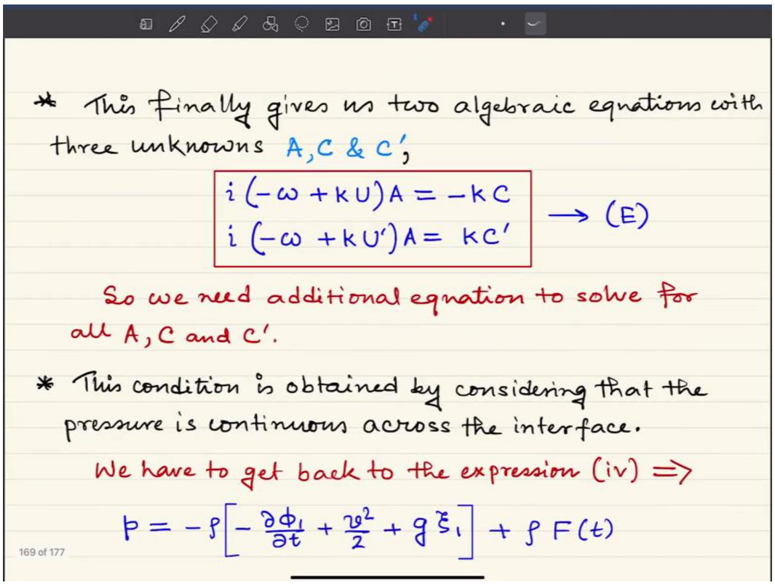

So, we continue our discussion on the Instabilities in the Interface of two fluids, and as previously, we saw that finally using plane wave type of solution for the perturbations we got 2 equations,

$$i(-\omega + kU)A = -kC,$$

$$i(-\omega + kU')A = -kC'$$

These 2 equations are obtained from substituting in the basic equations.

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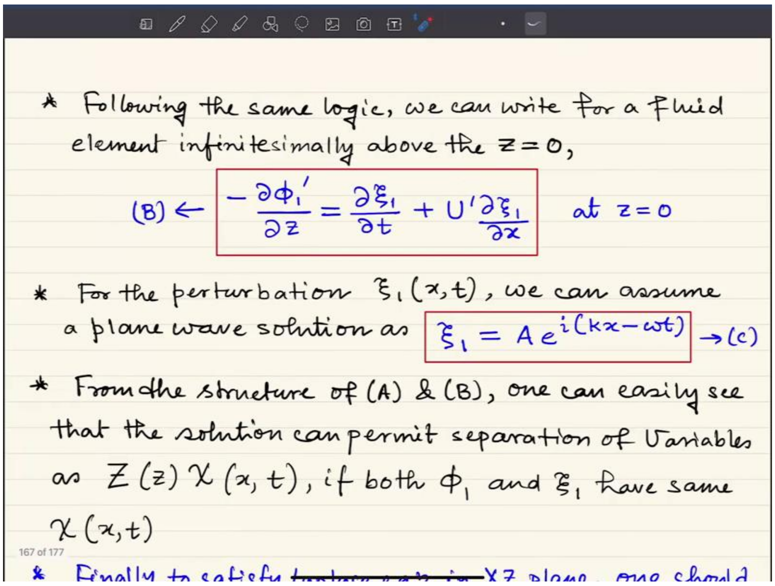

Where we equated the Eulerian velocity component in the *z* direction, and the Lagrangian velocity component of a fluid particle situated at the same position, both for the fluid above and for the fluid below.

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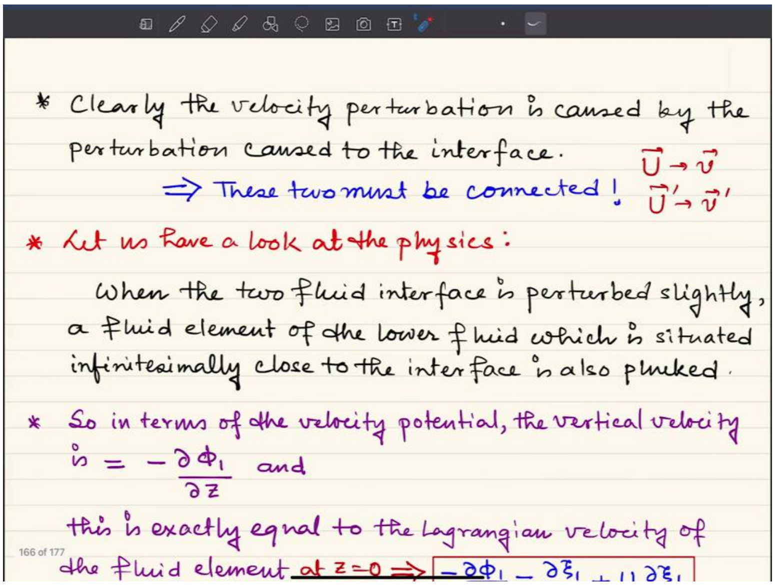

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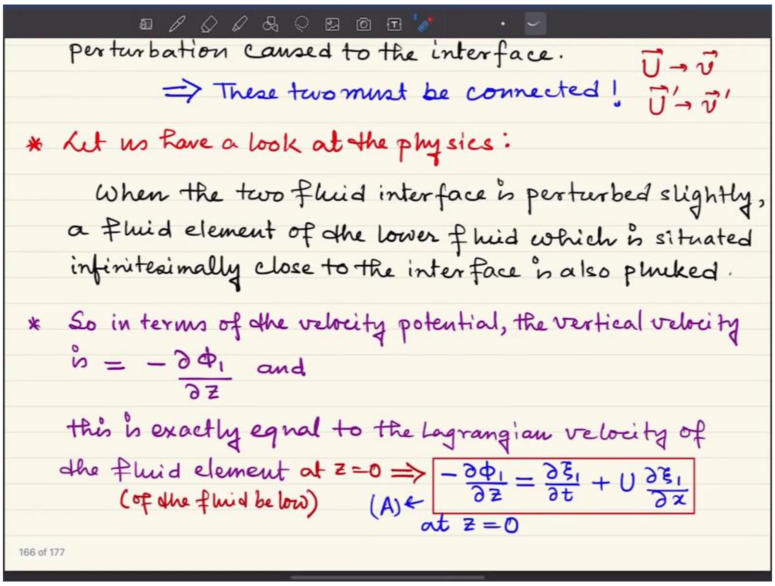

If you remember correctly, then using all these things finally, we got these above equations and we said that we have now 2 equations, but 3 unknown constants. So, of course, we cannot completely eliminate 3 constants from these equations. So, we need any other supplementary or additional equation or relation between this A, C, and $C^{'}$.

So, this condition is obtained by considering that the pressure is continuous across the interface, this is a very crucial consideration. If pressure is not equalized across the interface, then the interface will be ruptured or the distorted that we do not know. There can be smooth perturbations, there can be curves, but very small perturbation. So, it cannot rupture or it cannot damage the interface that the interface should not be affected by the whole perturbation process, that is the whole story behind it.

So, the pressure should be continuous and if the pressure is continuous then we have to look back when we derived some generalized type of Bernoulli theorem.

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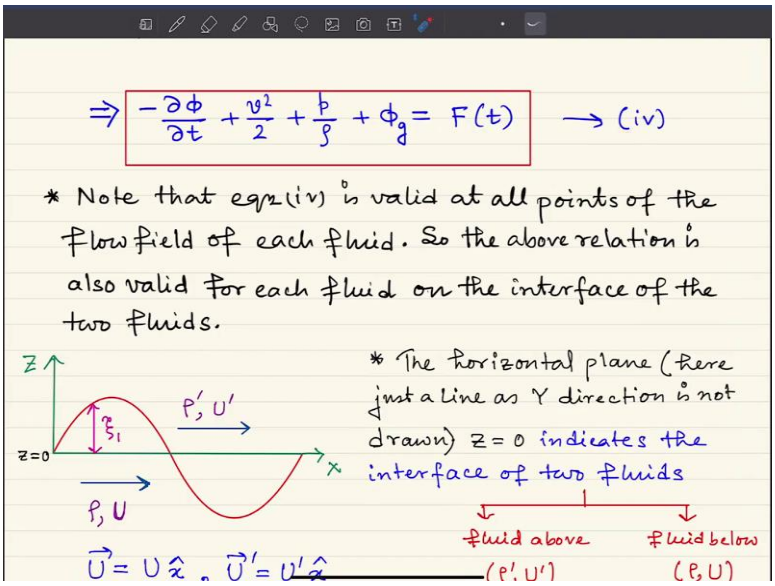

If you remember, this type of thing in the equation

$$-\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} + \varphi_g = F(t) \qquad \text{(iv)}$$

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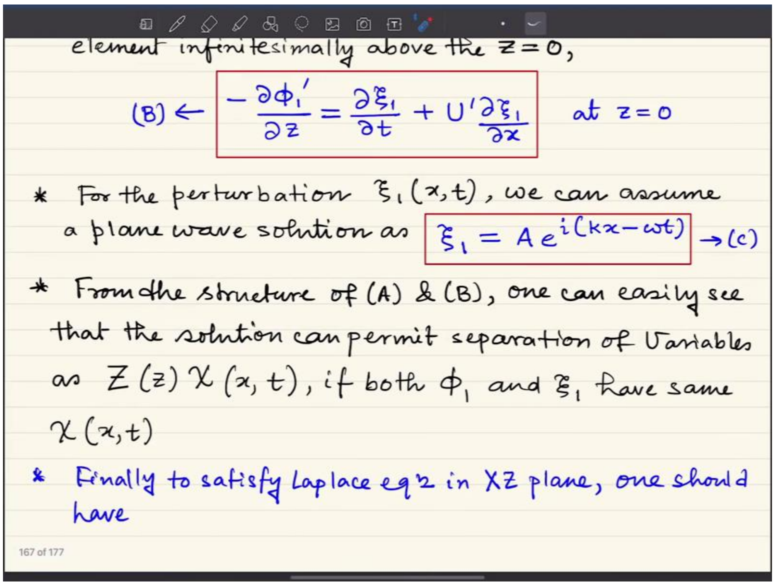

And we said that this is true for any points in the flow field in our case, because the vorticity is identically 0 for this fluid.

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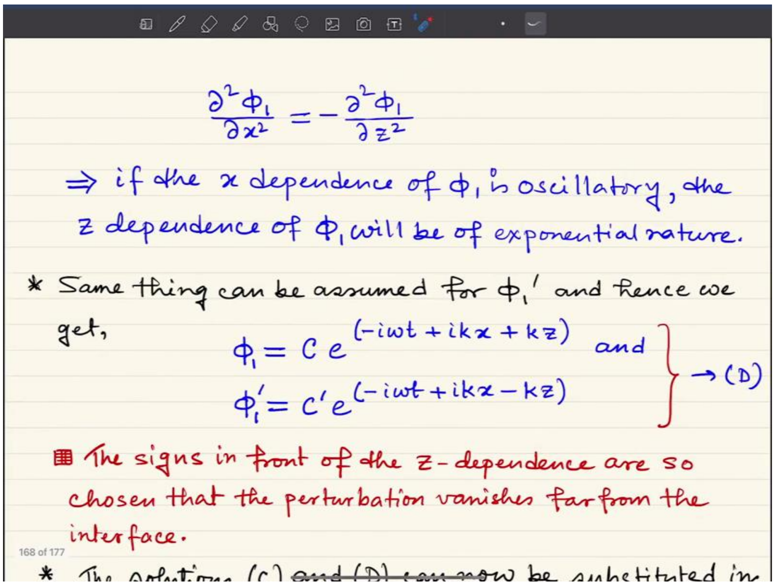

And if this is the case, then we can p is equal to

$$p = -\rho\left[-\frac{\partial \varphi_1}{\partial t} + \frac{v^2}{2} + g\xi_1\right] + \rho F(t)$$

Of course, here do not forget that ρ is just a constant and g is the gravitational acceleration ξ_1 is the displacement.

So, this is actually giving the proxy for the gravitational potential. So, not proxy I mean this is now, represents the term of the gravitational potential.

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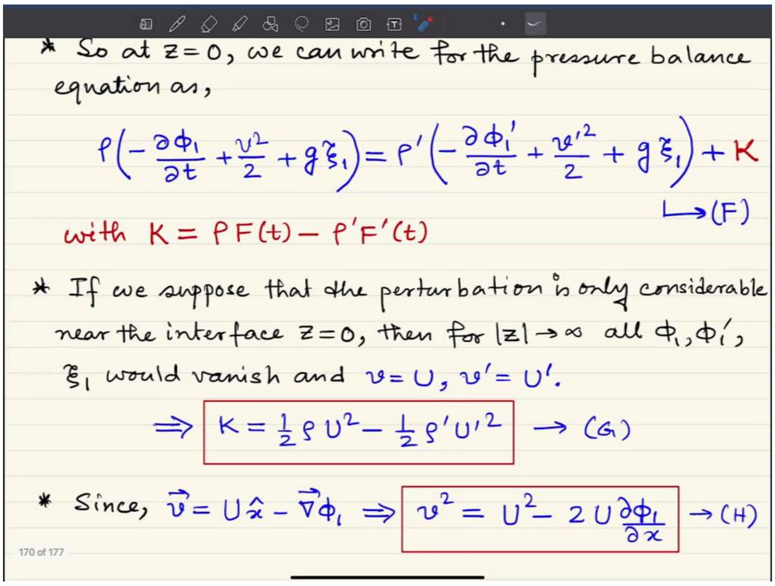

So, you can easily also understand that for any point that is true. Now, we are interested for our current analysis at z is equal to 0, that is the interface and we can write the pressure balance equation.

So, for pressure balance equation this is the pressure $-\rho\left[-\frac{\partial \varphi_1}{\partial t} + \frac{v^2}{2} + g\xi_1\right] + \rho F(t)$, at interface due to the fluid below the interface and this $-\rho'\left[-\frac{\partial \varphi_1{}'}{\partial t} + \frac{v'^2}{2} + g\xi_1\right] + \rho' F'(t)$ is the pressure at interface due to the fluid above the interface.

$$-\rho\left[-\frac{\partial \varphi_1}{\partial t} + \frac{v^2}{2} + g\xi_1\right] = -\rho'\left[-\frac{\partial \varphi_1{}'}{\partial t} + \frac{v'^2}{2} + g\xi_1\right] + K \qquad \text{(F)}$$

But here actually, a part of the pressure with the unprimed coordinates is equal to a part of the pressure with prime coordinates plus K and this K contains $\rho F(t) - \rho' F'(t)$. These algebraic steps, you can do easily, please check that. Now, if we suppose that the perturbation is only considerable near the interface z is equal to 0, then of course, for z tends to infinity whether this is plus infinity or minus infinity.

All the φ_1, $\varphi_1{}'$ and ξ_1 would vanish and then basically, what we get is the original or initial steady state. That means, v is equal to U and v' is equal to U'. So, they are constants and all the part of quantities are 0.

So, now, you can just do a calculation and you can simply see that at initial condition or initial state K should be equal to

$$K = \frac{1}{2}\rho U^2 - \frac{1}{2}\rho' U'^2 \qquad \text{(G).}$$

Now, remember what is the total velocity in general that is the initial velocity $U\hat{x} - \vec{\nabla}\varphi_1$, this is the part of the velocity v_1, if you want to write like this. Then the square of this will simply be equal to $U^2 - 2U\frac{\partial \varphi_1}{\partial x}$.

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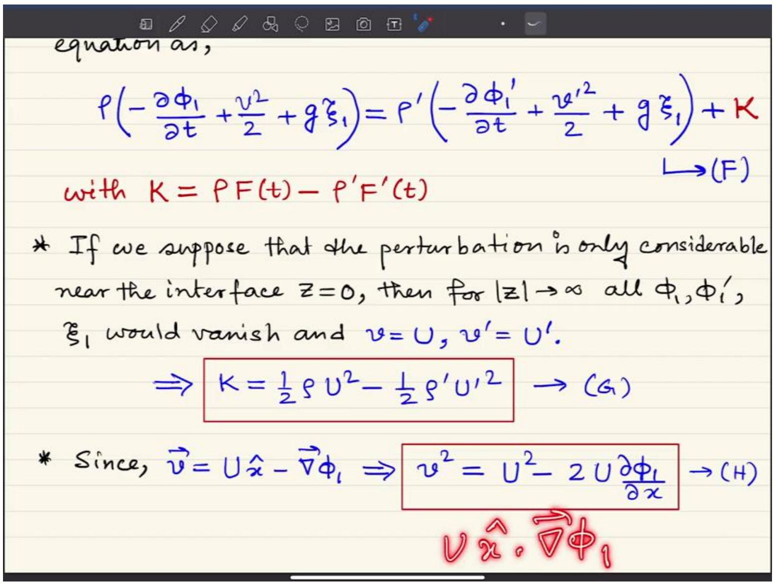

So, it will simply be $U\hat{x}.\vec{\nabla}\varphi_1$.

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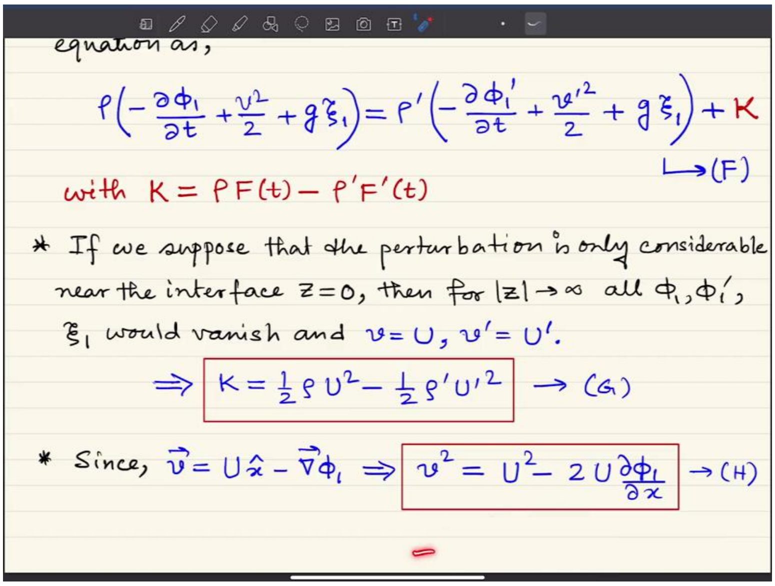

Of course, with a minus sign here.

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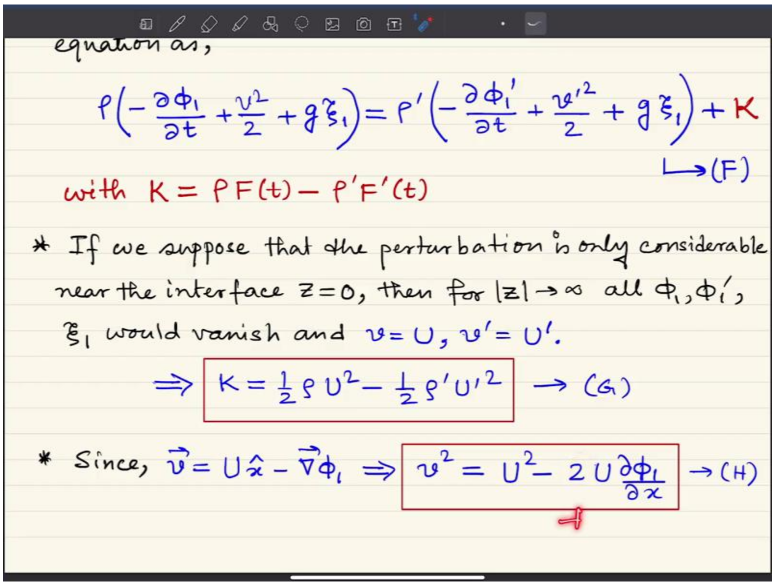

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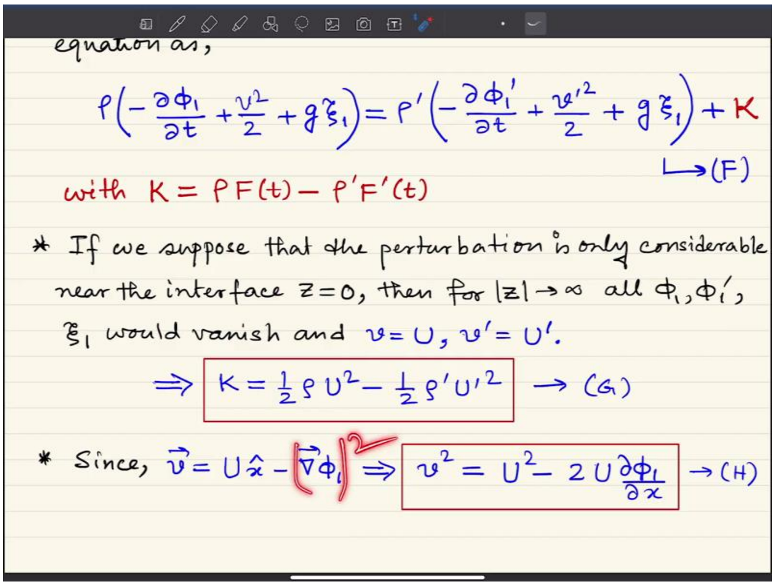

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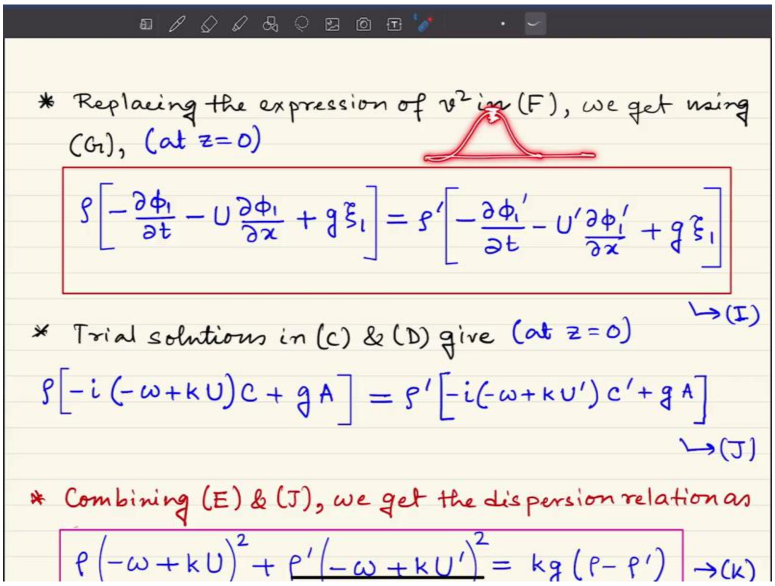

Now, replacing the expression of v^2 in equation (F), what we can get? Because, here for obtaining this one you have to use the expression of K, first write the expression of K which you obtain from (G) and finally, you replace v^2.

If you write, you will see that the term with U^2 in K will be cancelling the terms in U^2 which comes due to v^2 and v'^2. If you do that finally, you have a compact relation like

$$\rho\left[-\frac{\partial\varphi_1}{\partial t} - U\frac{\partial\varphi_1}{\partial x} + g\xi_1\right] = \rho'\left[-\frac{\partial\varphi_1{}'}{\partial t} - U'\frac{\partial\varphi_1{}'}{\partial x} + g\xi_1\right] \qquad \text{(I)}$$

Because, when the interface is plucked, let us say from here to here, both the fluid particles which are infinitesimally above and below the interface both are lifted by a same distance ξ_1 and also both are lifted against equations (G). So, that is why this term is similar, but of course, do not forget that this is multiplied with ρ' and this is multiplied with ρ. This type of subtleties should not be neglected.

Now, finally, if we write the trial solutions in equations (C) and (D), and if we replace all the solutions in our newly obtained condition due to the pressure balance at the interface. We finally, get a relation super interesting relation like

$$\rho[-i(-\omega + kU)C + gA] = \rho'[-i(-\omega + kU')C' + gA] \qquad \text{(J)}$$

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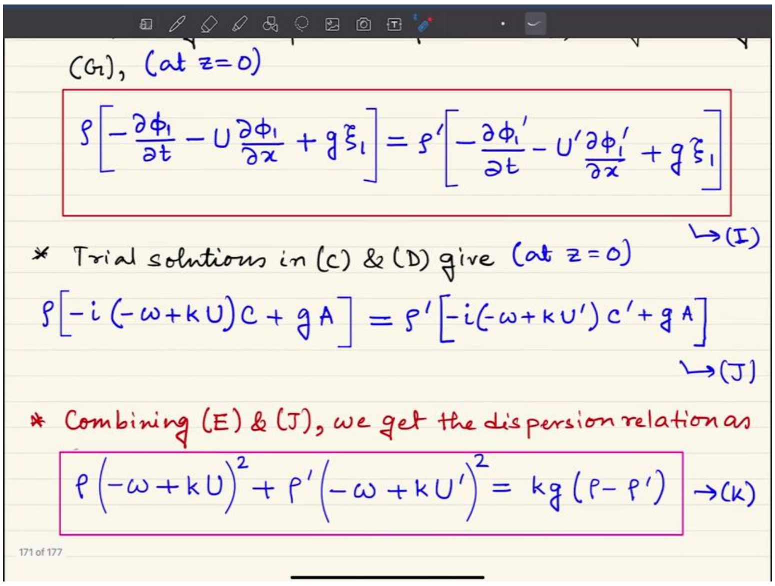

So, finally, our purpose is solved we have obtained another algebraic relation between C, A and C'.

So, now, we have 3 equations this one equation (J) and the other 2 are here in equation (E), and now, we can totally eliminate all these constants of integration, I mean they are not constants of integration, all these constant amplitudes of the perturbation to obtain, the relation between the frequency of each Fourier mode and the corresponding wave numbers. So, if you do that correctly. This is a doable homework a bit 2 or 3 steps are there, but if you do systematically you can easily get there.

So, this dispersion relation finally, comes out to be

$$\rho(-\omega + kU)^2 + \rho'(-\omega + kU')^2 = kg(\rho - \rho') \qquad \text{(K)}$$

So, we just think that if the two fluids are having very matching close density, then this is the order of 0, then what happens to this dispersion relation.

For example, if U and U', they are same then what happens to this dispersion relation ok? So, think whenever in physics you obtain some analytical relation try to understand different type of limits, if your analytical result is correct. There is no problem in calculation or in assumptions, then it should give you correct results or intuitive results in the known limits. So, that I leave for you to check and verify. If you have any question, of course, we can discuss in one of the sessions.

So, this equation number (K) is the parent equation of all our following discussions, this is nothing but the dispersion relation of the linear wave mode.

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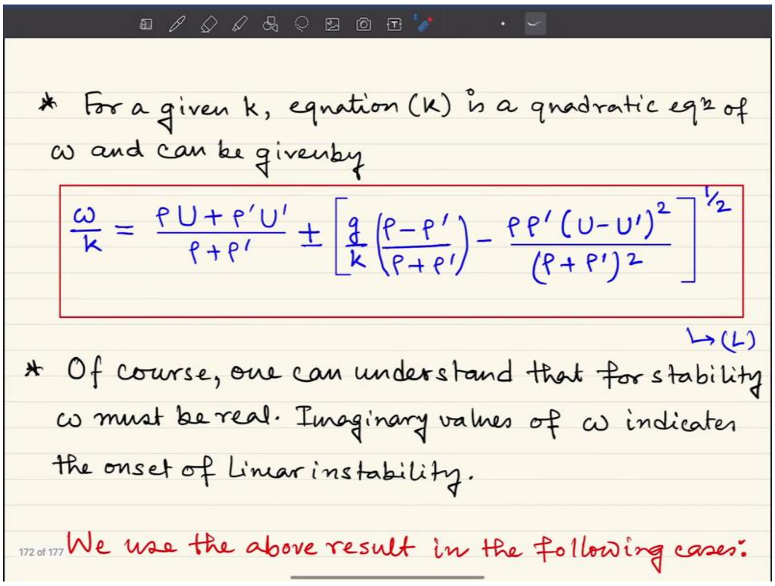

If the dispersion relations frequency can be written in this way as equation (L), of course, you can easily understand that if we consider that the wave vector is given then this is nothing but, quadratic equation for ω, and ω can be written in this way that $\frac{\omega}{k}$ is equal to

$$\frac{\omega}{k} = \frac{\rho U + \rho' U'}{(\rho + \rho')} \pm \left[\frac{g}{k}\left(\frac{\rho - \rho'}{\rho + \rho'}\right) - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}\right]^{1/2} \qquad \text{(L)}$$

Finally, you can easily check that the ω at least that can be real.

So, whether this is positive or negative that actually can be absorbed in dispersion relation with k sometimes. So, that is another thing but at least here for this part there is no chance for ω being complex this is a real part. Now, the question that whether this will give us a stable solution or not, for that, we have to check under which condition ω has a real value, if it is a real value then we have a stability condition.

That means, the system responds in terms of linear wave modes, if ω has a complex part, which should come from this part $[\frac{g}{k}\left(\frac{\rho-\rho'}{\rho+\rho'}\right) - \frac{\rho\rho'(U-U')^2}{(\rho+\rho')^2}]^{1/2}$, of course, you understand that there is a square root and there are 2 terms. So, the interplay of the 2 terms will tell us that whether this is imaginary or this is real.

So, if this has an imaginary part ω, then the system will lead to an instability, of course, linear instability. Now, we just using this equation, we will just discuss very briefly the following interesting cases.

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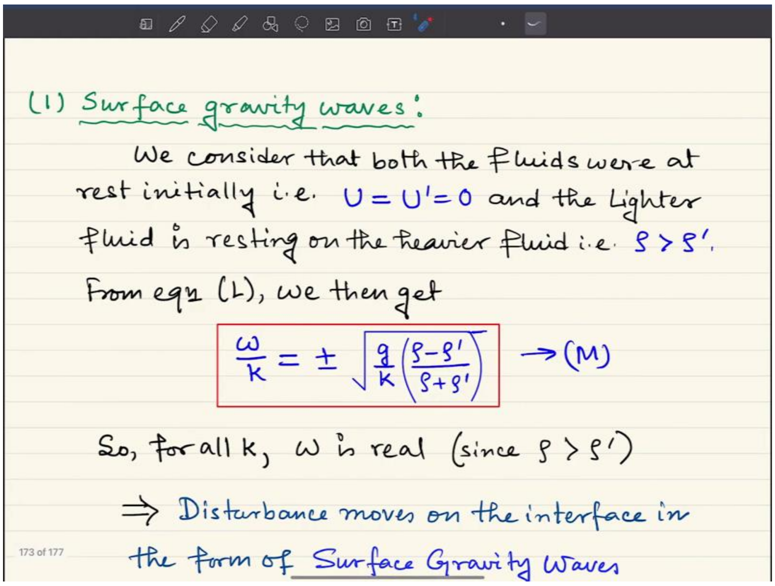

So, one case is known as the Surface gravity waves, so, that means that we have two fluids, which are initially at rest. That means, both U and U' are 0 and the lighter fluid is resting on the heavier fluid. So, ρ is greater than ρ'. So, ρ' is the fluid above. So, this one is lighter this one is heavier.

So, the dispersion relation will be

$$\frac{\omega}{k} = \pm\sqrt{\frac{g}{k}\left(\frac{\rho-\rho'}{\rho+\rho'}\right)} \qquad \text{(M)}$$

Of course, this will be positive because square root of this positive thing will be a real thing.

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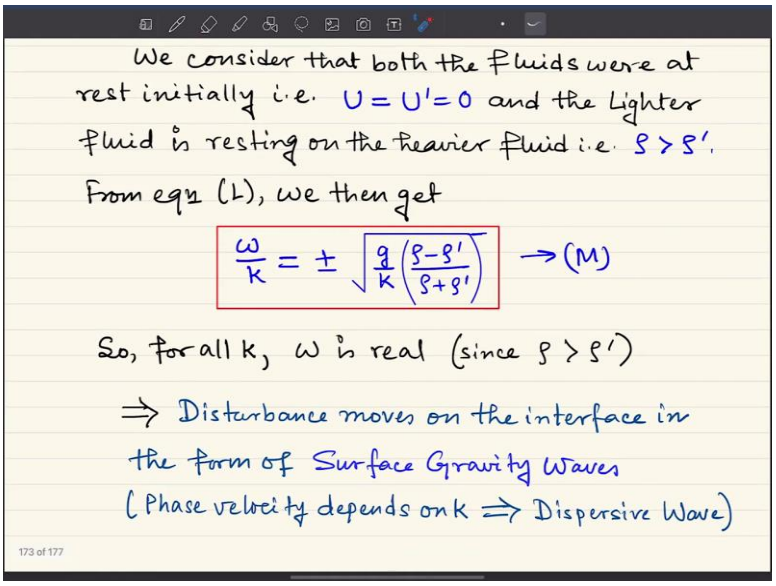

So, in this case, we will see that the system is at rest, two fluid system was initially at rest actually, gives birth to a linear wave mode. So, any weak perturbation caused to this interface to fluid interface would give birth to a linear mode, whose phase velocity is or the dispersion relation is given by equation (M).

So, finally, you can see that the phase velocity of the mode is just this equation (M). So, for all k, we do not care what is the value of k, ω is always real, since, ρ is greater than ρ'. Now, it simply says that the disturbance, in this case, moves on the interface in the form of a linear wave mode, which is called the surface gravity wave, and here, you can see for this mode unlike the sound speed the phase velocity actually depends on k.

So, this is a Dispersive wave. So, the phase velocity is not a constant, it depends on k. So, this is the story of one case.

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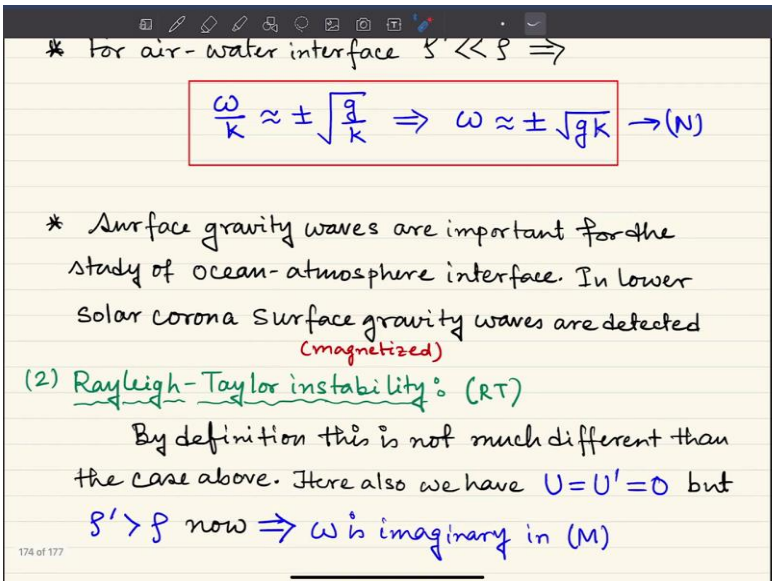

Now, if we just consider a very special case, what I will say that it is always good to check the limits. For example, air water interface, so where ρ' is very less than ρ, the density of air is very less than density of water, air is compressible but it is true that it is fairly incompressible with respect to the other gaseous thing.

So, reasonably we can approximately use our current framework that is true. So, if we do that then finally, what happens this both $\rho - \rho'$ and $\rho + \rho'$ both will be approximately equal to ρ and finally, this will get cancelled and we will have simply

$$\frac{\omega}{k} \approx \pm\sqrt{\frac{g}{k}} => \omega \approx \pm\sqrt{gk} \qquad \text{(N)}$$

Now, surface gravity waves are mainly important for the study of ocean atmosphere interface, that is in oceanography. For the astrophysical framework, we can have also interesting thing with the surface gravity wave. Because, in lower solar corona, it is observed actually surface gravity waves are there and it is true that then actually,

We cannot model the whole thing using a neutral fluid, then plasma comes into play and the surface gravity waves will be a generalized form of this surface gravity waves which will be the magnetized or magneto acoustic surface gravity waves which I will talk a bit later, when I will talk about like magneto hydrodynamic waves.

After that another very interesting thing, so this was the story of a stability condition where the system responds in terms of a linear wave mode. Now, we see the inverse situation, but always with the same initial condition, by definition, this is another condition, which is known as Rayleigh Tylor instability or RT instability. So, the initial condition is exactly the same, that means, U is equal to U' is equal to 0.

But one thing is changed there now, whenever you have two fluids then the relative configurations or the relative situation of one fluid with respect to the other is very important. So, here this is true that this is exactly equal to the previous case, but now, ρ' is greater than ρ. So, in that aspect the initial condition is not exactly the same.

So, $\rho - \rho'$ must be negative and so, ω should be imaginary in the equation (M). So, this part will add finally to the instability, and this is known as the Rayleigh Taylor instability.

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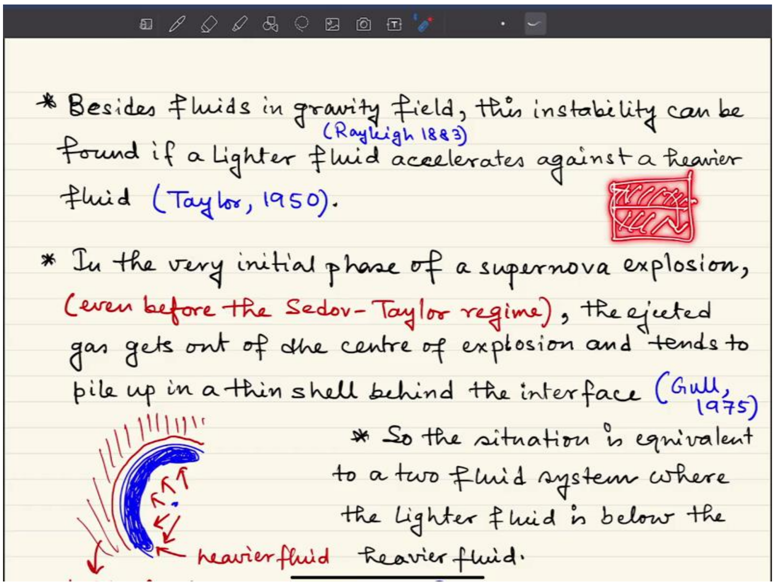

Now, just instead of seeing this as a very exotic thing, just think very clearly that let us say I make somehow by force or something clamping or some set up two fluids at rest, of course, in the Earth's gravity field.

Now, they are rest vertically, so, the lighter fluid is below and the heavier fluid is on the upper side. What will happen if we just lift the constraint? So, this heavier fluid will come down and this one will try to go in right, of course, not in the convective manner.

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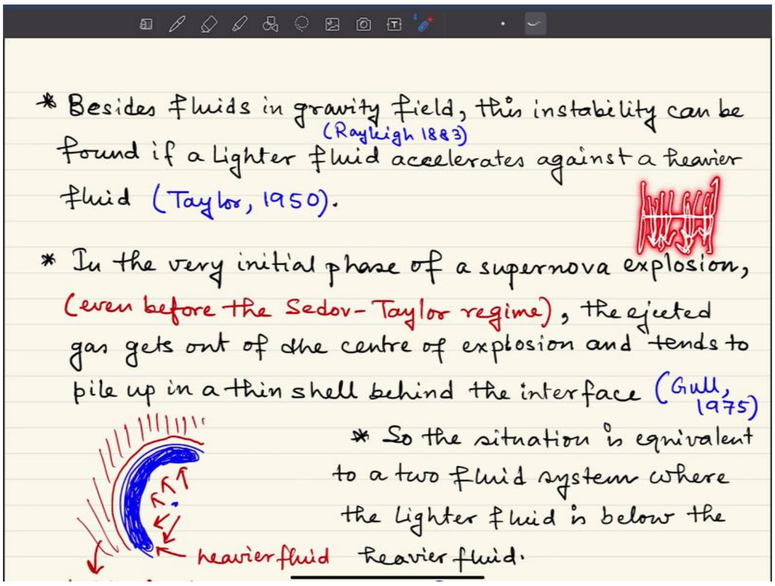

But, in general, this fluid is heavier fluid actually, comes down in the structure of fingertip type of thing. So, it penetrates this fluid gradually, and the final destination is finally trying to goes down and below the lighter fluid.

Of course, this is very interesting case for the fluids in gravity field which was observed for gravity field by Rayleigh in the year 1883. But this instability can also be found if a lighter fluid accelerates against a heavier fluid. So, the gravity fluid can make gravity field can be any other accelerating field right, of course, the accelerating field is acting as a conservative body force field that is something very important.

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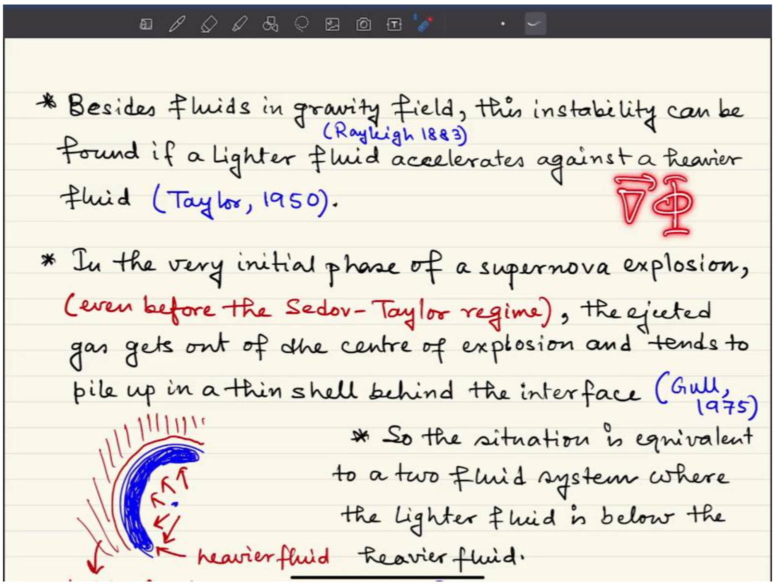

Otherwise, that cannot be written as a gradient of some potential function and the generalized form of the Bernoulli equation cannot be proved easily. That was done by Taylor 67 year later in 1950.

So, this is the same Taylor, who worked on the Sedov-Taylor instability during the supernova explosion I mean, sorry, using the blast waves during the second world war for the atomic bomb explosion. Now, this type of Rayleigh Taylor instability is much importance for the astrophysicist, because in many instances one can see RT stability.

For example, I here just want to mention one very useful, which we have already discussed that is the very initial phase of supernova explosion. When the expression just takes place in the explosion does not enter into the self-similar phase that is the phase of Sedov-Taylor, before that the ejected gas gets out of the center of the explosion and tends to pile up in a thin shell behind the interface. That was observed by Gull in the year 1975.

So, the situation is equivalent to a two-fluid system where the lighter fluid is below the heavier fluid and the acceleration is outward. So, the acceleration is basically due to the energy of this fluid. So, the fluid is now piling up here, this is the interface and there is very dilute intense ambient medium.

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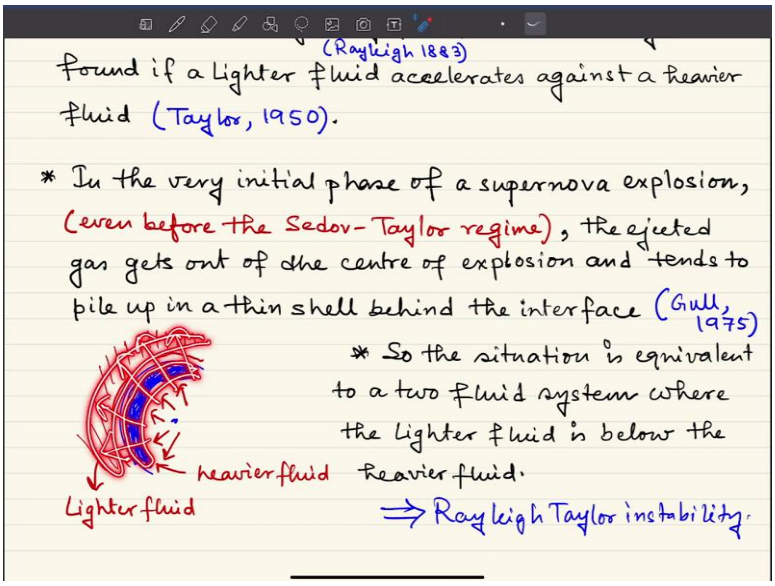

If you just see that this acceleration is acting in this direction, then it is exactly equivalent to saying that the heavier fluid is above and the lighter fluid is below. So, we can expect, of course, a Rayleigh Taylor type of instability and actually they can see that.

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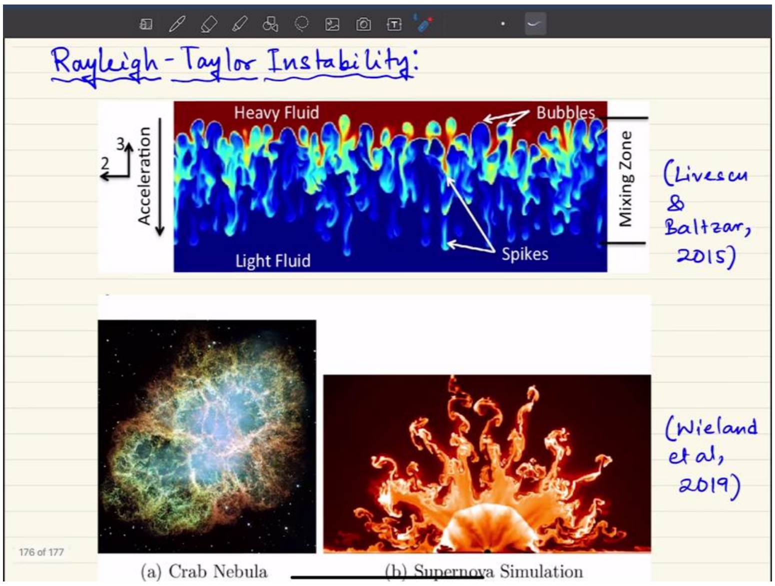

So, here, in this slide, I will try to show you different observational instances of Rayleigh Taylor instability. So, one is taken from a paper by Livescu and Baltzar in 2015. So, for a numerical simulation, you can see that the heavier fluid basically becomes penetrating in the lighter fluid, now, you see this brownish fingertip. So, whenever you see this type of fingertips are coming, this is a clear signature of Rayleigh Taylor instability.

Of course, in this case the two fluids should be at rest initially or very low velocity, otherwise this is not possible. This is a picture by crab nebula where you can also see finger type of structures to the ambient dilute medium, which can also indicate towards a possible instability.

Now, finally, for the supernova simulation, this is a very early phase supernova simulation, where this is not self-similar and spherically symmetric till now, and you can see that the blast wave material actually tries to penetrate to the very dilute ambient medium just by the virtue of RT instability. This was taken from a paper, recent paper by Wieland et.al. 2019.

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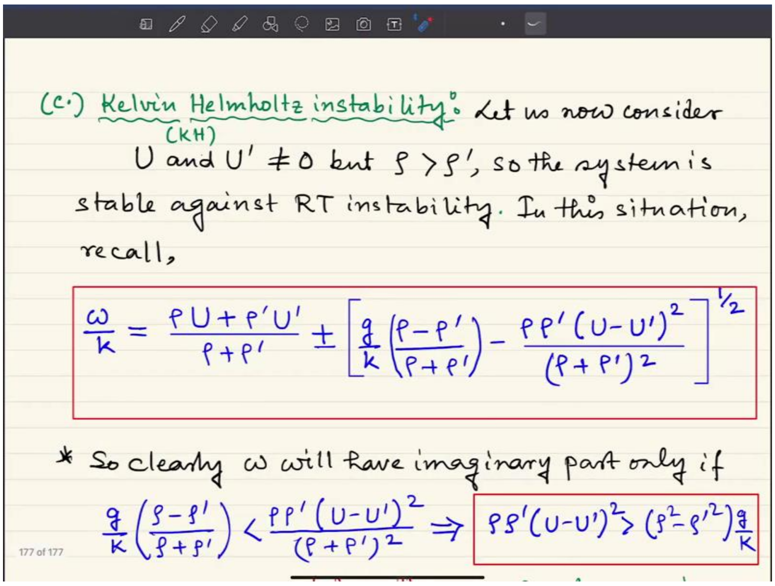

So, the last one is called Kelvin Helmholtz instability. So, what is that? For Kelvin Helmholtz instability, U and U' are not 0, but if they are not 0 and they are equal then again you can see this $U - U'$ term will be 0. So, we will always have some stability, there is nothing interesting.

So, we will not only make them non zero, but make them different and ρ' is less than ρ. So, the system should be already stable in terms of Rayleigh Taylor instability. So, it is the heavier fluid, at the bottom and the lighter fluid is on the top.

But now, both the fluids are flowing with non-zero velocity, that is the initial steady condition, but with uniform velocity. Then what happens? Then of course, this will have an imaginary part only, when this part $[\frac{g}{k}\left(\frac{\rho-\rho'}{\rho+\rho'}\right) - \frac{\rho\rho'(U-U')^2}{(\rho+\rho')^2}]^{1/2}$ is negative and that is simply true when this part $\frac{g}{k}\left(\frac{\rho-\rho'}{\rho+\rho'}\right)$ is less than this part $\frac{\rho\rho'(U-U')^2}{(\rho+\rho')^2}$. So, just by algebraic manipulation, you can see that $\rho + \rho'$ will cancel out.

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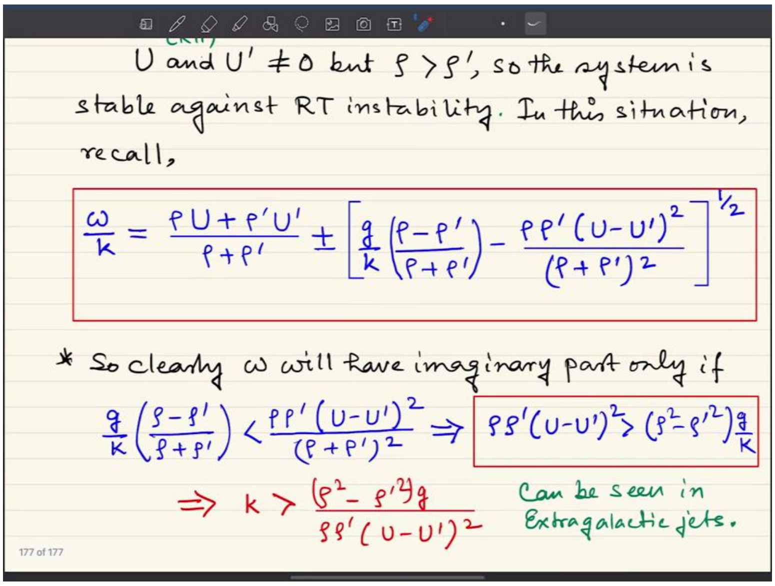

Finally, we have

$$\rho\rho'(U-U')^2 > (\rho^2 - {\rho'}^2)\frac{g}{k},$$

and then the k should be there, till now we did not have any k dependence in the criterion of the instability. Now, we will have a k dependence.

$$k > \frac{(\rho^2 - {\rho'}^2)g}{\rho\rho'(U-U')^2}$$

So, if k is greater than this value, I will have a Kelvin Helmholtz type of instability and Kelvin Helmholtz type of instability is also important for solar physics. Maybe, I will come to that later and also in the case, where you can see that some flowing river and on just on the river surface, some wind is also blowing, then the surface is actually undulating, and that is the clear signature of Kelvin Helmholtz type of instability.

In case of astrophysics, Kelvin Helmholtz type of instability can be seen in extra galactic jets, where some fluids are spitted by a galaxy from very localized region in a galaxy and tries to mix or tries to go to the very dilute intergalactic medium. So, here you can see that k is greater than some critical value means the length scale of the perturbation should be less than to some minimum value.

For the instability, l should be less than some maximum value. So, of course, one very important thing in whole treatment, we have taken into account the incompressibility for water and air surface we have discussed, but we have neglected the effect of surface tension. Actually, the introduction of surface tension majorly modifies the results and you can see that the k will have also the lower bound, then actually $U - U'$ should have another type of bound depending on the nature of the surface tension.

So, this type of exercise you can actually see on internet, how surface tension can actually modify Kelvin Helmholtz instability, how magnetic field can modify Kelvin Helmholtz instability, those things are subject of very interesting and ongoing research.

So, that is something I can inspire you to think and just try to do some more general analytical approach. So, in this lecture, we have seen analytically, how to tackle the instabilities or the very weak perturbation and the response of the two-fluid medium. Instead, to this perturbation whether in terms of linear modes or in terms of instability and we actually also discussed that why and in which circumstances Rayleigh Taylor instability, Kelvin Helmholtz instability this type of two-fluid interfacing stabilities are important.

In the next discussion that I will give next week, I will talk about a very interesting topic which is known as the Physics of Stellar Oscillations, and actually, you can see it is observed that the luminosity of the stars including the sun, they are oscillating. So, that the luminosity changes periodically, and in this way actually it is called the oscillation of the sun or the oscillation of the corresponding star.

In fact, one can easily understand by proper analysis that there is a proper mechanical oscillation type of thing of the Taylor body corresponding to the periodic change in the luminosity. So, that I will discuss in the next week.

Thank you very much.