Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

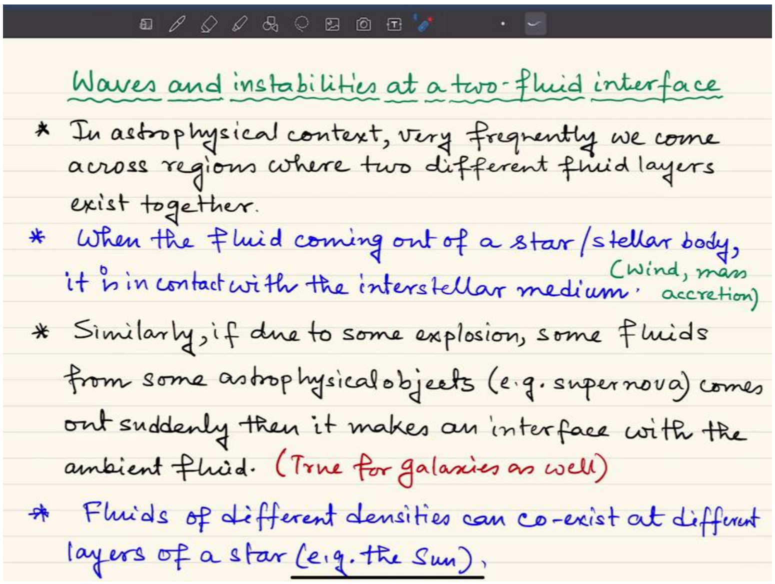
Lecture - 33
Waves and instabilities in a two-fluid Interface I

Hello, and welcome to another session of Introduction to Astrophysical Fluids. Previously, we discussed about convective instability, relevant convection problem, we also discussed about Jeans instability. In this discussion we will focus on the instabilities in the interface of two fluids. These types of instabilities have a very important role in astrophysics as I will say during the explanation of different type of instabilities and the corresponding analytical treatment.

So, the question is that if two fluids are existing together having some common interface then some very small or weak perturbation is made to that interface, now, this is a system of two fluids, how do the two fluids respond to that perturbation, depending on their initial configuration depending on their various kinetic and dynamic properties and also depending on the nature of the perturbation of course, we can understand that there can be the possibility of some linear modes, linear wave mode if the initial condition is stable.

If not then it will lead to some instability, this type of instability sometime it is called configurational instability or two fluid interface instability.

(Refer Slide Time: 01:51)

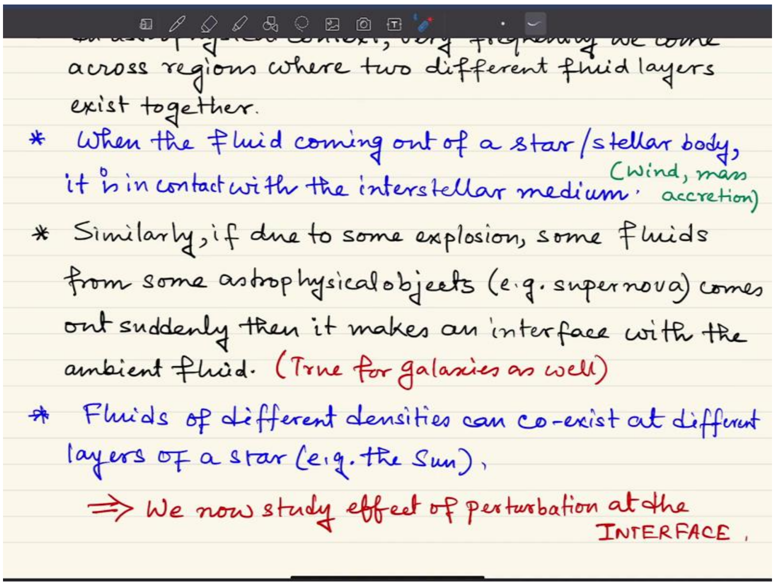


So, in astrophysical context, very frequently we come across regions where two different fluid layers they coexist and that is why they define what we called a two-fluid interface. Now when the fluid coming out of a star or a stellar body, for example, as a stellar wind or due to the mass accretion, during the mass accretion from a less star to a compact star, for example.

Then that fluid is in contact for a certain time interval that fluid is in contact with the interstellar medium right, and that makes actually another possibility of a two-fluid interface. Similarly, another instance of astrophysical to fluid interface is that, if let us say due to some explosion now we know due to our previous discussion of supernova that supernova explosion is one of the most important explosions in the framework of astrophysics.

So, due to that explosion some fluids that always come out from that astrophysical objects for example, supernova and when it comes out suddenly then it makes also an interface with the ambient fluid and now you know that actually it makes a shock or discontinuous type of thing most frequently, but if let us say that the explosion is in a such a phase that the shock is not yet formed, and then actually you can still talk about an interface of the blast wave or the explosion wave fluid to the ambient fluid, and that is also true for galaxies that means, if some fluid is spitted from some galaxy due to some instability or some explosion, some mass eruption takes place then this fluid will also make some interface with the intergalactic medium.

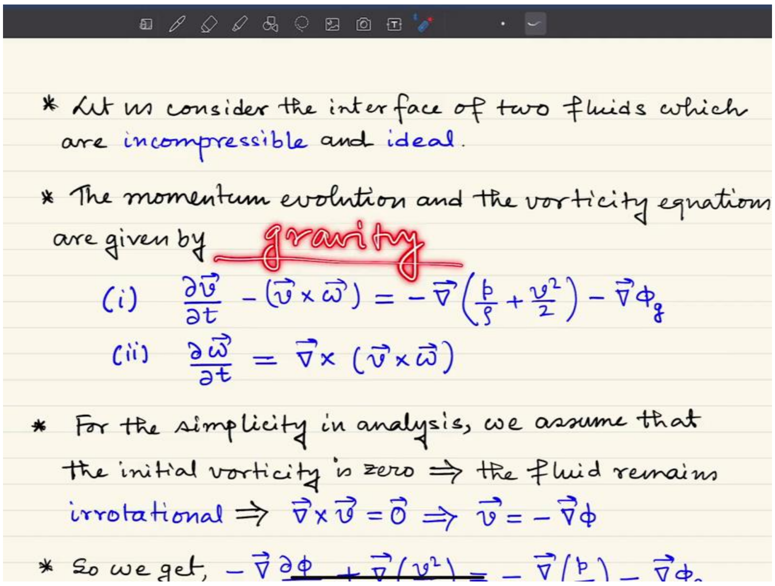
(Refer Slide Time: 04:22)



Now, fluid of different densities can coexist at different layers of a star as well. So, till now we were discussing the story of some fluid which are coming out of the star or some galaxy, now even inside of a fluid in its different layers you can actually see that fluids of various densities can coexist, and that defines another type of two fluid interface inside a star and one very easy and prominent example is the sun.

Now in this discussion, we will study mainly the effect of perturbation at those interfaces. Of course, you can understand that analytically handling all these issues the fluids energy, the discontinuity, the compressibility, the effect of viscosity, the effect of rotation which basically is not possible in the scope of this course and this is actually the subject of active research.

(Refer Slide Time: 05:44)



For this course, we will do something very simple and analytically, very primary however, the basic methodology will remain the same even if you try to sophisticate your treatment. Now, let us consider, the interface of two fluids which are already incompressible and ideal.

So, if now, for example, we are talking about the two-fluid interface and the corresponding instabilities for example, for a highly compressible intergalactic medium and some galactic fluid, then this is not quite correct analysis which we will do here, but once again that is just for the analytical as a first step in analytical approach you can start with taking very simple examples incompressible fluid and ideal fluid.

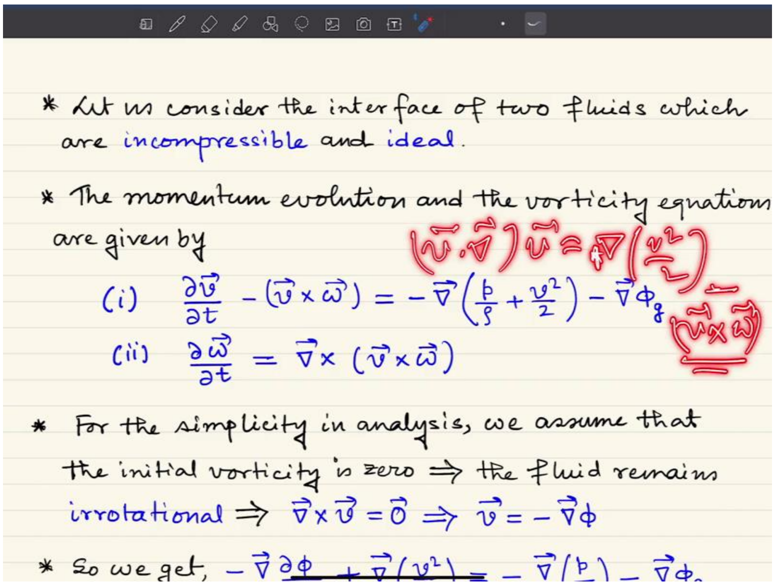
Now, just remember the momentum evolution and the vorticity evolution equation for this type of fluids which are in gravity field that is very important here. We are considering that the fluid is in a body force field which is the gravity. Then what happens? The two equations will be given by

$$\frac{\partial \vec{v}}{\partial t} - (\vec{v} \times \vec{\omega}) = -\vec{\nabla}\left(\frac{p}{\rho} + \frac{v^2}{2}\right) - \vec{\nabla}\varphi_g \qquad \text{(i)}$$

So, φ_g is nothing but the gravitational potential.

So, $-\vec{\nabla}\varphi_g$ is nothing but g if you remember. Now, ρ is a constant because this fluid is incompressible, so, it can actually come inside the gradient.

(Refer Slide Time: 07:50)

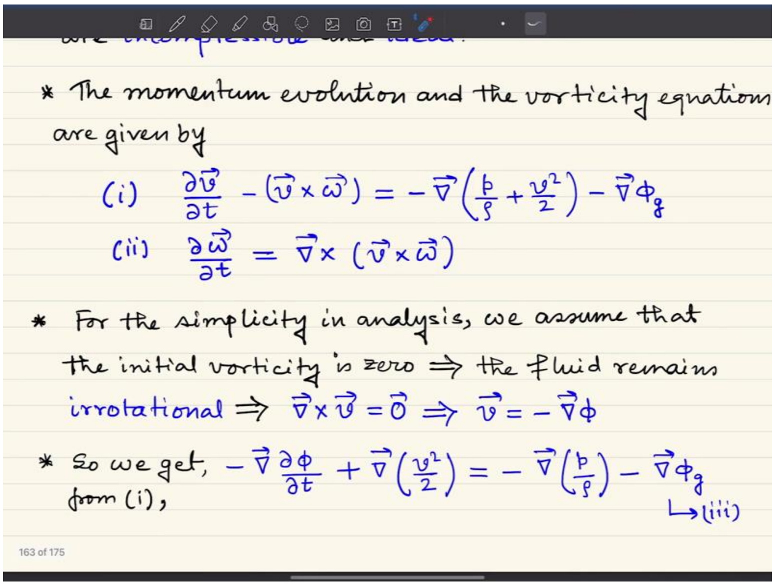


where $\vec{v} \times \vec{\omega}$ is nothing but the minus of the lamp vector. Again, if you take the curl of this equation (i), you know that these two terms $-\vec{\nabla}\left(\frac{p}{\rho} + \frac{v^2}{2}\right) - \vec{\nabla}\varphi_g$ will go away and you will finally, have

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega}) \qquad \text{(ii)}$$

ideal fluid and incompressible fluid no viscosity is there.

(Refer Slide Time: 08:36)



Now, for simplicity we know that as a property of such fluid that if you start with zero vorticity, for example, at t is equal to 0. Your vorticity is 0 then $\frac{\partial \vec{\omega}}{\partial t}$ will be 0 and your vorticity will always remain 0. So, we also assume for the simplicity and analysis that in our case, the initial vorticity is zero and that is actually a bit very much simplistic with respect to the realistic cases.

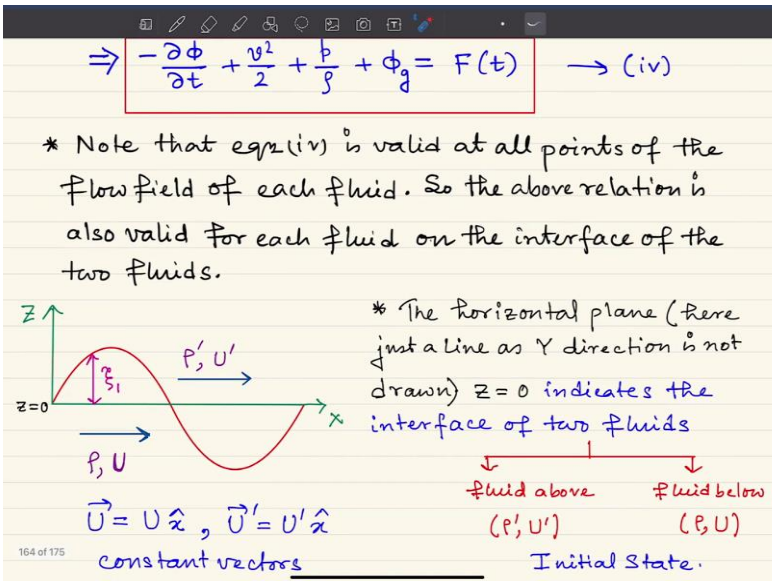
For the realistic cases, initially, the fluids interfaces always have some vorticity and the system is very complex, in general. Our system is very simple and for this case what happens that our fluid velocity is not only incompressible or divergence less, but it will now have a zero curl as well.

So, this velocity field is divergence less and curl less. So, v can be written as a gradient of a scalar velocity potential that is exactly what is written over here. So, v is equal to $-\vec{\nabla}\varphi$. So, finally, we get from equation (i) that

$$-\vec{\nabla}\frac{\partial \varphi}{\partial t} + \vec{\nabla}\frac{v^2}{2} = -\vec{\nabla}\left(\frac{p}{\rho}\right) - \vec{\nabla}\varphi_g \qquad \text{(iii)}$$

So, finally, our equation can be written in terms of φ, although v^2 I am not changing here just I keep it here.

(Refer Slide Time: 10:32)



So, the whole thing under the gradient will be a function of time only because its gradient is 0 and that is exactly what I have written over here

$$-\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} + \varphi_g = F(t) \qquad \text{(iv)}$$

Now note that, this looks like Bernoulli's theorem, but for Bernoulli's theorem we use a steady flow in general, but this is a generalized version of Bernoulli's theorem which is true for non-steady flow as well and here we do not need to make it true along the stream lines this is true at any point because now the omega is 0 identically.

So, you do not have to really take the dot products with the dl along the stream line, so, that finally, you can make the whole thing 0. If you remember the trick of Bernoulli equation, if you cannot please have a look again. Here equation (iv) is valid at all points of the flow field whether we are tracing out a stream line or not. So, the above relation is also valid for each fluid on the interface of the two fluids and that is very important and this one we will use later.

Now, coming back to our initial question that what happens if we create some disturbance in the two-fluid interface? How would the system of two fluid respond now this is not only depending on one single fluid, but there are two fluids.

So, here I am just taking again for simplicity a planar picture. So, along the X direction you have the intersect interface and so this green line is the interface of the two fluid. So, one fluid is above Z is equal to 0, another fluid is below Z is equal to 0 and the vertical upward direction is of course, Z as I already said. Now when this is not perturbed then this interface is like that.

And at that point, we can say that both the fluids are having constant densities because of incompressibility ρ' and ρ and also have some non zero uniform velocity along X direction that is quite ok, actually the fluid is flowing on $X-Y$ plane, but along Y, there is no considerable change or variation and that is why simply this does not create any considerable change in the whole treatment. So, Y dimension we just keep aside for the time being.

So, $\vec{U}$ is equal to $U\hat{x}$ and $\vec{U}'$ is equal to $U'\hat{x}$ and these two are constant vectors because their magnitude is constant, their directions are constants. So, these two are constant vectors and actually as you can see that they will also satisfy the initial condition of a steady state.

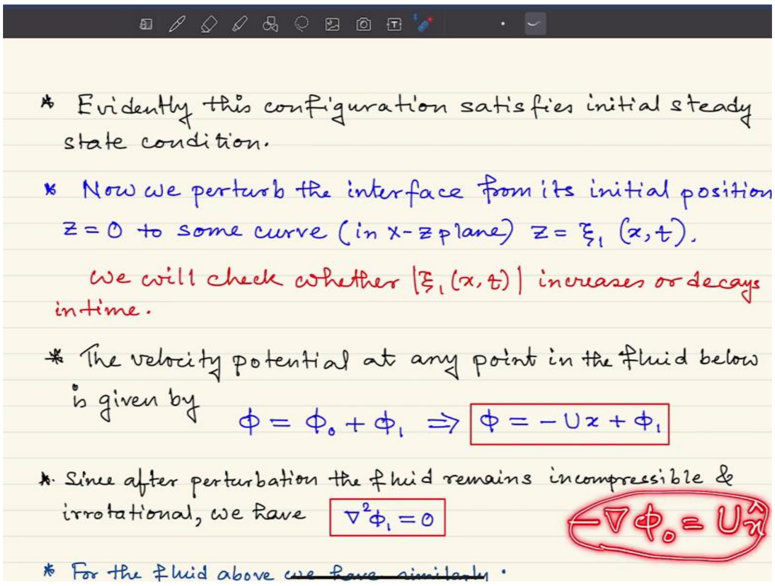
As I said at the very beginning of the discussion, in some previous lectures when I started discussing about the waves and instability in the fluid, I said that the initial condition with respect to the system should be perturbed must be a steady condition, steady state and sometimes it can be steady and static state. Here we are talking about steady and non-static, but moving with the uniform velocity and that is also a steady state.

Now, this line along at Z is equal to 0 defines the interface of the two fluids, and so as I have already said that the fluid above is characterized by ρ' and U' that mean the primed coordinates quantities, and the fluid below this is characterized by the unprimed quantity that is ρ and U, this is the initial state. Now, we perturb slightly, although this looks a bit big, this is actually just for simplicity and this is for exaggeration so that you can understand this is perturbed.

Otherwise, this perturbation is very small that means, this amplitude is very small. So, in the interface the fluid at interface is perturbed a bit.

So, for example, the fluid from here is plugged to this point and the fluid from this is plugged in this direction up to this point then what happens, would the perturbation grow in time or the perturbation decay in time or the perturbation oscillate in time that is our question?

(Refer Slide Time: 16:14)



So, as I said that the configuration satisfies the initial steady state condition. So, we are all set for analyzing the perturbation around it. Now, we perturb the interface from its initial position Z is equal to 0 to some curve which is given by Z is equal to $\xi_1(x, t)$. So, this curve is only depending on the x and t. So, along X, it can have different values according to the different values of X and it can also have an explicit time dependence.

That is also possible that depends on the nature of perturbation, if I just perturb the system by some time depending oscillating element even for the X is constant. For example, here just the perturbation is made by some machine which is periodically oscillating or changing.

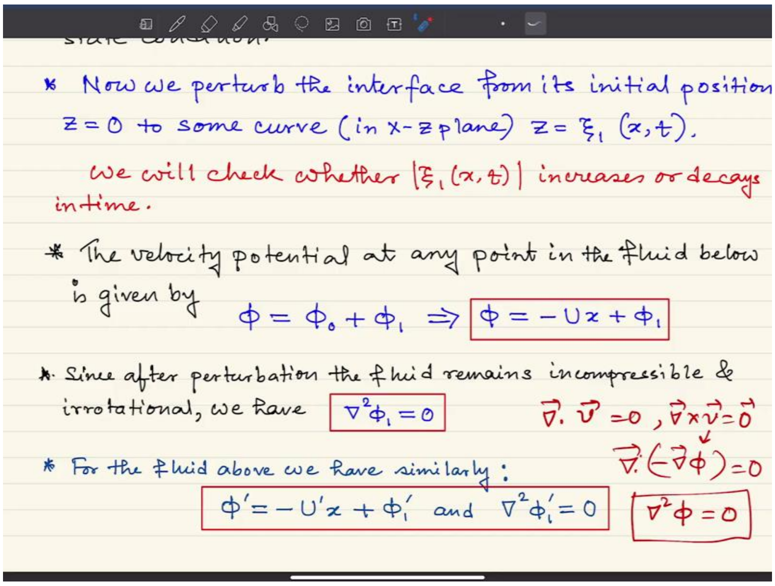
The perturbation value will change actually explicitly with time irrespective of the X value, that type of dependence is also included here. Now, we will check that whether this modulus of this perturbation increases or decays in time. Now, the velocity potential at any point in the fluid below with unprimed quantities is given by $\varphi = \varphi_0 + \varphi_1$.

So, φ_0 is the velocity potential at the initial steady state, φ_1 is the perturbed part of this velocity potential and this φ_0 is nothing but $-Ux$ because, at the initial position you only have gradient of φ_0 which is equal to $-U\hat{x}$.

So, if you correctly integrate that you will have only one component $\frac{d\varphi_0}{dx}$ and that will be simply equal to $-U$. So, if you integrate properly, you will see that this will give you $-Ux$

and this is the initial part, this φ_1 is the perturbed part. So, the total part should be written as this $-Ux + \varphi_1$ as simple as that.

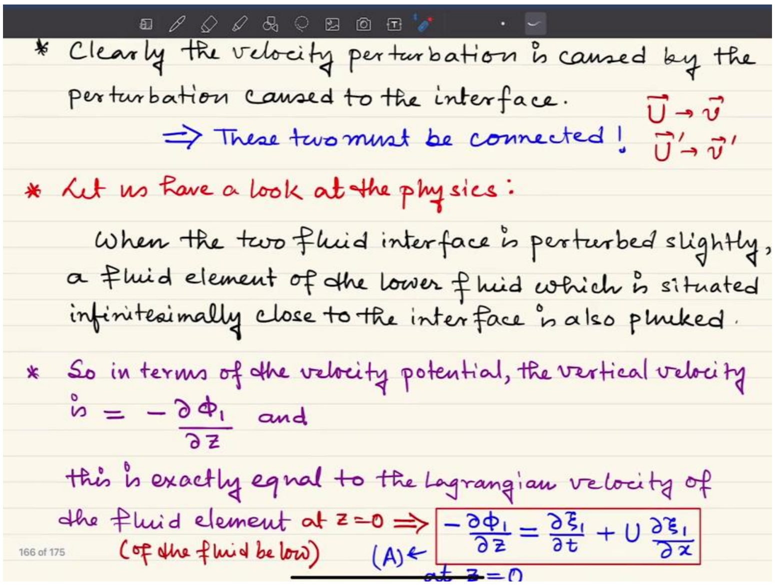
(Refer Slide Time: 18:50)



Now after perturbation the fluid remains incompressible that is, of course, true otherwise there will be a big problem in our assumption and also irrotational. So, if you know that for a system like the v such that divergence of v is 0 and curl of v is 0 then of course, v is equal to $-\vec{\nabla}\varphi$ and since divergence of this is 0 you can actually write this is 0.

So, the φ actually satisfies the Laplace equation. So, the perturbed part of the velocity potential also satisfies a Laplace's equation. Same type of argument can be done for the fluid above and for that we have similarly φ' is equal to $U'x + \varphi_1{}'$ and $\nabla^2\varphi_1{}'$ is also equal to 0 exactly what we have for the unprimed fluid, but this is the fluid above the interface.

(Refer Slide Time: 20:25)



Now clearly, we have some elements of analysis. Now, we have to consider some physical aspects. So, clearly the velocity perturbation is caused by the perturbation caused to the interface. Of course, the velocity perturbation of the fluid very close to the interface occurs.

That means, the uniform velocity U is now getting to some v actually.

So, the velocity of the fluid below the interface becomes from U to v and for the fluid above the interface becomes U' to v' and that is because that the perturbation has been caused to the interface and these two must be connected and that is the key of the whole treatment. So, now, how to connect them for that we have to think a bit physics of that.

So, when the two-fluid interface is perturbed slightly a fluid element of the lower fluid which is situated infinitesimally close to the interface is also plucked. What is the meaning of that? Just go to this image. So, when this interface is now plucked to this position actually what happens due to plucking some fluid element which is infinitesimally close to this interface and which was the part of this fluid actually comes to this fluid.

Actually, you can see the same happens for the fluid above, some part of the fluid above actually some fluid elements would come into this part right because you see that is very easy. Because now this is the new interface, so, new interface simply says that this will be then the fluid above and this will be then the fluid below. So, the fluid below actually penetrates a bit in the previous region of the fluid above and here the opposite thing is happening, I think this part is clear.

So, in terms of the velocity potential the vertical velocity component is nothing but $-\frac{\partial \varphi_1}{\partial z}$. So, this is nothing, but $-\vec{\nabla}\varphi$ and along the z direction. So, that is the velocity component of the fluid in terms of its velocity potential in the vertical direction, and that should be exactly equal to the Lagrangian velocity of the fluid elements.

So, this is because the fluid configuration is perturbed. So, that the Eulerian velocity potential field is perturbed is displaced. Now, along with the Eulerian field is perturbed the fluid particle is displaced actually right that is the physical consequence.

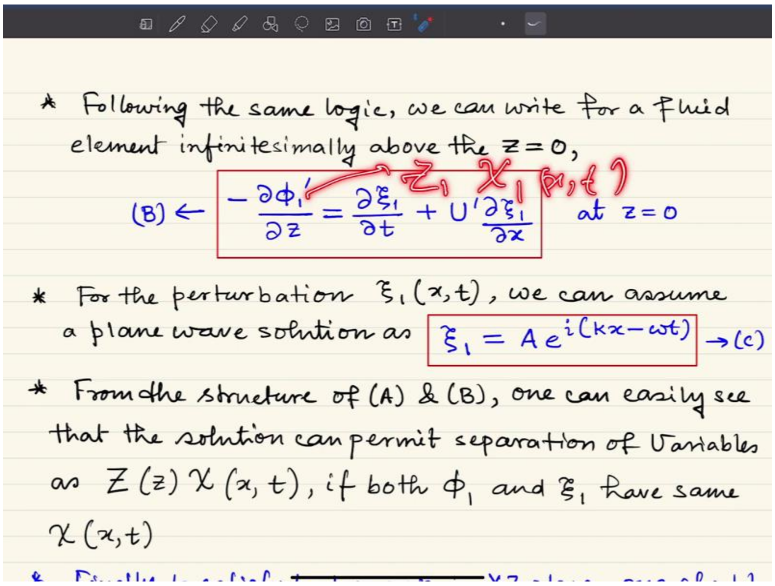
And with that fluid particle, we can actually say that the Eulerian velocity field at some point at z is equal to 0 will exactly be equal to the, due to perturbation, Lagrangian velocity of a fluid particle situated at that point. So, once again, when the surface is displaced here with this surface of the fluid particle which was situated here now comes here.

So, now this is infinitesimally distant. So, you can actually think that the vertical velocity component will be exactly equal here and here and that is nothing, but $\frac{\partial \varphi_1}{\partial z}$ that is the Lagrangian velocity. So, they only have one component and that is the vertical component because the fluid element is only moving in the vertical direction because the perturbation is made along the z direction and then these two must be equal.

So, $\frac{\partial \varphi_1}{\partial z}$ have two parts $\frac{\partial \xi_1}{\partial t}$ and $U\frac{\partial \xi_1}{\partial x}$ So, why only this x component is there? Because U only have the x component and this is true only for z is equal to 0 or in the neighborhood of the old interface.

$$-\frac{\partial \varphi_1}{\partial z} = \frac{\partial \xi_1}{\partial t} + U\frac{\partial \xi_1}{\partial x} \quad \text{(A)}$$

(Refer Slide Time: 25:49)



If this is true, then the same thing we can also write for the fluid above. So, at interface it will also have this type of equality of the Eulerian field and the Lagrange Eulerian velocity field and the Lagrangian velocity field of the corresponding fluid particle which was situated at that point. Now, you have two equalities (A) and (B).

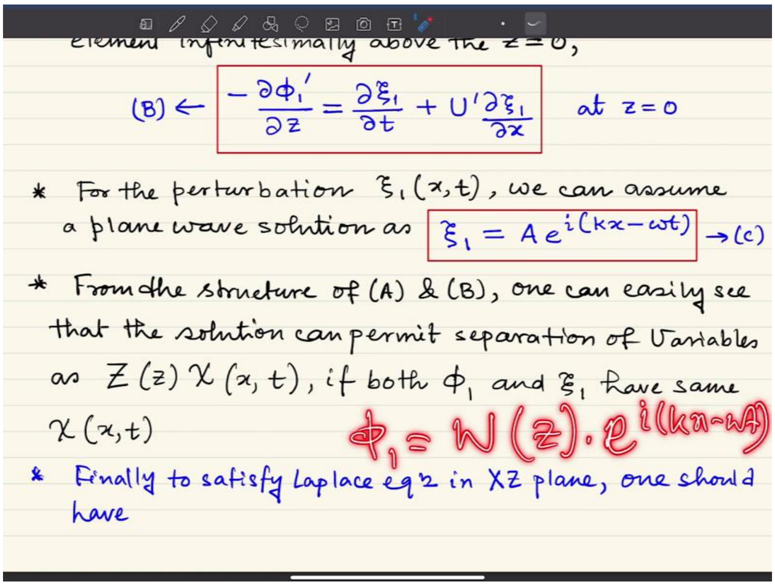
$$-\frac{\partial \varphi_1}{\partial z} = \frac{\partial \xi_1}{\partial t} + U' \frac{\partial \xi_1}{\partial x} \qquad \text{(B)}$$

So, for the perturbation, we now have to guess some solution for this perturbed quantity ξ_1 and φ_1. For ξ_1, this is quite easy because we can see that this is just a perturbation, this perturbation can only be a function of x and t and both in x and t, we do not have any others source of disturbance or something. So, this is plucked and then it is totally released. So, we can simply assume a plane wave type of solution.

Because finally, once again these equations, where you can see the ξ_1 is linear in nature. So, once again we can assume that they can be thought to be composed of Fourier components and we take any one Fourier component which looks like this. So, ξ_1 is equal to $Ae^{i(kx-\omega t)}$.

$$\xi_1 = Ae^{i(kx-\omega t)} \qquad \text{(C)}$$

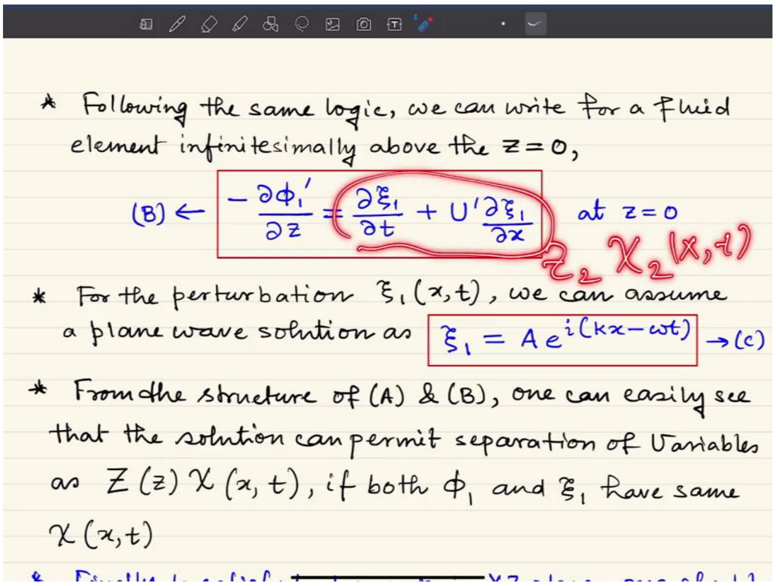
(Refer Slide Time: 27:20)



Now from the structure of (A) and (B) that is something we have to go a bit slower. For the structure of (A) and (B), so this is the part (A) and this is the part for (B). So, just a minute, one can actually see that the solutions can permit separation of variables as in the form of some Z, which is a function of small z times a function χ, which is a function of x and t, if in this form one can see that the solution can be permitted, if both φ_1 and ξ_1 have the same x and t dependence part.

If not, you can actually try. So, you start with for example, for φ_1, you start with some $Z_1\chi_1(x,t)$.

(Refer Slide Time: 28:28)

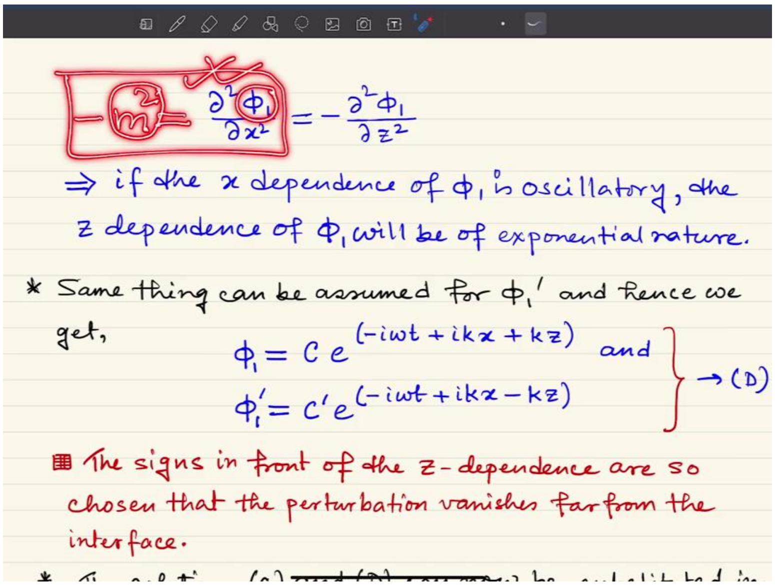


And for ξ_1, you start with some $Z_2\chi_2(x,t)$ and you will see that when χ_1 and χ_2, they are exactly equal then only you can treat the whole system in terms of the separation of variables. And actually, to be very honest, there are no perturbation is made along the Z direction.

That is true. So, in the x and t direction there is no very prominent difference between the structure of φ_1 perturbation and ξ_1 perturbation or even φ_1'. So, they can have same type of dependence. So, that is true that there is no hard and fast rule or hard and fast reason for which we can have this dependence identical just because this analytics part becomes simple or suitable for separation of variable method.

Now finally, once we can say that we can actually have for both φ_1, φ_1' and ξ_1 for all this three this x and t dependent part will be exactly equal. So, then for φ_1 and φ_2, we can actually use some $W(z)$ times $e^{i(kx-\omega t)}$ this type of thing right. So, because this part $e^{i(kx-\omega t)}$ would not change and same thing for φ_1'. Finally, do not forget that φ_1 has to satisfy also a Laplace's equation.

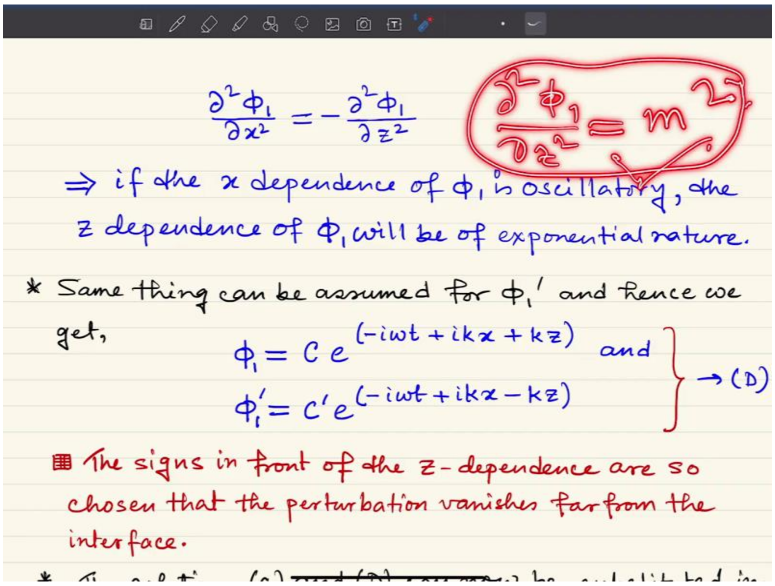
(Refer Slide Time: 30:38)



So, finally, to satisfy the Laplace's equation in $X - Z$ plane, one should have this one $\frac{\partial^2 \varphi_1}{\partial x^2}$ is equal to $-\frac{\partial^2 \varphi_1}{\partial z^2}$ because we are not taking into account the y direction variation. So, if you see that the x the dependence of φ_1 is of the type of oscillatory here as we have already said.

If that is oscillatory then that is exactly equal to some $-m^2$. So, where m^2 is some positive quantity. So, that is the condition that φ_1 can have an oscillatory solution in a function of x.

(Refer Slide Time: 31:29)



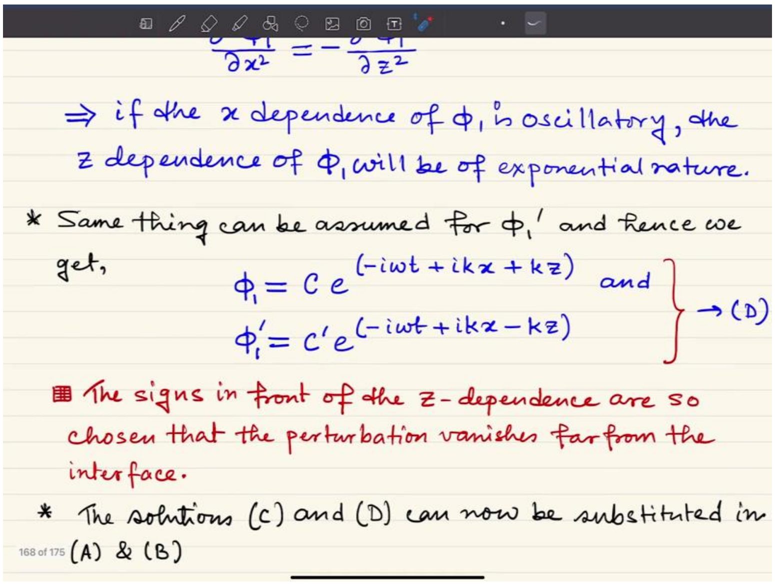
Then $\frac{\partial^2 \varphi_1}{\partial z^2}$ must be exactly equal to simply m^2 and which gives you an exponential dependence either growing or decaying of φ_1 with respect to z, and that is exactly exploited here to write this whole expression. So, φ_1 can be written then is equal to

$$\varphi_1 = Ce^{(-i\omega t+ikx+kz)}$$

$$\varphi_1' = C'e^{(-i\omega t+ikx-kz)} \qquad \text{(D)}$$

Now, we can easily understand why there is only kz and $-kz$, but why there is plus and there is minus that we cannot understand now.

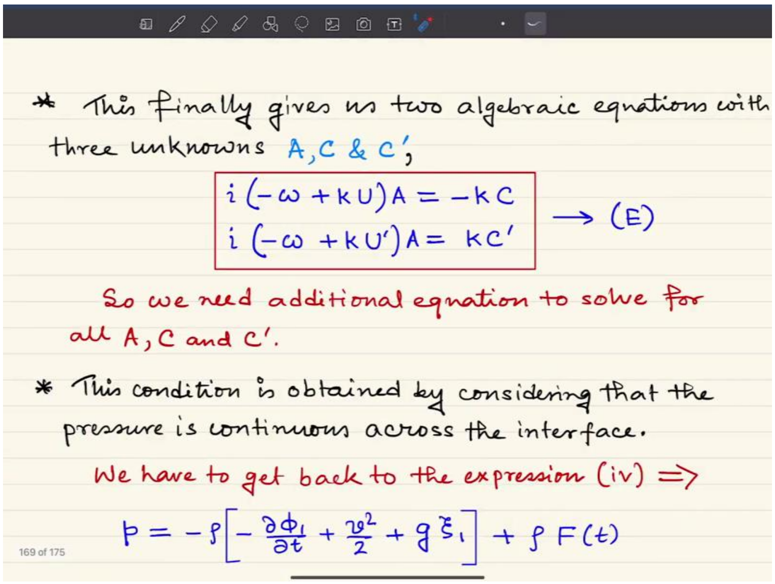
(Refer Slide Time: 32:17)



For understanding that we have to realize that this type of perturbation φ_1 and φ_1' should vanish if we go really far away from the interface that means, when z value is very large then both φ_1 and φ_1', z dependence should go away.

And what is the solution for that? We just write in this way. So, that for φ_1 this is the fluid below so it deals with negative z values. So, when z is very very large, this term will go away will be tending to 0. For φ_1', it deals with z positive values. So, φ_1' has a minus sign in z term. So, when z is very large that z dependence will vanish. So, that is the justification then combining all our elements finally, the solution (C) and (D) can now be substituted in (A) and (B) equation.

(Refer Slide Time: 33:34)



If we do that properly we can see that finally, we have two algebraic equations like this

$$i(-\omega + kU)A = -kC$$

$$i(-\omega + kU')A = -kC'$$

So, we have two algebraic equations, but how many unknown variables we have now? We have A, C, and C'.

So, we cannot eliminate A, C, and C' from these equations that means, we cannot solve these to get an equation or a relation between ω and k, which is known as the dispersion relation. So, for having or obtaining the dispersion relation, we need another supplementary equation or relation and that will be discussed in the next discussion.

Thank you.